

# Violating the Singular Cardinals Hypothesis Without Large Cardinals\*

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## Abstract

We extend a transitive model  $V$  of ZFC + GCH cardinal preservingly to a model  $N$  of ZF + “GCH holds below  $\aleph_\omega$ ” + “there is a surjection from the power set of  $\aleph_\omega$  onto  $\lambda$ ” where  $\lambda$  is an arbitrarily high fixed cardinal in  $V$ . The construction can roughly be described as follows: add  $\aleph_{n+1}$  many COHEN subsets of  $\aleph_{n+1}$  for every  $n < \omega$ , and adjoin  $\lambda$  many subsets of  $\aleph_\omega$  which are unions of  $\omega$ -sequences of those COHEN subsets; then let  $N$  be a choiceless submodel generated by equivalence classes of the  $\lambda$  subsets of  $\aleph_\omega$  modulo an appropriate equivalence relation.

In [1], ARTHUR APTER and the second author constructed a model of ZF + “GCH holds below  $\aleph_\omega$ ” + “there is a surjection from  $[\aleph_\omega]^\omega$  onto  $\lambda$ ” where  $\lambda$  is an arbitrarily high fixed cardinal in the ground model  $V$ . This amounts to a strong *surjective* violation of the *singular cardinals hypothesis* SCH. The construction assumed a measurable cardinal in the ground model. It was also shown in [1] that a measurable cardinal in some inner model is necessary for that combinatorial property, using the DODD-JENSEN covering theorem [2].

The first author then noted that one can work without measurable cardinals if one considers surjections from  $\mathcal{P}(\aleph_\omega)$  onto  $\lambda$  instead of surjections from  $[\aleph_\omega]^\omega$  onto  $\lambda$ :

**Theorem 1.** *Let  $V$  be any ground model of ZFC + GCH and let  $\lambda$  be some cardinal in  $V$ . Then there is a cardinal preserving model  $N \supseteq V$  of the theory ZF + “GCH holds below  $\aleph_\omega$ ” + “there is a surjection from  $\mathcal{P}(\aleph_\omega)$  onto  $\lambda$ ”.*

Note that in the presence of the axiom of choice (AC) the latter theory for  $\lambda \geq \aleph_{\omega+2}$  has large consistency strength and implies the existence of measurable cardinals of high MITCHELL orders in some inner model (see [3] by the first author). The pcf-theory of SAHARON SHELAH [4] shows that the situation for  $\lambda \geq \aleph_{\omega_4}$  is incompatible with AC. Hence Theorem 1 yields a choiceless violation of pcf-theory without the use of large cardinals.

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## The forcing

Fix a ground model  $V$  of ZFC + GCH and let  $\lambda$  be some regular cardinal in  $V$ . We first present two building blocks of our construction. The forcing  $P_0 = (P_0, \supseteq, \emptyset)$  adjoins one COHEN subset of  $\aleph_{n+1}$  for every  $n < \omega$ .

$$P_0 = \{p \mid \exists (\delta_n)_{n < \omega} (\forall n < \omega: \delta_n \in [\aleph_n, \aleph_{n+1}) \wedge p: \bigcup_{n < \omega} [\aleph_n, \delta_n) \rightarrow 2)\}.$$

A standard product analysis proves that forcing with  $P_0$  does not collapse cardinals. Adjoining one COHEN subset of  $\aleph_{n+1}$  for every  $n < \omega$  is equivalent to adjoining  $\aleph_{n+1}$ -many by the following “two-dimensional” forcing  $(P_*, \supseteq, \emptyset)$ :

$$P_* = \{p_* \mid \exists (\delta_n)_{n < \omega} (\forall n < \omega: \delta_n \in [\aleph_n, \aleph_{n+1}) \wedge p_*: \bigcup_{n < \omega} [\aleph_n, \delta_n)^2 \rightarrow 2)\}.$$

For  $p_* \in P_*$  and  $\xi \in [\aleph_0, \aleph_\omega)$  let  $p_*(\xi) = \{(\beta, p_*(\xi, \zeta)) \mid (\xi, \zeta) \in \text{dom}(p_*)\}$  be the  $\xi$ -th section of  $p_*$ .

The forcing employed in the subsequent construction is a kind of finite support product of  $\lambda$  copies of  $P_0$  where the factors are eventually coupled via  $P_*$ .

**Definition 2.** Define the forcing  $(P, \leq_P, \emptyset)$  by:

$$P = \{(p_*, (a_i, p_i)_{i < \lambda}) \mid \exists (\delta_n)_{n < \omega} \exists D \in [\lambda]^{< \omega} (\forall n < \omega: \delta_n \in [\aleph_n, \aleph_{n+1}), \\ p_*: \bigcup_{n < \omega} [\aleph_n, \delta_n)^2 \rightarrow 2, \\ \forall i \in D: p_i: \bigcup_{n < \omega} [\aleph_n, \delta_n) \rightarrow 2 \wedge p_i \neq \emptyset, \\ \forall i \in D: a_i \in [\aleph_\omega \setminus \aleph_0]^{< \omega} \wedge \forall n < \omega: \text{card}(a_i \cap [\aleph_n, \aleph_{n+1})) \leq 1, \\ \forall i \notin D (a_i = p_i = \emptyset)\}.$$

If  $p = (p_*, (a_i, p_i)_{i < \lambda}) \in P$  then  $p \in P_*$  and  $p_i \in P_0$ , with all but finitely many  $p_i$  being  $\emptyset$ . Extending  $p_i$  is controlled by linking ordinals  $\xi \in a_i$ . More specifically extending  $p_i$  in the interval  $[\aleph_n, \aleph_{n+1})$  is controlled by  $\xi \in a_i \cap [\aleph_n, \aleph_{n+1})$  if that intersection is nonempty. Let  $\text{supp}(p) = \{i < \lambda \mid p_i \neq \emptyset\}$  be the support of  $p = (p_*, (a_i, p_i)_{i < \lambda})$ , i.e., the set  $D$  in the definition of  $P$ .  $P$  is partially ordered by

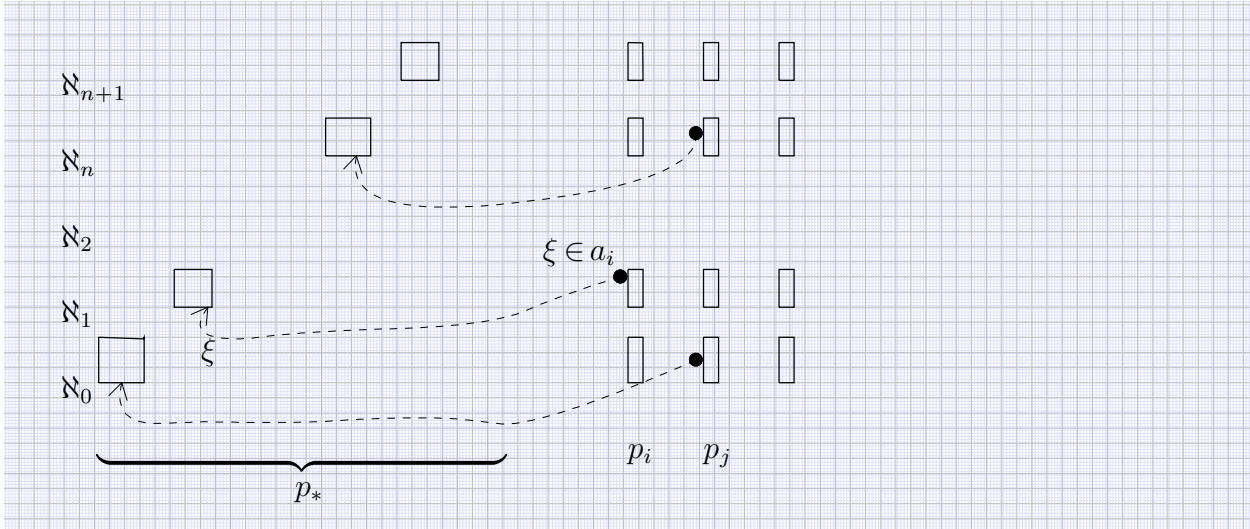
$$p' = (p'_*, (a'_i, p'_i)_{i < \lambda}) \leq_P (p_*, (a_i, p_i)_{i < \lambda}) = p$$

iff

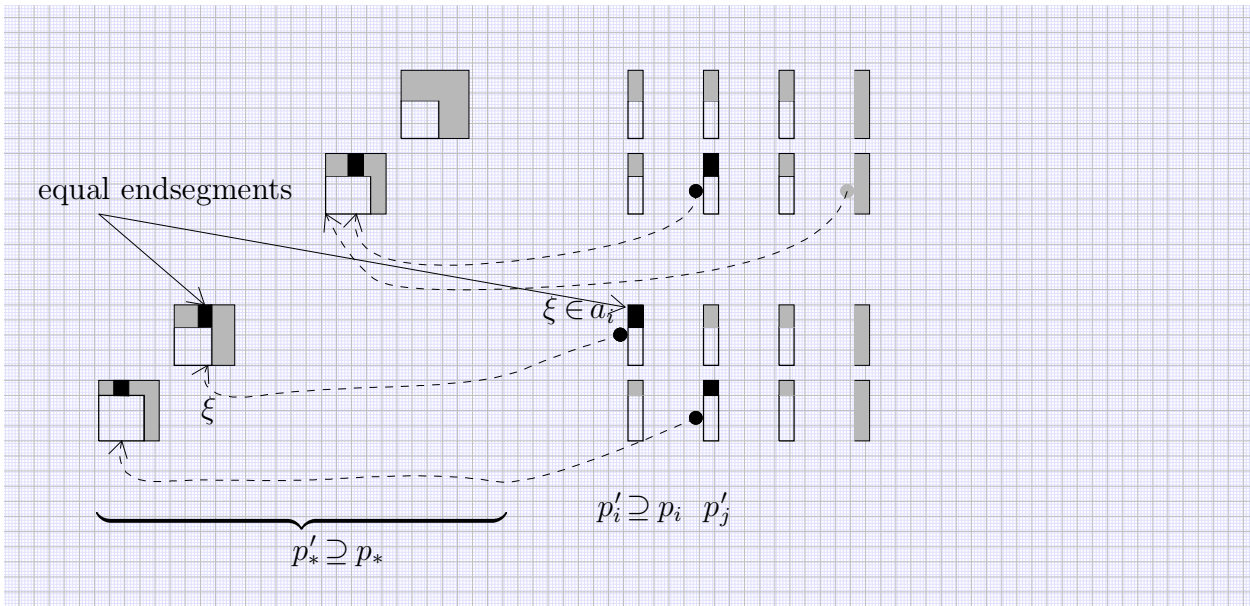
- a)  $p'_* \supseteq p_*, \forall i < \lambda (a'_i \supseteq a_i \wedge p'_i \supseteq p_i)$ ,
- b) (Linking property)  $\forall i < \lambda \forall n < \omega \forall \xi \in a_i \cap [\aleph_n, \aleph_{n+1}) \forall \zeta \in \text{dom}(p'_i \setminus p_i) \cap [\aleph_n, \aleph_{n+1})$ :  
 $p'_i(\zeta) = p'_*(\xi)(\zeta)$ , and
- c) (Independence property)  $\forall j \in \text{supp}(p): (a'_j \setminus a_j) \cap \bigcup_{i \in \text{supp}(p), i \neq j} a'_i = \emptyset$ .

$1 = (\emptyset, (\emptyset, \emptyset)_{i < \lambda})$  is the maximal element of  $P$ .

One may picture a condition  $p \in P$  as



and an extension  $(p'_*, (a'_i, p'_i)_{i < \lambda}) \leq_P (p_*, (a_i, p_i)_{i < \lambda})$  as



The gray areas indicate new 0-1-values in the extension  $p' \leq_P p$ , and the black areas indicate equality of new values forced by linking ordinals  $\xi$ .

Let  $G$  be a  $V$ -generic filter for  $P$ . Several generic objects can be extracted from  $G$ . It is easy to see that the set

$$G_* = \{p_* \in P_* \mid (p_*, (a_i, p_i)_{i < \lambda}) \in G\}$$

is  $V$ -generic for the partial order  $P_*$ . Set  $A_* = \bigcup G_*: \bigcup_{n < \omega} [\aleph_n, \aleph_{n+1}]^2 \rightarrow 2$ . For  $\xi \in [\aleph_n, \aleph_{n+1})$  let

$$A_*(\xi) = \{(\zeta, A_*(\xi, \zeta)) \mid \zeta \in [\aleph_n, \aleph_{n+1})\}: [\aleph_n, \aleph_{n+1}) \rightarrow 2$$



we may assume that there is  $p_* \in P_*$  such that  $\forall j < \aleph_{\omega+2}: p_*^j = p_*$ . We may assume that the supports  $\text{supp}((p_*, (a_i^j, p_i^j)_{i < \lambda})) \subseteq \lambda$  form a  $\Delta$ -system with a finite kernel  $I \subseteq \lambda$ . Finally we may assume that there are  $(a_i, p_i)_{i \in I}$  such that  $\forall j < \aleph_{\omega+2} \forall i \in I: (a_i^j, p_i^j) = (a_i, p_i)$ . Then  $(p_*^0, (a_i^0, p_i^0)_{i < \lambda})$  and  $(p_*, (a_i^1, p_i^1)_{i < \lambda})$  are compatible since

$$(p_*, (a_i^0 \cup a_i^1, p_i^0 \cup p_i^1)_{i < \lambda}) \leq_P (p_*^0, (a_i^0, p_i^0)_{i < \lambda})$$

and

$$(p_*, (a_i^0 \cup a_i^1, p_i^0 \cup p_i^1)_{i < \lambda}) \leq_P (p_*^1, (a_i^1, p_i^1)_{i < \lambda}).$$

□

By Lemma 3, cardinals  $\geq \aleph_{\omega+2}^V$  are absolute between  $V$  and  $V[G]$ .

## Fuzzifying the $A_i$

We want to construct a model which contains all the  $A_i$  and a map which maps every  $A_i$  to its index  $i$ . An *injective* map  $\lambda \mapsto \mathcal{P}(\aleph_\omega)$  for some high  $\lambda$  would imply large consistency strength (see [1]). To disallow such maps, the  $A_i$  are replaced by their equivalence classes modulo an appropriate equivalence relation.

The *exclusive or* function  $\oplus : 2 \times 2 \rightarrow 2$  is defined by

$$a \oplus b = 0 \text{ iff } a = b.$$

Obviously,  $(a \oplus b) \oplus (b \oplus c) = a \oplus c$ . For functions

$$A, A': \text{dom}(A) = \text{dom}(A') \rightarrow 2$$

define the *pointwise exclusive or*  $A \oplus A': \text{dom}(A) \rightarrow 2$  by

$$(A \oplus A')(\xi) = A(\xi) \oplus A'(\xi).$$

For functions  $A, A': (\aleph_\omega \setminus \aleph_0) \rightarrow 2$  define an equivalence relation  $\sim$  by

$$A \sim A' \text{ iff } \exists n < \omega ((A \oplus A') \upharpoonright \aleph_{n+1} \in V[G_*] \wedge (A \oplus A') \upharpoonright [\aleph_{n+1}, \aleph_\omega] \in V).$$

This relation is clearly reflexive and symmetric. We show transitivity. Consider  $A \sim A' \sim A''$ . Choose  $n < \omega$  such that

$$(A \oplus A') \upharpoonright \aleph_{n+1} \in V[G_*] \wedge (A \oplus A') \upharpoonright [\aleph_{n+1}, \aleph_\omega] \in V$$

and

$$(A' \oplus A'') \upharpoonright \aleph_{n+1} \in V[G_*] \wedge (A' \oplus A'') \upharpoonright [\aleph_{n+1}, \aleph_\omega] \in V.$$

Then

$$(A \oplus A'') \upharpoonright \aleph_{n+1} = ((A \oplus A') \upharpoonright \aleph_{n+1} \oplus (A' \oplus A'') \upharpoonright \aleph_{n+1}) \in V[G_*]$$

and

$$(A \oplus A'') \upharpoonright [\aleph_{n+1}, \aleph_\omega] = ((A \oplus A') \upharpoonright [\aleph_{n+1}, \aleph_\omega] \oplus (A' \oplus A'') \upharpoonright [\aleph_{n+1}, \aleph_\omega]) \in V.$$

Hence  $A \sim A''$ .

For  $A: (\aleph_\omega \setminus \aleph_0) \rightarrow 2$  define the  $\sim$ -equivalence class  $\tilde{A} = \{A' \mid A' \sim A\}$ .

## The symmetric submodel

Our final model will be a model generated by the following parameters and their constituents

- $T_* = \mathcal{P}(< \kappa)^{V[A_*]}$ , setting  $\kappa = \aleph_\omega^V$ ;
- $\vec{A} = (\tilde{A}_i \mid i < \lambda)$ .

The model

$$N = \text{HOD}^{V[G]}(V \cup \{T_*, \vec{A}\} \cup T_* \cup \bigcup_{i < \lambda} \tilde{A}_i)$$

consists of all sets which, in  $V[G]$  are hereditarily definable from parameters in the transitive closure of  $V \cup \{T_*, \vec{A}\}$ . This model is *symmetric* in the sense that it is generated from parameters which are invariant under certain (partial) isomorphisms of the forcing  $P$ .

**Lemma 4.**  *$N$  is a model of ZF, and there is a surjection  $f: \mathcal{P}(\kappa) \rightarrow \lambda$  in  $N$ .*

**Proof.** Note that for every  $i < \lambda$ :  $A_i \in N$ .

(1) Let  $i < j < \lambda$ . Then  $A_i \approx A_j$ .

*Proof.* Assume instead that  $A_i \sim A_j$ . Then take  $n < \omega$  such that  $v = (A_i \oplus A_j) \upharpoonright [\aleph_{n+1}, \aleph_\omega] \in V$ . The set

$$D = \{(p_*, (a_k, p_k)_{k < \lambda}) \mid \exists \xi \in [\aleph_{n+1}, \aleph_\omega] (\xi \in \text{dom}(p_i) \cap \text{dom}(p_j) \wedge v(\xi) \neq p_i(\xi) \oplus p_j(\xi))\} \in V$$

is readily seen to be dense in  $P$ . Take  $(p_*, (a_k, p_k)_{k < \lambda}) \in D \cap G$ . Take  $\xi \in [\aleph_{n+1}, \aleph_\omega)$  such that

$$\xi \in \text{dom}(p_i) \cap \text{dom}(p_j) \wedge v(\xi) \neq p_i(\xi) \oplus p_j(\xi).$$

Since  $p_i \subseteq A_i$  and  $p_j \subseteq A_j$  we have  $v(\xi) \neq A_i(\xi) \oplus A_j(\xi)$  and  $v \neq (A_i \oplus A_j) \upharpoonright [\aleph_{n+1}, \aleph_\omega)$ . Contradiction. *qed*(1)

Thus

$$f(z) = \begin{cases} i, & \text{if } z \in \tilde{A}_i; \\ 0, & \text{else;} \end{cases}$$

is a well-defined surjection  $f: \mathcal{P}(\kappa) \rightarrow \lambda$ , and  $f$  is definable in  $N$  from the parameters  $\kappa$  and  $\vec{A}$ .  $\square$

The main theorem will be established by showing that, in  $N$ , the situation below  $\kappa$  is largely as in  $V$ , in particular  $\kappa = \aleph_\omega^N$ . This requires an analysis of sets of ordinals in  $N$ .

**Lemma 5.** *Every set  $X \in N$  is definable in  $V[G]$  in the following form: there are an  $\in$ -formula  $\varphi$ ,  $x \in V$ ,  $n < \omega$ , and  $i_0, \dots, i_{l-1} < \lambda$  such that*

$$X = \{u \in V[G] \mid V[G] \models \varphi(u, x, T_*, \vec{A}, A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \dots, A_{i_{l-1}})\}.$$

**Proof.** By the original definition, every set in  $N$  is definable in  $V[G]$  from finitely many parameters in

$$V \cup \{T_*, \vec{A}\} \cup T_* \cup \bigcup_{i < \lambda} \tilde{A}_i.$$

To reduce the class of defining parameters to

$$V \cup \{T_*, \vec{A}\} \cup \{A_* \upharpoonright (\aleph_{n+1}^V)^2 \mid n < \omega\} \cup \{A_i \mid i < \lambda\}$$

observe:

- Let  $x \in T_*$  be a bounded subset of  $\aleph_\omega^V$  with  $x \in V[A_*]$ . A standard product analysis of the generic extension  $V[G_*] = V[A_*]$  of  $V$  yields that  $x \in V[A_* \upharpoonright (\aleph_{n+1}^V)^2]$  for some  $n < \omega$ .
- Let  $y \in \tilde{A}_i$ . Then  $y \sim A_i$ , i.e.,

$$(y \oplus A_i) \upharpoonright \aleph_{m+1} \in V[G_*] \wedge (y \oplus A_i) \upharpoonright [\aleph_{m+1}, \aleph_\omega) \in V$$

for some  $m < \omega$ . Let  $z = (y \oplus A_i) \upharpoonright \aleph_{m+1} \in V[A_*]$ . By the previous argument  $z \in V[A_* \upharpoonright (\aleph_{n+1}^V)^2]$  for some  $n < \omega$ . Let  $z' = (y \oplus A_i) \upharpoonright [\aleph_{m+1}, \aleph_\omega] \in V$ . Then

$$\begin{aligned} y &= (y \upharpoonright \aleph_{m+1}) \cup (y \upharpoonright [\aleph_{m+1}, \aleph_\omega]) \\ &= ((z \oplus A_i) \upharpoonright \aleph_{m+1}) \cup ((z' \oplus A_i) \upharpoonright [\aleph_{m+1}, \aleph_\omega]) \\ &\in V[A_* \upharpoonright (\aleph_{n+1}^V)^2, A_i]. \end{aligned}$$

Finitely many parameters of the form  $A_* \upharpoonright (\aleph_{n+1}^V)^2$  can then be incorporated into a single such parameter taking a sufficiently high  $n < \omega$ .  $\square$

## Approximating $N$

Concerning sets of ordinals, the model  $N$  can be approximated by “mild” generic extensions of the ground model. Note that many set theoretic notions only refer to ordinals and sets of ordinals.

**Lemma 6.** *Let  $X \in N$  and  $X \subseteq \text{Ord}$ . Then there are  $n < \omega$  and  $i_0, \dots, i_{l-1} < \lambda$  such that*

$$X \in V[A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \dots, A_{i_{l-1}}].$$

**Proof.** By Lemma 5 take an  $\in$ -formula  $\varphi$ ,  $x \in V$ ,  $n < \omega$ , and  $i_0, \dots, i_{l-1} < \lambda$  such that

$$X = \{u \in \text{Ord} \mid V[G] \models \varphi(u, x, T_*, \vec{A}, A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \dots, A_{i_{l-1}})\}.$$

By taking  $n$  sufficiently large, we may assume that

$$\forall j < k < l \forall m \in [n, \omega) \forall \delta \in [\aleph_m, \aleph_{m+1}): A_{i_j} \upharpoonright [\delta, \aleph_{m+1}) \neq A_{i_k} \upharpoonright [\delta, \aleph_{m+1}).$$

For  $j < l$  set

$$a_{i_j}^* = \{\xi \mid \exists m \leq n \exists \delta \in [\aleph_m, \aleph_{m+1}): A_{i_j} \upharpoonright [\delta, \aleph_{m+1}) = A_*(\xi) \upharpoonright [\delta, \aleph_{m+1})\}$$

where  $A_*(\xi) = \{(\zeta, A_*(\xi, \zeta)) \mid (\xi, \zeta) \in \text{dom}(A_*)\}$ . By the properties of  $Q$ ,  $a_{i_j}^* \subseteq \aleph_{n+1}$  is finite and  $\forall m \leq n: \text{card}(a_{i_j}^* \cap [\aleph_m, \aleph_{m+1})) = 1$ .

Now define

$$\begin{aligned} X' = \{u \in \text{Ord} \mid &\text{there is } p = (p_*, (a_i, p_i)_{i < \lambda}) \in P \text{ such that} \\ &p_* \upharpoonright (\aleph_{n+1}^V)^2 \subseteq A_* \upharpoonright (\aleph_{n+1}^V)^2, \\ &a_{i_0} \supseteq a_{i_0}^*, \dots, a_{i_{l-1}} \supseteq a_{i_{l-1}}^*, \\ &p_{i_0} \subseteq A_{i_0}, \dots, p_{i_{l-1}} \subseteq A_{i_{l-1}}, \text{ and} \\ &p \Vdash \varphi(\check{u}, \check{x}, \sigma, \tau, \dot{A} \upharpoonright (\check{\aleph}_{n+1})^2, \dot{A}_{i_0}, \dots, \dot{A}_{i_{l-1}})\}, \end{aligned}$$

where  $\sigma, \tau, \dot{A}, \dot{A}_{i_0}, \dots, \dot{A}_{i_{l-1}}$  are canonical names for  $T_*, \vec{A}, A_*, A_{i_0}, \dots, A_{i_{l-1}}$  resp.

Then  $X' \in V[A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \dots, A_{i_{l-1}}]$ .

(1)  $X \subseteq X'$ .

*Proof.* Consider  $u \in X$ . Take  $p = (p_*, (a_i, p_i)_{i < \lambda}) \in G$  such that

$$p \Vdash \varphi(\check{u}, \check{x}, \sigma, \tau, \dot{A} \upharpoonright (\check{\aleph}_{n+1})^2, \dot{A}_{i_0}, \dots, \dot{A}_{i_{l-1}}).$$

Then  $p_* \upharpoonright (\aleph_{n+1}^V)^2 \subseteq A_* \upharpoonright (\aleph_{n+1}^V)^2$  and  $p_{i_0} \subseteq A_{i_0}, \dots, p_{i_{l-1}} \subseteq A_{i_{l-1}}$ . Using a density argument we may also assume that  $\text{card}(a_{i_0} \cap \aleph_{n+1}) = \dots = \text{card}(a_{i_{l-1}} \cap \aleph_{n+1}) = n$ . Then  $a_{i_0} \supseteq a_{i_0}^*, \dots, a_{i_{l-1}} \supseteq a_{i_{l-1}}^*$ . Thus  $u \in X'$ . *qed*(1)

The converse direction,  $X' \subseteq X$ , is more involved and uses an isomorphism argument. Suppose for a contradiction that there were  $u \in X' \setminus X$ . Then take a condition  $p = (p_*, (a_i, p_i)_{i < \lambda}) \in P$  as in the definition of  $X'$ , i.e.,

$$(2) p_* \upharpoonright (\aleph_{n+1}^V)^2 \subseteq A_* \upharpoonright (\aleph_{n+1}^V)^2,$$

$$(3) a_{i_0} \supseteq a_{i_0}^*, \dots, a_{i_{l-1}} \supseteq a_{i_{l-1}}^*,$$

$$(4) p_{i_0} \subseteq A_{i_0}, \dots, p_{i_{l-1}} \subseteq A_{i_{l-1}}, \text{ and}$$

$$(5) p \Vdash \varphi(\check{u}, \check{x}, \sigma, \tau, \dot{A} \upharpoonright (\check{\aleph}_{n+1})^2, \dot{A}_{i_0}, \dots, \dot{A}_{i_{l-1}}).$$

By  $u \notin X$  take  $p' = (p'_*, (a'_i, p'_i)_{i < \lambda}) \in G$  such that

$$(6) p' \Vdash \neg \varphi(\check{u}, \check{x}, \sigma, \tau, \dot{A} \upharpoonright (\check{\aleph}_{n+1})^2, \dot{A}_{i_0}, \dots, \dot{A}_{i_{l-1}}).$$

By genericity we may assume that

$$(7) p'_* \upharpoonright (\aleph_{n+1}^V)^2 \subseteq A_* \upharpoonright (\aleph_{n+1}^V)^2$$

$$(8) a'_{i_0} \supseteq a_{i_0}^*, \dots, a'_{i_{l-1}} \supseteq a_{i_{l-1}}^*, \text{ and}$$

$$(9) p'_{i_0} \subseteq A_{i_0}, \dots, p'_{i_{l-1}} \subseteq A_{i_{l-1}}.$$

By strengthening the conditions we can arrange that  $p$  and  $p'$  have similar “shapes” whilst preserving the above conditions (2) to (9):

$$(10) \text{ ensure that } \text{supp}(p) = \text{supp}(p'); \text{ choose some } \aleph_{m+1} \text{ such that } \forall i \in \text{supp}(p) (a_i \subseteq \aleph_{m+1} \wedge a'_i \subseteq \aleph_{m+1});$$

$$(11) \text{ extend the } a_i \text{ and } a'_i \text{ such that}$$

$$\forall i \in \text{supp}(p) \forall k \leq m: \text{card}(a_i \cap [\aleph_k, \aleph_{k+1})) = \text{card}(a'_i \cap [\aleph_k, \aleph_{k+1})) = 1;$$

(12) also extend the conditions so that they involve the same “linking” ordinals, possibly at different positions within the conditions:

$$\bigcup_{i < \lambda} a_i = \bigcup_{i < \lambda} a'_i$$

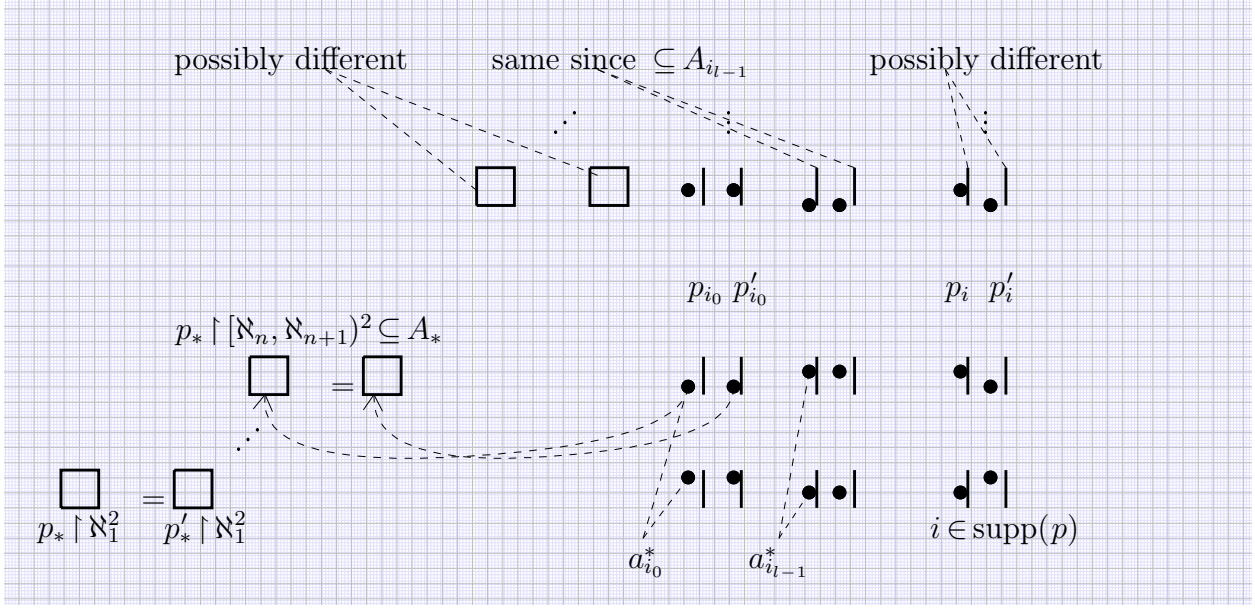
(13) extend the  $p_*$  and  $p_i$ 's in  $p$  and  $p'$  resp. so that for some sequence  $(\delta_k | k < \omega)$ :

$$\text{dom}(p_*) = \text{dom}(p'_*) = \bigcup_{k < \omega} [\aleph_k, \delta_k]^2$$

and

$$\forall i \in \text{supp}(p): \text{dom}(p_i) = \text{dom}(p'_i) = \bigcup_{k < \omega} [\aleph_k, \delta_k].$$

The following picture tries to capture some aspects of the shape similarity between  $p$  and  $p'$ ; corresponding components of  $p$  and  $p'$  are drawn side by side



Now define a map

$$\pi: (P \upharpoonright p, \leq_P) \rightarrow (P \upharpoonright p', \leq_P),$$

where the restricted partial orders are defined as  $P \upharpoonright p = \{q \in P \mid q \leq_P p\}$  and  $P \upharpoonright p' = \{q' \in P \mid q' \leq_P p'\}$ . For  $q = (q_*, (b_i, q_i)_{i < \lambda}) \leq_P (p_*, (a_i, p_i)_{i < \lambda}) = p$  define  $\pi(p) = \pi(q_*, (b_i, q_i)_{i < \lambda}) = (q'_*, (b'_i, q'_i)_{i < \lambda})$  by the following three conditions:

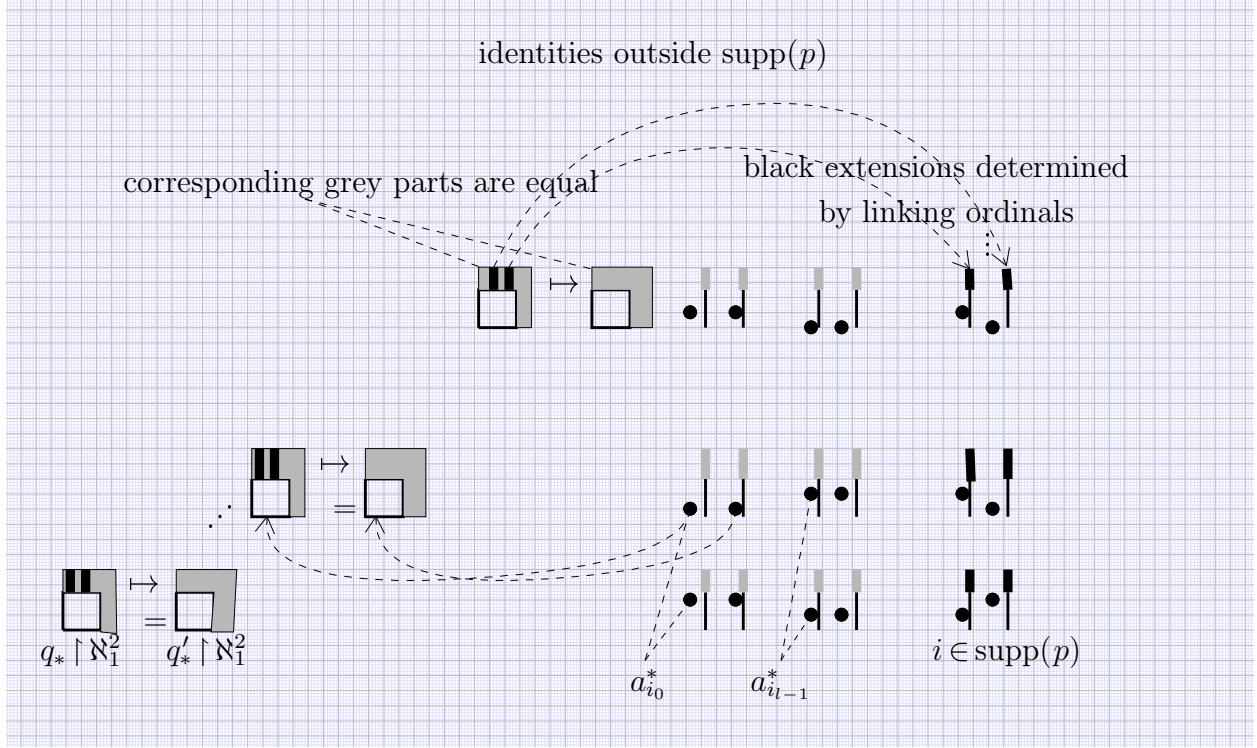
(14)  $q'_* = (q_* \setminus p_*) \cup p'_*$ ; note that this is a legitimate function since  $\text{dom}(p_*) = \text{dom}(p'_*)$ ;

(15) for  $i < \lambda$  let  $b'_i = (b_i \setminus a_i) \cup a'_i$ ; so if  $i \in \text{supp}(p)$ , the  $m + 1$  ordinals in  $a_i$  are substituted by the  $m + 1$  ordinals in  $a'_i$ ; if  $i \notin \text{supp}(p)$ , we have  $b'_i = b_i$ ;

(16) for  $i \in \lambda \setminus \text{supp}(q)$  let  $q'_i = \emptyset$ , and for  $i \in \text{supp}(q)$  define  $q'_i: \text{dom}(q_i) \rightarrow 2$  by

$$q'_i(\zeta) = \begin{cases} p'_i(\zeta), & \text{if } \zeta \in \text{dom}(p_i); \\ q_*(\xi', \zeta), & \text{if } \zeta \notin \text{dom}(p_i) \wedge \exists k < \omega: \zeta \in [\aleph_k, \aleph_{k+1}) \wedge a'_i \cap [\aleph_k, \aleph_{k+1}) = \{\xi'\}; \\ q_i(\zeta), & \text{if } \zeta \notin \text{dom}(p_i) \wedge \exists k < \omega: \zeta \in [\aleph_k, \aleph_{k+1}) \wedge a'_i \cap [\aleph_k, \aleph_{k+1}) = \emptyset. \end{cases}$$

Here is a picture of (some features of)  $\pi$ .



We verify that  $\pi: (P \upharpoonright p, \leq_P) \rightarrow (P \upharpoonright p', \leq_{P'})$  is an *isomorphism*.

(17)  $(q'_*, (b'_i, q'_i)_{i < \lambda}) \in P$ , since it has the same structure as  $(q_*, (b_i, q_i)_{i < \lambda})$ , with some function values altered.

(18)  $(q'_*, (b'_i, q'_i)_{i < \lambda}) \leq_P (p'_*, (a'_i, p'_i)_{i < \lambda})$ .

*Proof.*  $q'_* \supseteq p'_*$  since  $q'_* = (q_* \setminus p_*) \cup p'_*$ , see (14). Similarly we get  $b'_i \supseteq a'_i$  and  $q'_i \supseteq p'_i$ . To check the linking property (Definition 2, b)), consider  $i < \lambda$ ,  $n < \omega$ , and  $\xi' \in a'_i \cap [\aleph_n, \aleph_{n+1})$ . For  $\zeta \in \text{dom}(q'_i \setminus p'_i)$  we have

$$q'_i(\zeta) = q_*(\xi', \zeta) = q'_*(\xi', \zeta).$$

Finally we have to show the independence property (Definition 2, c)) within the linking ordinals. Consider  $j \in \text{supp}(p') = \text{supp}(p)$ . We claim that  $(b'_j \setminus a'_j) \cap \bigcup_{i \in \text{supp}(p'), i \neq j} b'_i = \emptyset$ . Assume for a contradiction that  $\xi' \in (b'_j \setminus a'_j) \cap b'_i$  for some  $i \in \text{supp}(p')$ ,  $i \neq j$ . Then  $\xi' \in (b_j \setminus a_j) \cap ((b_i \setminus a_i) \cup a'_i)$ . The case  $\xi' \in (b_j \setminus a_j) \cap (b_i \setminus a_i)$  is impossible by the independence property in  $q \leq_P p$ . And

$$(b_j \setminus a_j) \cap \bigcup_{i \in \text{supp}(p'), i \neq j} a'_i = (b_j \setminus a_j) \cap \bigcup_{i \in \text{supp}(p), i \neq j} a_i = \emptyset$$

by the independence property in  $q \leq_P p$  and by (12). *qed*(18)

(19)  $\pi$  is order-preserving.

*Proof.* Consider

$$r = (r_*, (c_i, r_i)_{i < \lambda}) \leq_P q = (q_*, (b_i, q_i)_{i < \lambda}) \leq_P p = (p_*, (a_i, p_i)_{i < \lambda})$$

and  $\pi(r) = r' = (r'_*, (c'_i, r'_i)_{i < \lambda})$ ,  $\pi(q) = q' = (q'_*, (b'_i, q'_i)_{i < \lambda})$ . We show that  $r' \leq_P q'$ . Concerning the inclusions:

- $r'_* = (r_* \setminus p_*) \cup p'_* \supseteq (q_* \setminus p_*) \cup p'_* = q'_*$ ;
- $c'_i = (c_i \setminus a_i) \cup a'_i \supseteq (b_i \setminus a_i) \cup a'_i = b'_i$ ;
- if  $i \in \lambda \setminus \text{supp}(q)$ , then  $q'_i = \emptyset$  and hence  $q'_i \subseteq r'_i$ . If  $i \in \text{supp}(q)$  then  $i \in \text{supp}(r)$ , and  $\text{dom}(r'_i) = \text{dom}(r_i)$  and  $\text{dom}(q'_i) = \text{dom}(q_i)$ . So we have

$$\text{dom}(p_i) = \text{dom}(p'_i) \subseteq \text{dom}(q'_i) \subseteq \text{dom}(r'_i).$$

For  $\zeta \in \text{dom}(q'_i)$  we have to show that  $q'_i(\zeta) = r'_i(\zeta)$ . In case  $\zeta \in \text{dom}(p_i)$  we have

$$q'_i(\zeta) = p'_i(\zeta) = r'_i(\zeta).$$

In case  $\zeta \notin \text{dom}(p_i) \wedge \exists k < \omega: \zeta \in [\aleph_k, \aleph_{k+1}) \wedge a'_i \cap [\aleph_k, \aleph_{k+1}) = \{\xi'\}$  we have

$$q'_i(\zeta) = q_*(\xi', \zeta) = r_*(\xi', \zeta) = r'_i(\zeta).$$

In case  $\zeta \in \text{dom}(p_i) \wedge \exists k < \omega: \zeta \in [\aleph_k, \aleph_{k+1}) \wedge a'_i \cap [\aleph_k, \aleph_{k+1}) = \emptyset$  we have

$$q'_i(\zeta) = q_i(\zeta) = r_i(\zeta) = r'_i(\zeta).$$

For the linking property consider  $i < \lambda$ ,  $n < \omega$ , and  $\xi' \in b'_i \cap [\aleph_n, \aleph_{n+1})$ . We have to check that  $\forall \zeta \in \text{dom}(r'_i \setminus q'_i) \cap [\aleph_n, \aleph_{n+1}): r'_i(\zeta) = r'_*(\xi', \zeta)$ . So consider  $\zeta \in \text{dom}(r'_i \setminus q'_i) \cap [\aleph_n, \aleph_{n+1})$ . Note that  $b'_i = (b_i \setminus a_i) \cup a'_i$ . In case  $\xi' \in a'_i$  we get:

$$r'_i(\zeta) = r_*(\xi', \zeta) = r'_*(\xi', \zeta).$$

If  $\xi' \in b_i \setminus a_i$ ,  $\xi' \notin a'_i$  and so  $a'_i \cap [\aleph_n, \aleph_{n+1}) = \emptyset$ . Hence

$$r'_i(\zeta) = r_i(\zeta) = r_*(\xi', \zeta) = r'_*(\xi', \zeta).$$

For the linking property consider  $j \in \text{supp}(q')$ . We have to show that

$$(c'_j \setminus b'_j) \cap \bigcup_{i \in \text{supp}(q'), i \neq j} c'_i = \emptyset.$$

Suppose for a contradiction that  $\xi' \in (c'_j \setminus b'_j) \cap \bigcup_{i \in \text{supp}(q'), i \neq j} c'_i$ . Then  $\xi' \in c'_j \setminus b'_j = c_j \setminus b_j$ . Take  $i \in \text{supp}(q')$ ,  $i \neq j$  such that  $\xi' \in c'_i$ . If  $i \in \text{supp}(p')$  this contradicts the property  $r' \leq_P p'$ . So  $i \in \text{supp}(q') \setminus \text{supp}(p')$ . Then  $c'_i = c_i$  and  $\xi' \in (c_j \setminus b_j) \cap c_i$ . But this contradicts the independence property for  $r \leq_P q$ . *qed*(19)

The definition of the map  $\pi$  only uses properties of  $p$  and  $p'$  which are the same for both of  $p$  and  $p'$ . So we can similarly define a map

$$\pi': (P \upharpoonright p', \leq_P) \rightarrow (P \upharpoonright p, \leq_P),$$

where for  $q' = (q'_*, (b'_i, q'_i)_{i < \lambda}) \leq_P (p'_*, (a'_i, p'_i)_{i < \lambda}) = p'$  the image  $\pi'(q') = (q_*, (b_i, q_i)_{i < \lambda})$  is defined by

$$(20) \quad q_* = (q'_* \setminus p'_*) \cup p_*;$$

$$(21) \quad \text{for } i < \lambda \text{ let } b_i = (b'_i \setminus a'_i) \cup a_i;$$

$$(22) \quad \text{for } i \in \lambda \setminus \text{supp}(q') \text{ let } q_i = \emptyset, \text{ and for } i \in \text{supp}(q') \text{ define } q_i: \text{dom}(q'_i) \rightarrow 2 \text{ by}$$

$$q_i(\zeta) = \begin{cases} p_i(\zeta), & \text{if } \zeta \in \text{dom}(p'_i), \\ q'_*(\xi, \zeta), & \text{if } \zeta \notin \text{dom}(p'_i) \wedge \exists k < \omega: \zeta \in [\aleph_k, \aleph_{k+1}) \wedge a_i \cap [\aleph_k, \aleph_{k+1}) = \{\xi\}, \\ q'_i(\zeta), & \text{if } \zeta \notin \text{dom}(p'_i) \wedge \exists k < \omega: \zeta \in [\aleph_k, \aleph_{k+1}) \wedge a_i \cap [\aleph_k, \aleph_{k+1}) = \emptyset. \end{cases}$$

The maps  $\pi$  and  $\pi'$  are inverses:

$$(23) \quad \pi' \circ \pi: (P \upharpoonright p, \leq_P) \rightarrow (P \upharpoonright p, \leq_P) \text{ is the identity on } (P \upharpoonright p, \leq_P).$$

*Proof.* Let  $(q_*, (b_i, q_i)_{i < \lambda}) \leq_P p = (p_*, (a_i, p_i)_{i < \lambda})$  and let  $\pi(q_*, (b_i, q_i)_{i < \lambda}) = (q'_*, (b'_i, q'_i)_{i < \lambda})$ . Concerning the first component,

$$q_* \xrightarrow{\pi} (q_* \setminus p_*) \cup p'_* \xrightarrow{\pi'} (((q_* \setminus p_*) \cup p'_*) \setminus p'_*) \cup p_* = q_*.$$

For  $i < \lambda$ ,

$$b_i \xrightarrow{\pi} (b_i \setminus a_i) \cup a'_i \xrightarrow{\pi'} (((b_i \setminus a_i) \cup a'_i) \setminus a'_i) \cup a_i = b_i.$$

For  $i \in \lambda \setminus \text{supp}(q)$ ,  $q_i = q'_i = \emptyset$  and so

$$q_i \xrightarrow{\pi} q'_i \xrightarrow{\pi'} q_i.$$

Now consider  $i \in \text{supp}(q) = \text{supp}(q')$ . Then  $\text{dom}(q_i) = \text{dom}(q'_i)$ . Let  $\zeta \in \text{dom}(q_i)$ . In case  $\zeta \in \text{dom}(p_i) = \text{dom}(p'_i)$  we have

$$q_i(\zeta) = p_i(\zeta) \xrightarrow{\pi} p'_i(\zeta) = q'_i(\zeta) \xrightarrow{\pi'} p_i(\zeta) = q_i(\zeta).$$

In case  $\zeta \notin \text{dom}(p_i) \wedge \exists k < \omega: \zeta \in [\aleph_k, \aleph_{k+1}) \wedge a_i \cap [\aleph_k, \aleph_{k+1}) = \{\xi\}$  let  $a'_i \cap [\aleph_k, \aleph_{k+1}) = \{\xi'\}$ . Then  $q_i(\zeta) = q_*(\xi, \zeta)$  and  $q'_i(\zeta) = q'_*(\xi', \zeta)$ . Then

$$q_i(\zeta) \xrightarrow{\pi} q'_*(\xi', \zeta) \xrightarrow{\pi'} q_*(\xi, \zeta) = q_i(\zeta).$$

Finally, if  $\zeta \notin \text{dom}(p_i) \wedge \exists k < \omega: \zeta \in [\aleph_k, \aleph_{k+1}) \wedge a_i \cap [\aleph_k, \aleph_{k+1}) = \emptyset$  then  $q_i(\zeta) = q_i(\zeta)$  and so

$$q_i(\zeta) \xrightarrow{\pi} q'_i(\zeta) \xrightarrow{\pi'} q_i(\zeta).$$

Thus

$$p \xrightarrow{\pi} p' \xrightarrow{\pi'} p.$$

*qed*(23)

Similarly,

$$(24) \quad \pi \circ \pi': (P' \upharpoonright p', \leq_P) \rightarrow (P' \upharpoonright p', \leq_P) \text{ is the identity on } (P' \upharpoonright p', \leq_P).$$

Hence  $\pi: (P \upharpoonright p, \leq_P) \rightarrow (P \upharpoonright p', \leq_P)$  is an isomorphism. Before we apply  $\pi$  to generic filters and objects defined from them, we note some properties of  $\pi$ .

$$(25) \quad \text{Let } q = (q_*, (b_i, q_i)_{i < \lambda}) \leq_P p \text{ and } \pi(p) = (q'_*, (b'_i, q'_i)_{i < \lambda}). \text{ Then } q'_* \upharpoonright (\aleph_{n+1}^V)^2 = q_* \upharpoonright (\aleph_{n+1}^V)^2, \text{ and } q'_{i_0} = q_{i_0}, \dots, q'_{i_{l-1}} = q_{i_{l-1}}.$$

Now let  $H_0$  be a  $V$ -generic filter for  $(P \upharpoonright p, \leq_P)$  with  $p \in H_0$ . Then

$$H = \{r \in P \mid \exists q \in H_0: q \leq_P r\}$$

is a  $V$ -generic filter for  $P$  with  $p \in H$ .

Moreover,  $H'_0 = \pi[H_0]$  is a  $V$ -generic filter for  $(P \upharpoonright p', \leq_P)$  with  $p' \in H'_0$  and

$$H' = \{r \in P \mid \exists q \in H'_0: q \leq_P r\}$$

is a  $V$ -generic filter for  $P$  with  $p' \in H'$ .

$$(26) \quad V[H] = V[H'] \text{ since the generic filters can be defined from each other using the isomorphism } \pi \in V.$$

Now define the parameters used in the definition of the model  $N$  from the generic filters  $H$  and  $H'$ :

$$H_* = \{q_* \in P_* \mid (q_*, (b_i, q_i)_{i < \lambda}) \in H\}, T_* = \sigma^H, \vec{A} = \tau^H, A_* = \dot{A}^H, \text{ and } A_i = \dot{A}_i^H \text{ for } i < \lambda$$

and

$$H'_* = \{q_* \in P_* \mid (q_*, (b_i, q_i)_{i < \lambda}) \in H'\}, T'_* = \sigma^{H'}, \vec{A}' = \tau^{H'}, A'_* = \dot{A}^{H'}, \text{ and } A'_i = \dot{A}_i^{H'} \text{ for } i < \lambda.$$

where  $\sigma, \tau, \dot{A}, \dot{A}_{i_0}, \dots, \dot{A}_{i_{l-1}}$  are the canonical names for  $T_*, \vec{A}, A_*, A_i$  resp. used before. The symbols  $G_*, \vec{A}, \dots$  are only used within the current proof, they do not conflict with their use before and after this proof.

$$(27) V[H_*] = V[H'_*].$$

*Proof.* Since  $H_*$  is  $V$ -generic for  $P_*$  and  $p_* \in P_*$ ,  $H_* \cap \{q_* \in P_* \mid q_* \supseteq p_*\}$  is, over the ground model  $V$ , equidefinable with  $H_*$ . Hence

$$\begin{aligned} V[H_*] &= V[H_* \cap \{q_* \in P_* \mid q_* \supseteq p_*\}] \\ &\supseteq V[\{(q_* \setminus p_*) \cup p'_* \mid q_* \in H_* \cap \{q_* \in P_* \mid q_* \supseteq p_*\}\}] \\ &= V[H'_* \cap \{q_* \in P_* \mid q_* \supseteq p'_*\}] \\ &= V[H'_*] \end{aligned}$$

*qed*(27)

This implies

$$(28) T_* = T'_*.$$

Let  $A_* = \bigcup H_*$  and  $A'_* = \bigcup H'_*$ .

$$(29) A_* \upharpoonright (\aleph_{n+1}^V)^2 = A'_* \upharpoonright (\aleph_{n+1}^V)^2.$$

*Proof.* Note that the map  $\pi$  is the identity on the  $*$ -component below  $\aleph_{m+1}^V$ . *qed*(29)

$$(30) \text{ For } i < \lambda: A_i \sim A'_i.$$

*Proof.* Recall  $A_i = \bigcup \{q_i \mid (q_*, (b_j, q_j)_{j < \lambda}) \in H\}: [\aleph_0, \aleph_\omega^V) \rightarrow 2$ . Since the map  $\pi$  maps the set  $b_i$  of linking ordinals to  $(b_i \setminus a_i) \cup a'_i$  the linking ordinals in the relevant sets  $b_i$  are equal to the linking ordinals in the sets  $b'_i$  with possibly finitely many exceptions. This means that the characteristic functions  $A_i$  and  $A'_i$  will be equal above  $p_i$  and  $p'_i$  respectively in all cardinal intervals  $[\aleph_k, \aleph_{k+1})$  with  $k > m$ ). In other words,

$$(A_i \oplus A'_i) \upharpoonright [\aleph_{m+1}^V, \aleph_\omega^V) \in V.$$

The functions  $A_i \upharpoonright \aleph_{m+1}^V$  and  $A'_i \upharpoonright \aleph_{m+1}^V$  are determined in the cardinal intervals  $[\aleph_k^V, \aleph_{k+1}^V)$  for  $k \leq m$  by  $p_i \upharpoonright [\aleph_k^V, \aleph_{k+1}^V)$  and  $p'_i \upharpoonright [\aleph_k^V, \aleph_{k+1}^V)$  and some cuts  $A_*(\xi)$  and  $A_*(\xi')$  respectively. Hence  $A_i \upharpoonright [\aleph_k^V, \aleph_{k+1}^V), A'_i \upharpoonright [\aleph_k^V, \aleph_{k+1}^V) \in V[A_* \upharpoonright (\aleph_{m+1}^V)^2] = V[A'_* \upharpoonright (\aleph_{m+1}^V)^2]$ . Thus

$$(A_i \oplus A'_i) \upharpoonright \aleph_{m+1}^V \in V[H_*] \text{ and } (A_i \oplus A'_i) \upharpoonright [\aleph_{m+1}^V, \aleph_\omega^V) \in V,$$

i.e.,  $A_i \sim A'_i$ . *qed*(30)

This implies immediately that the sequences of equivalence classes agree in both models:

$$(31) \vec{A} = \vec{A}'.$$

$$(32) A_{i_0} = A'_{i_0}, \dots, A_{i_{l-1}} = A'_{i_{l-1}}.$$

*Proof.* Note that the isomorphism  $\pi$  is the identity at the indices  $i_0, \dots, i_{l-1}$ . *qed*(32)

Since  $p \Vdash \varphi(\check{u}, \check{x}, \sigma, \tau, \dot{A} \upharpoonright (\check{\aleph}_{n+1})^2, \dot{A}_{i_0}, \dots, \dot{A}_{i_{l-1}})$  and  $p \in H$  we have

$$V[H] \models \varphi(u, x, T_*, \vec{A}, A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \dots, A_{i_{l-1}}).$$

Since  $p' \Vdash \neg\varphi(\check{u}, \check{x}, \sigma, \tau, \dot{A} \upharpoonright (\aleph_{n+1})^2, \dot{A}_{i_0}, \dots, \dot{A}_{i_{l-1}})$  and  $p' \in H'$  we have

$$V[H'] \models \neg\varphi(u, x, T'_*, \vec{A}', A'_* \upharpoonright P_* \upharpoonright (\aleph_{n+1}^V)^2, A'_{i_0}, \dots, A'_{i_{l-1}}).$$

But the various equalities proved above imply

$$V[H] \models \neg\varphi(u, x, T_*, \vec{A}, A_* \upharpoonright P_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \dots, A_{i_{l-1}}),$$

which is the desired contradiction.  $\square$

## Wrapping up

We show that the approximation models are mild generic extensions of  $V$ .

**Lemma 7.** *Let  $n < \omega$  and  $i_0, \dots, i_{l-1} < \lambda$ . Then cardinals are absolute between  $V$  and  $V[A^* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \dots, A_{i_{l-1}}]$ .*

**Proof.** Take  $p^0 = (p_*^0, (a_i^0, p_i^0)_{i < \lambda}) \in G$  such that  $\{i_0, \dots, i_{l-1}\} \subseteq \text{supp}(p^0)$ . Since the models  $V[A^* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \dots, A_{i_{l-1}}]$  are monotonely growing with  $n$  we may assume that  $n$  is large enough such that

$$\forall i \in \text{supp}(q) \forall \xi \in a_i^0: \xi \in \aleph_{n+1}.$$

Since every  $A_{i_j} \cap \aleph_{n+1}^V$  can be computed from  $A^* \upharpoonright (\aleph_{n+1}^V)^2$ , we have

$$V[A^* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \dots, A_{i_{l-1}}] = V[A^* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V], \dots, A_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V]].$$

Let  $P' = (P', \supseteq, \emptyset)$  be the forcing

$$P' = \{r \mid \exists (\delta_m)_{n < m < \omega} (\forall m \in (n, \omega): \delta_m \in [\aleph_m^V, \aleph_{m+1}^V] \wedge r: \bigcup_{n < m < \omega} [\aleph_m^V, \delta_m] \rightarrow 2)\},$$

which adjoins COHEN subsets to the  $\aleph_m$ 's with  $m > n$ .

(2)  $(A_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V], \dots, A_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V])$  is  $V$ -generic for  $(P')^l = \underbrace{P' \times \dots \times P'}_{l \text{ times}}$ .

*Proof.* Let  $D \subseteq (P')^l$  be dense open,  $D \in V$ . We have to show that  $D$  is met by  $(A_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V], \dots, A_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V])$ . Let

$$D' = \{(p_*, (a_i, p_i)_{i < \lambda}) \in Q \mid (p_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V], \dots, p_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V]) \in D\}.$$

This set is dense in  $P$  below  $p^0$ : consider  $p^1 = (p_*^1, (a_i^1, p_i^1)_{i < \lambda}) \leq_P (p_*^0, (a_i^0, p_i^0)_{i < \lambda}) = p^0$ . Take  $(\delta_m)_{n < m < \omega}$  such that

$$p_*^1 \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V]^2: \bigcup_{n < m < \omega} [\aleph_m^V, \delta_m]^2 \rightarrow 2.$$

Take  $p_{i_0}, \dots, p_{i_{l-1}}$  such that

$$(p_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V], \dots, p_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V]) \in D,$$

and such that  $p_{i_0}, \dots, p_{i_{l-1}}$  have the same domains. Through some ordinals in  $a_{i_0}^1, \dots, a_{i_{l-1}}^1$ , the choice of  $p_{i_0}, \dots, p_{i_{l-1}}$  determines some values of  $p_*$  by the linking property b) of Definition 2:

$$\forall i < \lambda \forall m < \omega \forall \xi \in a_i \cap [\aleph_m, \aleph_{m+1}) \forall \zeta \in \text{dom}(p_i \setminus p_i^1) \cap [\aleph_m, \aleph_{m+1}): p_i(\zeta) = p_*(\xi)(\zeta).$$

The independence property implies that the linking sets  $a_{i_0}^1, \dots, a_{i_{l-1}}^1$  are pairwise disjoint above  $\aleph_{n+1}^V$ , i.e., the sets

$$a_{i_0}^1 \cap [\aleph_{n+1}^V, \aleph_\omega^V), \dots, a_{i_{l-1}}^1 \cap [\aleph_{n+1}^V, \aleph_\omega^V)$$

are pairwise disjoint. So the linking requirements can be satisfied simultaneously. Then we can amend the definition of the other components of  $p \leq p^1$  and obtain  $p \in D'$ .

By the genericity of  $G$  take  $(p_*, (a_i, p_i)_{i < \lambda}) \in D' \cap G$ . Then

$$(p_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V), \dots, p_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V]) \in D$$

with

$$p_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V) \subseteq A_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V), \dots, p_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V) \subseteq A_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V),$$

as required. *qed*(2)

The forcing  $(P')^l$  is  $< \aleph_{n+2}$ -closed.  $A_* \upharpoonright (\aleph_{n+1}^V)^2$  is  $V$ -generic for the forcing

$$P_* \upharpoonright (\aleph_{n+1}^V)^2 = \{r \upharpoonright (\aleph_{n+1}^V)^2 \mid r \in P_*\}.$$

By the GCH in  $V$ ,  $\text{card}(P_* \upharpoonright (\aleph_{n+1}^V)^2) = \aleph_{n+1}$ . Hence every dense subset of  $P_* \upharpoonright (\aleph_{n+1}^V)^2$  which is in  $V[A_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V), \dots, A_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V)]$  is already an element of  $V$ . Thus  $A_* \upharpoonright (\aleph_{n+1}^V)^2$  is  $V[A_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V), \dots, A_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V)]$ -generic for  $P_* \upharpoonright (\aleph_{n+1}^V)^2$ .

This means that

$$V[A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V), \dots, A_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V)]$$

is equal to the two-stage iteration

$$V[A_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V], \dots, A_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V]][A_* \upharpoonright (\aleph_{n+1}^V)^2],$$

which does not destroy cardinals. □

**Lemma 8.** *Cardinals are absolute between  $N$  and  $V$ , and in particular  $\kappa = \aleph_\omega^V = \aleph_\omega^N$ .*

**Proof.** If not, then there is a function  $f \in N$  which collapses a cardinal in  $V$ . By Lemma 6,  $f$  is an element of some model  $V[A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \dots, A_{i_{l-1}}]$  as above. But this contradicts Lemma 7. □

**Lemma 9.** *GCH holds in  $N$  below  $\aleph_\omega$ .*

**Proof.** If  $X \subseteq \aleph_n$  and  $X \in N$  then  $X$  is an element of some model  $V[A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \dots, A_{i_{l-1}}]$  as above. Since  $A_{i_0}, \dots, A_{i_{l-1}}$  do not adjoin new subsets of  $\aleph_n$  we have that

$$X \in V[A_* \upharpoonright (\aleph_{n+1}^V)^2].$$

Hence  $\mathcal{P}(\aleph_n^V) \cap N \in V[A_* \upharpoonright (\aleph_{n+1}^V)^2]$ . GCH holds in  $V[A_* \upharpoonright (\aleph_{n+1}^V)^2]$ . Hence there is a bijection  $\mathcal{P}(\aleph_n^V) \cap N \leftrightarrow \aleph_{n+1}^V$  in  $V[A_* \upharpoonright (\aleph_{n+1}^V)^2]$  and hence in  $N$ . □

## Discussion and Remarks

The above construction straightforwardly generalises to other cardinals  $\kappa$  of cofinality  $\omega$ . In that extension, cardinals  $\leq \kappa$  are preserved, GCH holds below  $\kappa$ , and there is a surjection from  $\mathcal{P}(\kappa)$  onto some arbitrarily high cardinal  $\lambda$ . To work with singular cardinals  $\kappa$  of *uncountable* cofinality, finiteness properties in the construction have to be replaced by the property of being of cardinality  $< \text{cof}(\kappa)$ . This yields results like the following choiceless violation of SILVER's theorem [6].

**Theorem 10.** *Let  $V$  be any ground model of ZFC + GCH and let  $\lambda$  be some cardinal in  $V$ . Then there is a cardinal preserving model  $N \supseteq V$  of the theory ZF + “GCH holds below  $\aleph_{\omega_1}$ ” + “there is a surjection from  $\mathcal{P}(\aleph_{\omega_1})$  onto  $\lambda$ ”. Moreover, the axiom of dependent choices DC holds in  $N$ .*

Note that in [5], SAHARON SHELAH studied uncountably singular cardinal arithmetic under DC, without assuming AC. The “local” GCH below  $\aleph_{\omega_1}$  in the conclusion of the above Theorem cannot be changed to the property  $\text{card}(\bigcup_{\alpha < \aleph_{\omega_1}} \mathcal{P}(\alpha)) = \aleph_{\omega_1}$  since Theorem 4.6 of [5] basically implies that then  $\mathcal{P}(\aleph_{\omega_1})$  would be wellorderable of ordertype  $\geq \lambda$ . By results of [1] an *injective* failure of SCH with big  $\lambda$  has high consistency strength. But here we are working without assuming any large cardinals.

## Bibliography

- [1] Arthur Apter and Peter Koepke. The consistency strength of choiceless failures of SCH. *To appear in: The Journal of Symbolic Logic*, 75, 2010.
- [2] Anthony Dodd and Ronald Jensen. The covering lemma for  $K$ . *Annals of Mathematical Logic*, 22:1–30, 1982.
- [3] Moti Gitik. The strength of the failure of the singular cardinal hypothesis. *Annals of Pure and Applied Logic*, 51:215–240, 1991.
- [4] Saharon Shelah. *Cardinal Arithmetic*. Oxford Logic Guides. Oxford University Press, 1994.
- [5] Saharon Shelah. Set theory without choice: not everything on cofinality is possible. *Archive for Mathematical Logic*, 36:81–125, 1996.
- [6] Jack Silver. On the singular cardinals problem. In *Proceedings of the International Congress of Mathematicians (Vancouver, British Columbia 1974)*, volume 1, pages 265–268, 1975.