A set $X \subseteq \delta$ is homogeneous for a partition $F$: $[\delta]^{<\omega} \rightarrow 2$ iff
\[
\forall n |F[[X]^n]| = 1;
\]
the partition property $\delta \rightarrow (\alpha)_2^{<\omega}$ is defined as
\[
\forall F: [\delta]^{<\omega} \rightarrow \exists X \subseteq \kappa (\text{otp}(X) \geq \alpha \wedge X \text{ is homogeneous for } F).
\]

An infinite cardinal $\kappa$ is $\alpha$-Erdős iff $\kappa \rightarrow (\alpha)_2^{<\omega}$,
it is Ramsey iff $\kappa \rightarrow (\kappa)_2^{<\omega}$.

**Definition 1.** An infinite cardinal $\kappa$ is almost Ramsey iff
\[
\forall \alpha < \kappa \kappa \rightarrow (\alpha)_2^{<\omega}.
\]
For any uncountable almost RAMSEY cardinal \( \kappa \) the following substructure property holds: if \( \lambda, \kappa', \lambda' \) are infinite cardinals satisfying \( \lambda \leq \kappa, \lambda' \leq \kappa' < \kappa \), and \( \lambda' \leq \lambda \) then \((\kappa, \lambda) \Rightarrow (\kappa', \lambda')\), which means that every first-order structure \((\kappa, \lambda, \ldots)\) with a countable language has an elementary substructure \(X \prec (\kappa, \lambda, \ldots)\) with \(|X| = \kappa'\) and \(|X \cap \lambda| = \lambda'\).

\textbf{Theorem 2.} \(\text{Con}(\text{ZFC} + \text{There exist cardinals } \kappa < \lambda \text{ such that } \kappa \text{ is } 2^\lambda \text{ supercompact where } \lambda \text{ is the least regular almost RAMSEY cardinal greater than } \kappa) \Rightarrow \text{Con}(\text{ZF + AC} + \text{Every successor cardinal is regular + Every (well-ordered) uncountable cardinal is almost RAMSEY}).\)

\textbf{Theorem 3.} Assume ZF and that every infinite cardinal is almost RAMSEY. Then there exists an inner model with a strong cardinal.

\textbf{Theorem 4.} The following theories are equiconsistent

   a) ZFC + There is a proper class of regular almost RAMSEY cardinals;

   b) ZF + All infinite cardinals except possibly successors of singular limit cardinals are almost RAMSEY.
Definition 5. For $\alpha \in \text{Ord}$ let $\kappa(\alpha)$ be the least $\kappa$ such that $\kappa \rightarrow (\alpha)^{\omega} \downarrow$, if such a $\kappa$ exists.

Proposition 6. (ZF) An infinite cardinal $\kappa$ is almost Ramsey iff $\kappa(\alpha)$ is defined for all $\alpha < \kappa$ and $\kappa = \bigcup_{\alpha < \kappa} \kappa(\alpha)$.

Proposition 7. (ZFC) Assume $\kappa$ is almost Ramsey. Then
\begin{itemize}
  \item[a)] $\forall \alpha < \kappa \ \kappa(\alpha) < \kappa$;
  \item[b)] $\kappa$ is a strong limit cardinal.
\end{itemize}

Proposition 8. Let $M$ be a transitive model of “ZF + $\kappa$ is almost Ramsey”. Let $N \supseteq M$ be a transitive model of ZFC such that $\forall \delta < \kappa \ \mathcal{P}(\delta) \cap M = \mathcal{P}(\delta) \cap N$. Then $\kappa$ is almost Ramsey in $N$.

Proof. Let $\alpha < \kappa$. By Proposition 7, $\kappa(\alpha)^M < \kappa$. $\mathcal{P}(\kappa(\alpha)^M) \cap M = \mathcal{P}(\kappa(\alpha)^M) \cap N$ implies that $\kappa(\alpha)^N = \kappa(\alpha)^M$. Hence $\kappa = \bigcup_{\alpha < \kappa} \kappa(\alpha)^N$ and $\kappa$ is almost Ramsey in $N$. \qed

Proposition 9. (ZFC)
\begin{itemize}
  \item[a)] Assume $\lambda$ is a Ramsey cardinal. Then the class of almost Ramsey cardinals is closed unbounded below $\lambda$ and the class of regular almost Ramsey cardinals is stationary below $\lambda$.
  \item[b)] Assume $\kappa$ is an uncountable regular almost Ramsey cardinal. Then the class of almost Ramsey cardinals is closed unbounded below $\lambda$.
\end{itemize}

Proposition 10. ZFC + There exists an uncountable regular almost Ramsey cardinal $\vdash \text{Con}(\text{ZFC} + \text{There exists a proper class of (singular) almost Ramsey cardinal})$. 

Proposition 11. (ZF) For infinite ordinals $\kappa$ the partition property $\kappa \rightarrow (\alpha)^{\leq \omega}_2$ is equivalent to: for any first-order structure $\mathcal{M} = (M, ...)$ in a countable language $S$ with $\kappa \subseteq M$ there is a set $X \subseteq \kappa$, $\text{otp}(X) \geq \alpha$ of indiscernibles, i.e., for all $S$-formulas $\varphi(v_0, ..., v_{n-1})$, $x_0, ..., x_{n-1} \in X$, $x_0 < ... < x_{n-1}$, $y_0, ..., y_{n-1} \in X$, $y_0 < ... < y_{n-1}$ holds

$$\mathcal{M} \models \varphi(x_0, ..., x_{n-1}) \iff \mathcal{M} \not\models \varphi(y_0, ..., y_{n-1}).$$

Proposition 12. (ZF) Assume $\kappa \rightarrow (\alpha)^{\leq \omega}_2$ where $\alpha$ is a limit ordinal. Then for any first-order structure $\mathcal{M} = (M, ...)$ in a countable language $S$ with $\kappa \subseteq M$ there is a set $X \subseteq \kappa$, $\text{otp}(X) \geq \alpha$ of good indiscernibles, i.e., for all $S$-formulas $\varphi(v_0, ..., v_{m-1}, w_0, ..., w_{n-1})$, $x_0, ..., x_{n-1} \in X$, $x_0 < ... < x_{n-1}$, $y_0, ..., y_{n-1} \in X$, $y_0 < ... < y_{n-1}$, and $a_0 < ... < a_{m-1} < \min(x_0, y_0)$ holds

$$\mathcal{M} \models \varphi(a_0, ..., a_{m-1}, x_0, ..., x_{n-1}) \iff \mathcal{M} \not\models \varphi(a_0, ..., a_{m-1}, y_0, ..., y_{n-1}).$$
Proof. We may assume that the structure $\mathcal{M}$ contains a unary predicate $\text{Ord}$ for the ordinals in $\mathcal{M}$ ($= \kappa$) and a collection of Skolem functions for ordinal-valued existential statements, i.e., for every $S$-formula $\varphi(v, \bar{w})$ there is a function $f$ of $\mathcal{M}$ such that

$$M \models \forall \bar{w} (\exists v (\text{Ord}(v) \land \varphi(v, \bar{w})) \rightarrow \varphi(f(\bar{w}), \bar{w})).$$

Choose a set $X \subseteq \kappa$, $\text{otp}(X) \geq \alpha$ of indiscernibles for $\mathcal{M}$ such that the minimum $\min(X)$ is minimal for all such sets of indiscernibles. Assume for a contradiction that $X$ is not good. Then there is an $S$-formula $\varphi(v_0, \ldots, v_{n-1})$, $x_0, \ldots, x_{n-1} \in X$, $x_0 < \ldots < x_{n-1}$, $y_0, \ldots, y_{n-1} \in X$, $y_0 < \ldots < y_{n-1}$ and $a_0 < \ldots < a_{m-1} < \min(x_0, y_0)$ such that

$$M \models \varphi(a_0, \ldots, a_{m-1}, x_0, \ldots, x_{n-1}) \quad \text{and} \quad M \not\models \varphi(a_0, \ldots, a_{m-1}, y_0, \ldots, y_{n-1}).$$

Since $\alpha$ is a limit ordinal we can take $z_0, \ldots, z_{n-1} \in X$, $z_0 < \ldots < z_{n-1}$ such that $x_{n-1} < z_0$ and $y_{n-1} < z_0$. In case $M \models \varphi(a_0, \ldots, a_{m-1}, z_0, \ldots, z_{n-1})$, one has

$$M \models \neg \varphi(a_0, \ldots, a_{m-1}, y_0, \ldots, y_{n-1}) \quad \text{and} \quad M \models \varphi(a_0, \ldots, a_{m-1}, z_0, \ldots, z_{n-1})$$

where $y_0 < \ldots < y_{n-1} < z_0 < \ldots < z_{n-1}$.

In case $M \models \neg \varphi(a_0, \ldots, a_{m-1}, z_0, \ldots, z_{n-1})$, one has

$$M \models \varphi(a_0, \ldots, a_{m-1}, x_0, \ldots, x_{n-1}) \quad \text{and} \quad M \not\models \varphi(a_0, \ldots, a_{m-1}, z_0, \ldots, z_{n-1})$$

where $x_0 < \ldots < x_{n-1} < z_0 < \ldots < z_{n-1}$. So in both cases we have an ascending $2n$-tuple of indiscernibles, such that the first half behaves differently from the second half with respect to the formula $\varphi$ and the parameters $a_0, \ldots, a_{m-1}$. So without loss of generality we may assume that $x_0 < \ldots < x_{n-1} < y_0 < \ldots < y_{n-1}$ and

$$M \models \varphi(a_0, \ldots, a_{m-1}, x_0, \ldots, x_{n-1}) \quad \text{and} \quad M \not\models \varphi(a_0, \ldots, a_{m-1}, y_0, \ldots, y_{n-1}).$$
Write \( \vec{x} = x_0, \ldots, x_{n-1} \) and \( \vec{y} = y_0, \ldots, y_{n-1} \). Since \( M \) contains Skolem functions there are functions \( f_0, \ldots, f_{m-1} \) of \( M \) which compute parameters like \( a_0, \ldots, a_{m-1} \):

\[
M \models \exists v_1 < x_0 \exists v_2 < x_1 \ldots \exists v_{m-1} < x_0 (f_0(\vec{x}, \vec{y}) < x_0 \land \varphi(f_0(\vec{x}, \vec{y}), v_1, \ldots, v_{m-1}, \vec{y}))
\]

\[
M \models \exists v_2 < x_0 \ldots \exists v_{m-1} < x_0 (f_0(\vec{x}, \vec{y}) < x_0 \land f_1(\vec{x}, \vec{y}) < x_0 \land \varphi(f_0(\vec{x}, \vec{y}), f_1(\vec{x}, \vec{y}), \ldots, v_{m-1}, \vec{x})) \land \neg \varphi(f_0(\vec{x}, \vec{y}), f_1(\vec{x}, \vec{y}), \ldots, v_{m-1}, \vec{y}))
\]

\[
M \models f_0(\vec{x}, \vec{y}) < x_0 \land \ldots \land f_{m-1}(\vec{x}, \vec{y}) < x_0 \land \varphi(f_0(\vec{x}, \vec{y}), f_1(\vec{x}, \vec{y}), \ldots, f_{m-1}(\vec{x}, \vec{y}), \vec{x}) \land \neg \varphi(f_0(\vec{x}, \vec{y}), f_1(\vec{x}, \vec{y}), \ldots, f_{m-1}(\vec{x}, \vec{y}), \vec{y})
\]

Now consider \( \vec{z} = z_0, \ldots, z_{n-1} \in X \), \( z_0 < \ldots < z_{n-1} \) such that \( y_{n-1} < z_0 \).

(1) There is \( k < m \) such that \( f_k(\vec{x}, \vec{y}) \neq f_k(\vec{y}, \vec{z}) \).

**Proof.** Assume not. Set \( \xi_0 = f_0(\vec{x}, \vec{y}), \ldots, \xi_{m-1} = f_{m-1}(\vec{x}, \vec{y}) \). Then

\[
M \models \varphi(\xi_0, \xi_1, \ldots, \xi_{m-1}, \vec{x}) \land \neg \varphi(\xi_0, \xi_1, \ldots, \xi_{m-1}, \vec{y})
\]

and

\[
M \models \varphi(\xi_0, \xi_1, \ldots, \xi_{m-1}, \vec{y}) \land \neg \varphi(\xi_0, \xi_1, \ldots, \xi_{m-1}, \vec{z}).
\]

In particular

\[
M \models \varphi(\xi_0, \xi_1, \ldots, \xi_{m-1}, \vec{y}) \land \neg \varphi(\xi_0, \xi_1, \ldots, \xi_{m-1}, \vec{y}),
\]

which is a contradiction. \( \square \)

So take \( k < m \) such that

(2) \( f_k(\vec{x}, \vec{y}) \neq f_k(\vec{y}, \vec{z}) \).

Let \( (\nu_i | i < \alpha) \) be a strictly increasing enumeration of the set \( X \) of indiscernibles, and let \( (\vec{x}^{(i)} | i < \alpha) \) with

\[
\vec{x}^{(i)} = \nu_{n-i}, \nu_{n-i+1}, \ldots, \nu_{n-i+n-1}
\]
be a partition of $X$ into ascending sequences of length $n$.

(3) $f_k(\vec{\alpha}^{(0)}, \vec{\alpha}^{(1)}) < f_k(\vec{\alpha}^{(1)}, \vec{\alpha}^{(2)})$.

**Proof.** By indiscernibility, (2) implies that $f_k(\vec{\alpha}^{(0)}, \vec{\alpha}^{(1)}) \neq f_k(\vec{\alpha}^{(1)}, \vec{\alpha}^{(2)})$. Assume for a contradiction that $f_k(\vec{\alpha}^{(0)}, \vec{\alpha}^{(1)}) > f_k(\vec{\alpha}^{(1)}, \vec{\alpha}^{(2)})$. Then again by indiscernibility we would obtain a *decreasing* sequence

$$f_k(\vec{\alpha}^{(0)}, \vec{\alpha}^{(1)}) > f_k(\vec{\alpha}^{(1)}, \vec{\alpha}^{(2)}) > f_k(\vec{\alpha}^{(2)}, \vec{\alpha}^{(3)}) > \ldots,$$

contradiction. □

But then

$$f_k(\vec{\alpha}^{(0)}, \vec{\alpha}^{(1)}) < f_k(\vec{\alpha}^{(2)}, \vec{\alpha}^{(3)}) < f_k(\vec{\alpha}^{(4)}, \vec{\alpha}^{(5)}) < \ldots$$

is an *ascending* $\alpha$-sequence of indiscernibles for $M$ with smallest element $f_k(\vec{\alpha}^{(0)}, \vec{\alpha}^{(1)}) < \nu_0$ which contradicts the minimal choice of $\min(X)$.

□

**Lemma 13. (ZF)** Let $\kappa^+$ be almost Ramsey. Then $(\kappa^+)^\text{HOD} < \kappa^+$.

**Proof.** Assume for a contradiction that $(\kappa^+)^\text{HOD} = \kappa^+$. For $\gamma \in [\kappa, \kappa^+]$ choose the $<_\text{HOD}$-least bijection $f_\gamma : \gamma \leftrightarrow \kappa$. Define $F : [\kappa]^3 \to 2$ by

$$F(\{\alpha, \beta, \gamma\}) = \begin{cases} 0 & \text{iff } f_\gamma(\alpha) < f_\gamma(\beta) \\ 1 & \text{iff } f_\gamma(\alpha) > f_\gamma(\beta) \end{cases}, \text{ for } \alpha < \beta < \gamma.$$
Take $X \subseteq \kappa^+$ homogeneous for $F$ with $\text{otp}(X) = \kappa + 2$. Let $\gamma = \max(X)$. Then define $h: \kappa + 1 \rightarrow \kappa$ by $h(\xi) = f_\gamma(\alpha_\xi)$ where $\alpha_\xi$ is the $\xi$-th element of $X$. 

**Case 1:** $\forall x \in [X]^3 F(x) = 0$. Then for $\xi < \zeta < \kappa + 1$ we have: $\alpha_\xi < \alpha_\zeta < \gamma$, $\{\alpha_\xi, \alpha_\zeta, \gamma\} \in [X]^3$, $F(\{\alpha_\xi, \alpha_\zeta, \gamma\}) = 0$, and so

$$h(\xi) = f_\gamma(\alpha_\xi) < f_\gamma(\alpha_\zeta) = h(\zeta).$$

Thus $h: \kappa + 1 \rightarrow \kappa$ is order preserving, which is impossible.

**Case 2:** $\forall x \in [X]^3 F(x) = 1$. Then for $\xi < \zeta < \kappa + 1$ we have: $\alpha_\xi < \alpha_\zeta < \gamma$, $\{\alpha_\xi, \alpha_\zeta, \gamma\} \in [X]^3$, $F(\{\alpha_\xi, \alpha_\zeta, \gamma\}) = 1$, and so

$$h(\xi) = f_\gamma(\alpha_\xi) > f_\gamma(\alpha_\zeta) = h(\zeta).$$

Thus $h: \kappa + 1 \rightarrow \kappa$ is a strictly descending $\kappa + 1$ chain in the ordinals, contradiction. \hfill \Box

Let $K^{\text{DJ}}$ be the canonical term for the Dodd-Jensen core model.

**Proposition 14.** (ZF) Let $a \subseteq \text{HOD}$ be a set. Then

a) $\text{HOD}[a]$ is a set-generic extension of $\text{HOD}$, so $\text{HOD}[a] \models \text{ZFC}$.

b) $(K^{\text{DJ}})^{\text{HOD}} = (K^{\text{DJ}})^{\text{HOD}[a]}$; moreover this equality holds for every level of the hierarchy, i.e., $(K^{\text{DJ}}_\alpha)^{\text{HOD}} = (K^{\text{DJ}}_\alpha)^{\text{HOD}[a]}$ for every $\alpha \in \text{Ord}$.

By the proposition we may define $K^{\text{DJ}} = (K^{\text{DJ}})^{\text{HOD}}$ in models without choice.

**Proposition 15.** Let $\kappa$ be an infinite cardinal and suppose $A \in K^{\text{DJ}} \cap P(K^{\text{DJ}}_\kappa)$, and that there is $I$, an infinite good set of indiscernibles for $A = (K^{\text{DJ}}_\kappa, A)$ and that $\text{cof}(\text{otp}(I)) > \omega$. Then there is $I' \in K^{\text{DJ}}$, $I' \supseteq I$ a set of good indiscernibles for $A$. 
Lemma 16. (ZF) Let $\kappa > \aleph_1$ be almost Ramsey. Then $\kappa$ is almost Ramsey in $K^{\text{DJ}}$.

Proof. Let $F: [\kappa]^{<\omega} \to 2$, $F \in K^{\text{DJ}}$ be a partition. Let $\alpha < \kappa$. Then $\alpha + \aleph_1 < \kappa$. By Proposition 12, take a set $X \subseteq \kappa$ of good indiscernibles for the structure $M = (K^{\text{DJ}}_\kappa, F)$ with $\text{otp}(X) \geq \alpha + \aleph_1$. Let $X'$ be the initial segment of $X$ of order type $(\alpha + \aleph_1)^{\text{HOD}(X)}$. In the model $\text{HOD}(X)$, $X'$ is a good set of indiscernibles for $M$ such that $\text{cof}(\text{otp}(X')) > \omega$. By the indiscernibles lemma applied inside $\text{HOD}(X)$ there is a set $Y \supseteq X'$, $Y \in K$ which is a good set of indiscernibles for $M$. Then $Y$ is also homogeneous for the partition $F$ of ordertype $\geq \alpha$. \qed

We are now able to prove the inner model direction of Theorem 4:

Lemma 17. $\text{Con}(ZF + \text{All infinite cardinals except possibly successors of singular limit cardinals are almost Ramsey})$ implies $\text{Con}(ZFC + \text{There is a proper class of regular almost Ramsey cardinals})$. 
Proof. Assume Con(ZF + All infinite cardinals except possibly successors of singular limit cardinals are almost Ramsey). If there is a proper class of regular almost Ramsey cardinals, we are done. So assume that this is not the case, and let the cardinal $\theta$ be an upper bound for the set of regular almost Ramsey cardinals. Then $\theta^{++}$ and $\theta^{+++}$ are not successors of limit cardinals. By assumption, $\theta^{++}$ and $\theta^{+++}$ are almost Ramsey. By the definition of $\theta$, $\theta^{++}$ and $\theta^{+++}$ must be singular. By [Sc99], this implies consistency strength far above Ramsey cardinals. \hfill $\square$

In the following we apply the core model below a strong cardinal, denoted by the class term $K$. As for the Dodd-Jensen core model we get:

**Proposition 18.** (ZF) Let $a \subseteq \text{HOD}$ be a set. Then $K^{\text{HOD}} = K^{\text{HOD}[a]}$.

If there is no inner model with a strong cardinal and the axiom of choice holds then the core model $K$ satisfies the weak covering theorem, i.e., for sufficiently large singular cardinals $\kappa$ we have $\kappa^+ = (\kappa^+)^K$. 
Lemma 19. (ZF) Let $\kappa^+$ be almost Ramsey where $\kappa$ is a singular cardinal $\geq \aleph_2$. Then there is an inner model with a strong cardinal.

Proof. Assume that there is no inner model with a strong cardinal. By Lemma 13, $(\kappa^+)^{\text{HOD}} < \kappa^+$. Since $K \subseteq \text{HOD}$, $(\kappa^+)^K < \kappa^+$. Choose a bijection $f: \kappa \leftrightarrow (\kappa^+)^K$ and a cofinal subset $Z \subseteq \kappa$ such that otp$(Z) < \kappa$. The class HOD$(f, Z)$ is a model of ZFC and it satisfies that $\kappa$ is a singular cardinal such that $(\kappa^+)^K < \kappa^+$. But this contradicts the covering theorem below $0^{\text{pistol}}$ inside the model HOD$(f, Z)$. □

Lemma 20. Assume ZF and that every infinite cardinal is almost Ramsey. Then there exists an inner model with a strong cardinal.

Proof. By assumption, $\aleph_{\omega+1}$ is almost Ramsey and the successor of the singular cardinal $\aleph_\omega \geq \aleph_2$. Now use Lemma 19. □