Combining Linguistics and Proof Checking of Mathematical Texts

by Peter Koepke

Mathematical Institute, University of Bonn, Germany

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The Princeton Companion to Mathematics, ed. Timothy Gowers: "The Language and Grammar of Mathematics":

1. Every nonempty set of positive integers has a least element.

2. $\forall A \subset \mathbb{N}$
   $[(\exists n \in \mathbb{N} n \in A)$
   $\Rightarrow (\exists x \in A \forall y \in A ((y > x) \vee (y = x))]$.

The Princeton Companion to Mathematics, ed. Timothy Gowers: "The Language and Grammar of Mathematics":

> "The ideal is to write in as friendly and approachable a way as possible, while making sure that the reader (who, one assumes, has plenty of experience and training in how to read mathematics) *can see easily how what one writes could be made more formal if it became important to do so.* And sometimes it does become important: when an argument is difficult to grasp it may be that the only way to convince oneself that it is correct is to rewrite it more formally. [Emphases by PK]

The Princeton Companion to Mathematics, ed. Timothy Gowers: "The Language and Grammar of Mathematics":

If we wish to translate this into a more formal language we need to strip it of words and phrases such as "nonempty" and "has". But this is easily done. To say that a set A of positive integers is nonempty is simply to say that there is a positive integer that belongs to A. This can be stated symbolically:

$$(16)\ \exists n \in \mathbb{N}\, n \in A.$$

What does it mean to say that A has a least element? [...] This formulation is again ready to be translated into symbols:

$$(17)\ \exists x \in A\, \forall y \in A\ (y > x) \vee (y = x).$$

[...] Thus it can be written symbolically as follows:

$$\begin{aligned}(18)\ \ & \forall A \subset \mathbb{N} \\ & \quad [(\exists n \in \mathbb{N}\, n \in A) \\ & \quad\quad \Rightarrow (\exists x \in A \forall y \in A\, ((y > x) \vee (y = x))].\end{aligned}$$

The Princeton Companion to Mathematics, ed. Timothy Gowers, "The Language and Grammar of Mathematics":

> In practice, there are many different levels of formality, and mathematicians are adept at switching between them. It is this that makes it possible to feel completely confident in the correctness of a mathematical argument even when it is not presented in the manner of (18) – though it is also this that allows mistakes to slip through the net from time to time.

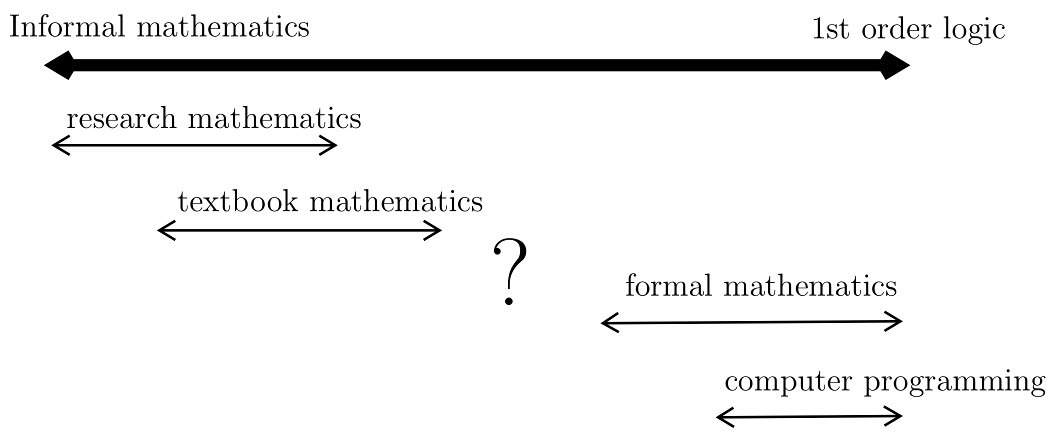
Serge Lang, Mathematics - Form and Function:

> As to precision, we have now stated an absolute standard of rigor: A mathematical proof is rigorous when it is (or could be) written out in the first-order predicate language $L(\in)$ as a sequence of inferences from the axioms ZFC, each inference made according to one of the stated rules. [...] When a proof is in doubt, its repair is usually a partial approximation to the fully formal version.

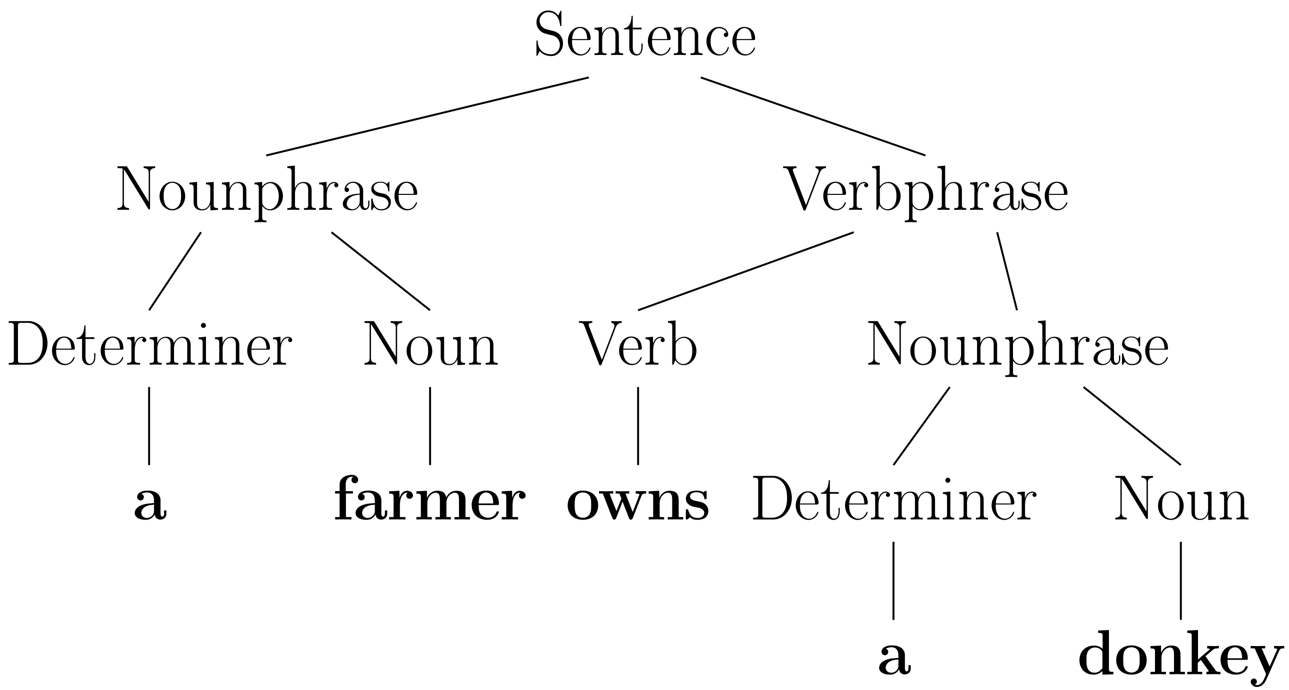
Jody Azzouni, The Derivation-Indicator View of Mathematical Practice:

ABSTRACT. A version of Formalism is vindicated: Ordinary mathematical proofs indicate (one or another) mechanically checkable derivation of theorems from the assumptions those ordinary mathematical proofs presuppose. The indicator view explains why mathematicians agree so readily on results established by proofs in ordinary language that are (palpably) not mechanically checkable. Mechanically checkable derivations in this way structure ordinary mathematical practice without its being the case that ordinary mathematical proofs can be 'reduced to' such derivations.

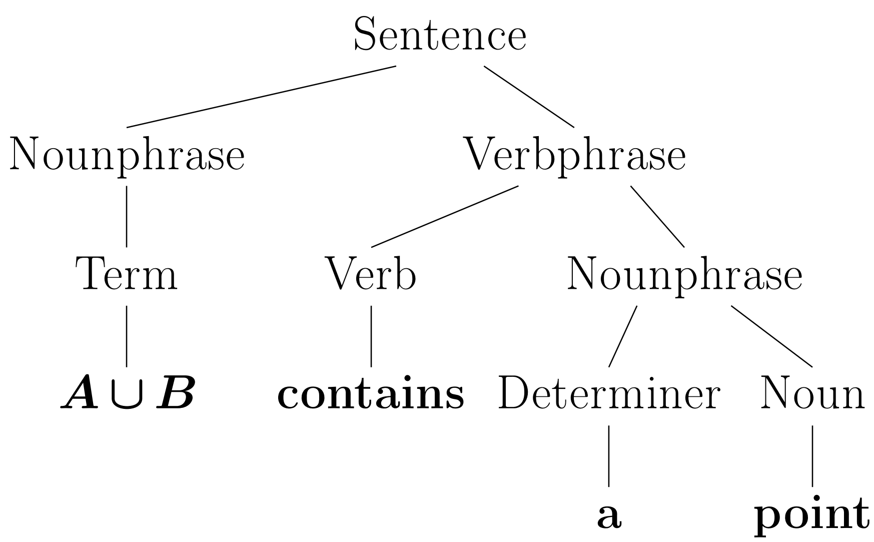
Various Registers of Formality



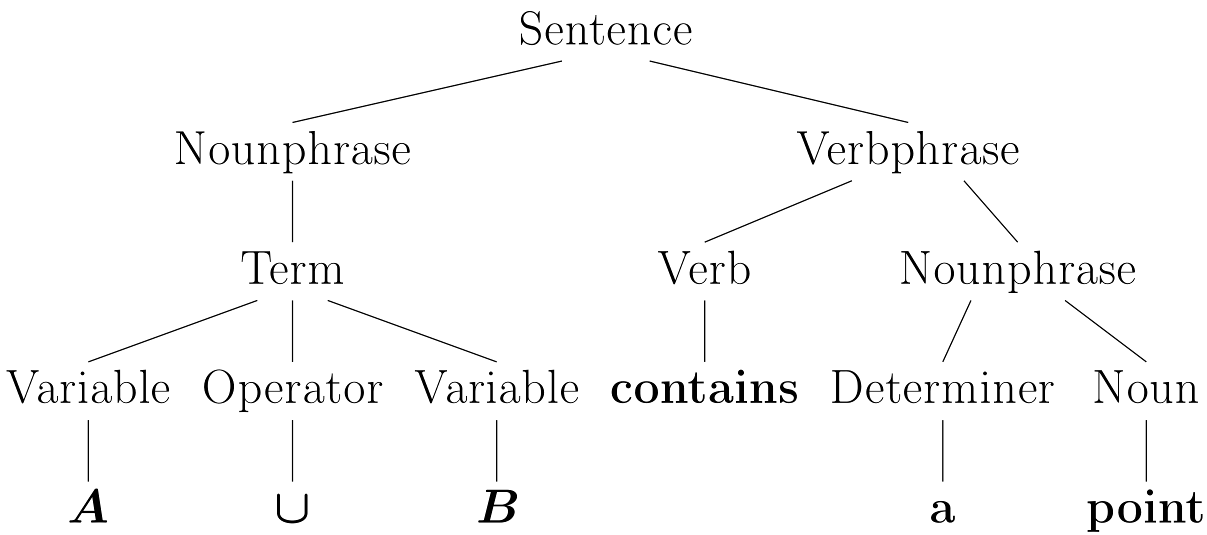
Linguistic Phrase Structure Parsing



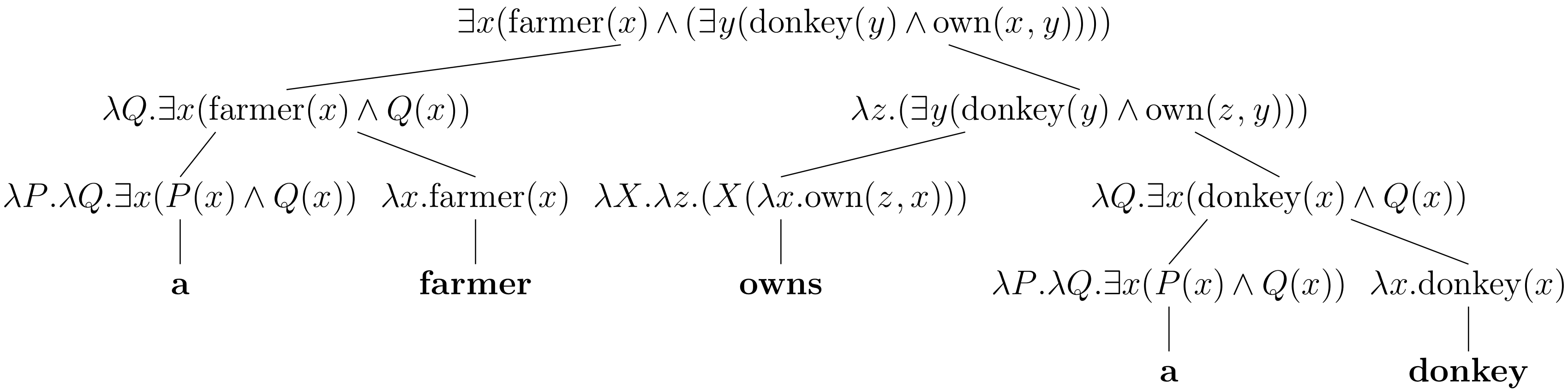
Linguistic Phrase Structure Parsing



Phrase Structure Parsing and Formula Parsing



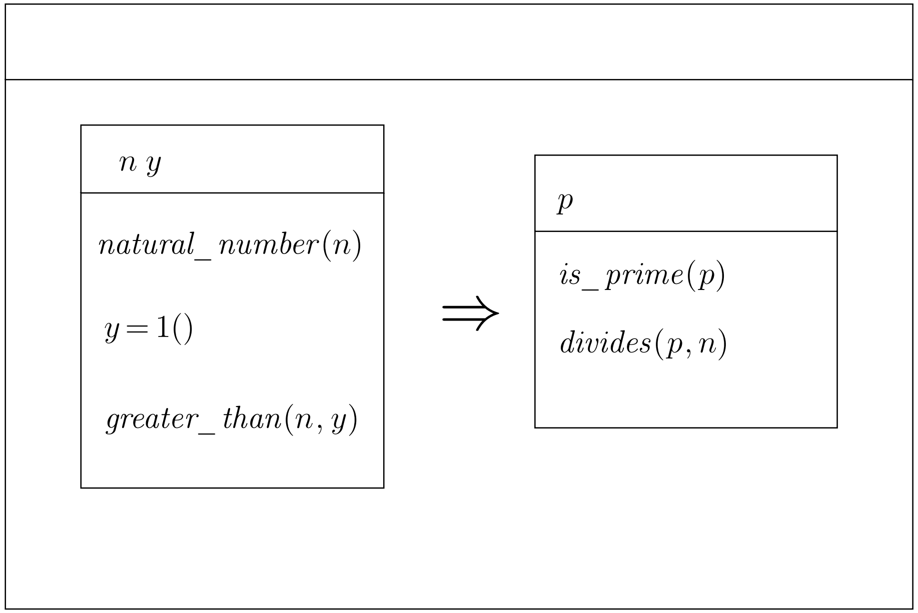
First-Order Compositional Semantics



Hypothesis: Adequate first-order renderings of mathematical statements can be obtained by standard linguistic techniques, in particular by phrase structure grammars and compositional semantics.

Discourse Relation Theory and Discourse Relation Structures

"Let n be a natural number which is greater than 1. Then there is a prime p such that $p|n$."



Hypothesis: Adequate first-order renderings of mathematical texts can be obtained by techniques like Discourse Representation Theory.

Stanislaw Jaskowski, Natural Deduction

1.	$((P \supset Q) \& (\sim R \supset \sim Q))$	Supposition
2.	P	Supposition
3.	$((P \supset Q)) \& (\sim R \supset \sim Q))$	1. Repeat
4.	$(P \supset Q)$	3, Simplification
5.	Q	2,4 Modus Ponens
6.	$(\sim R \supset \sim Q)$	3, Simplification
7.	$\sim R$	Supposition
8.	$(\sim R \supset \sim Q)$	6, Repeat
9.	$\sim Q$	7,8 Modus Ponens
10.	Q	5, Repeat
11.	R	7-10 Reductio ad Absurdum
12.	$P \supset R$	2-11 Conditionalization
13.	$(((P \supset Q) \& (\sim R \supset \sim Q)) \supset (P \supset R))$	1-12 Conditionalization

Formal Mathematics:

– Avigad et al: Prime Number Theorem (elementary proof)

– John Harrison: Prime Number Theorem (analytic proof)

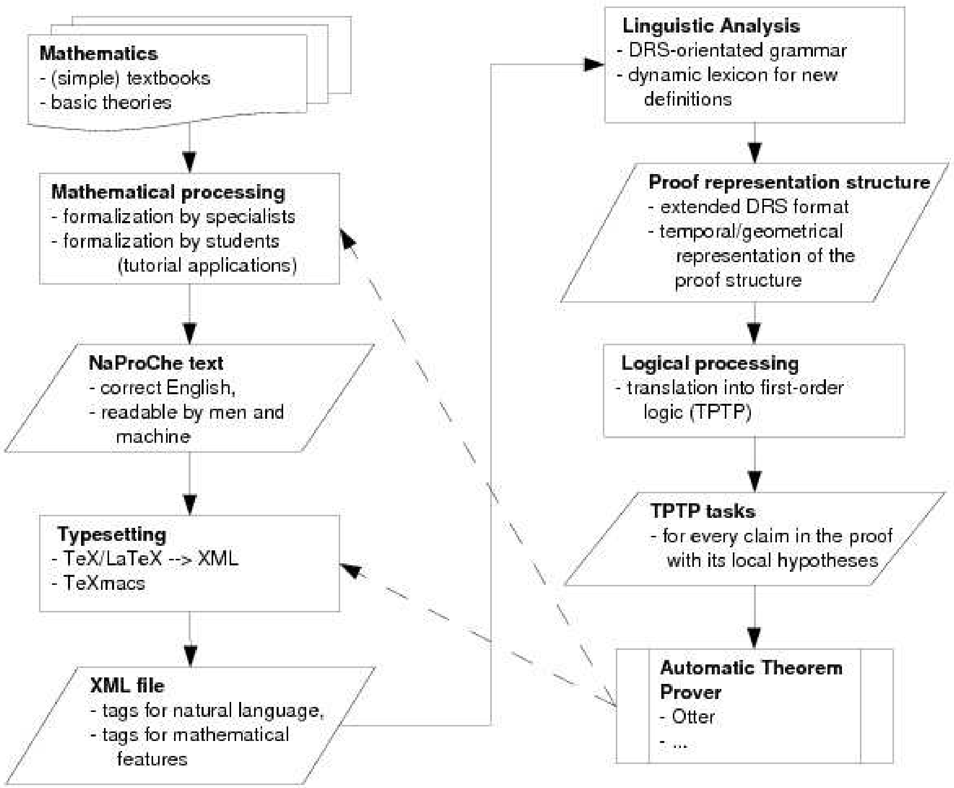
– Thomas Hales: Jordan Curve Theorem

– Georges Gonthier: Four-Colour Theorem

– ...

– ...

Naproche: Natural Language Proof Checking



Naproche Example:

Theorem 31 (Associative Law of Multiplication):

$$(xy)z = x(yz).$$

Proof: Fix x and y, and let $\mathfrak{M}$ be the set of all z for which the assertion holds true.

I) $(xy) \cdot 1 = xy = x(y \cdot 1);$

hence 1 belongs to $\mathfrak{M}$.

II) Let z belong to $\mathfrak{M}$. Then

$$(xy)z = x(yz).$$

and therefore, using Theorem 30,

$$(xy)z' = (xy)z + xy = x(yz) + xy = x(yz + y) = x(yz'),$$

so that z' belongs to $\mathfrak{M}$. Therefore $\mathfrak{M}$ contains all natural numbers.

Theorem 31: For all x, y, z, $(x * y) * z = x * (y * z)$.
Proof:
Fix x, y. Then $(x * y) * 1 = x * y = x * (y * 1)$.
Now suppose $(x * y) * z = x * (y * z)$.
Then by theorem 30, $(x * y) * z' = ((x * y) * z) + (x * y) = (x * (y * z)) + (x * y) = x * ((y * z) + y) = x * (y * z')$.
Thus by induction, for all z $(x * y) * z = x * (y * z)$. Qed.

Theses

- In the near future, formal methods will allow to formulate substantial mathematical texts in a humanly acceptable language which is translatable and checkable by computer.

- Formal mathematics will become more widespread and will have (controlled) natural language interfaces with LaTeX-quality typesetting.

- This may lead to a new kind of formalism, in which natural language mathematical texts are automatically augmented by formal translations and verifications. Such a "fortified" formalism requires sophisticated formal methods and strong computing power.

- Fortified formalism may be viewed as a vindication and realisation of the derivation indicator view.

Theses:

- Research towards these goals is shedding light on the human processing of mathematical ideas and arguments. Formal methods like discourse representation or unification and resolution in automatic proving may model mental actions in formulating and checking mathematics.

- If the advantages of computer processing and checking outway the restrictions, mathematicians will accept to formulate (parts of their) arguments in sufficiently rich controlled natural languages instead of free natural language.

- Formal mathematics will have a revolutionary impact on practical mathematical work in research, teaching and applications.

- In the long run practical formal mathematics will influence many issues the Philosophy of Mathematical Practice is concerned with.