The Influence of Felix Hausdorff on the Early Development of Descriptive Set Theory

Abstract

The rôle of Felix Hausdorff within descriptive set theory is often underestimated. After a brief review of earlier work in descriptive set theory we give an overview of Hausdorff's contributions to the field, including the presentation of descriptive set theory in his book *Grundzüge der Mengenlehre* and his result on the cardinalities of Borel sets. We conclude with a discussion of the importance of Hausdorff's work for the development of the field.

1 The origins of descriptive set theory

The term *descriptive set theory* was first used in print in the *Topologie* of Paul Alexandroff and Heinz Hopf [32], p.19 where they define the subject concisely as:

> Zur Topologie gehört auch die „*deskriptive Punktmengenlehre*“ (Théorie descriptive des ensembles), d.h. im wesentlichen die Theorie der Borelschen Mengen, der A-Mengen und der projektiven Mengen.[1]

Georg Cantor's investigations on the uniqueness of the coefficients of a Fourier expansion of a function [5] can be seen as the common origin of *general* and of *descriptive* set theory. Cantor defines and classifies "small" sets of real numbers which can be neglected for the uniqueness result. Later Cantor [6] uses transfinite ordinal numbers to prove that every closed set of reals is a union of a perfect set and an at most denumerably infinite set. These results are paradigmatic for the aims of descriptive set theory: to characterize and classify natural classes of pointsets, i.e., sets of real numbers. Cantor was confident that his classification result for closed sets could eventually be extended to all sets of reals ([6], p.488):

> Dass dieser merkwürdige Satz eine weitere Gültigkeit auch für *nicht abgeschlossene* lineare Punktmengen und ebenso auch für alle n-dimensionalen Punktmengen hat, wird in späteren Paragraphen bewiesen werden.[2]

The emergence of descriptive set theory as a mathematical field in its own right may be dated to the turn of the 19th to the 20th century when the french school of function theory extended their measure-theoretic and analytic studies to larger classes of sets and functions. Implicit in Èmile Borel's [4] definition of the *Borel measure* is the definition of a class of measurable sets which were later called *Borel sets* by Hausdorff [12], p.305 and others. The Borel sets are those which can be generated from real intervals by the operations of set difference and countable unions. René Baire [3] also used *countable* operations to define his hierarchy of functions which comprises the smallest class of real functions which contains the continuous functions and is closed under pointwise limits of functions.

The key work of early descriptive set theory is Henri Lebesgue's *Sur les fonctions représentables analytiquement* [29], whose significance was described by Alexandroff and Hopf [32], p.20:

> Die deskriptive Mengenlehre wurde (anschließend an Baires Arbeiten über unstetige Funktionen) von Lebesgue 1905 begründet.[3]

1. Topology includes *"descriptive point set theory"* (Théorie descriptive des ensembles), i.e., essentially the theory of Borel sets, A-sets and projective sets.
Today, A-sets are usually called *analytic sets* or Σ^1_1-sets.

2. It will be shown in later paragraphs that this remarkable theorem holds true also for *non-closed* linear point sets and also for all n-dimensional point sets.
We know now that Cantor's conjecture does not hold if one assumes the axiom of choice (AC).

LEBESGUE classifies Borel sets by the number of infinitary operations necessary for their definition. He shows that the resulting transfinite hierarchy of Borel sets corresponds to the Baire hierarchy of functions. Every level of the hierarchy contains objects not classified at earlier stages. LEBESGUE claims that the system of Baire functions and Borel sets is closed under *analytic* definitions and constructions and is thus sufficient for the needs of analysis. LEBESGUE's mémoire characterises the state of art at the time when HAUSDORFF takes up his work in descriptive set theory.

The origins of descriptive set theory were strongly influenced by the foundational crisis in mathematics in the first years of the 20th century. The famous *Cinq lettres sur la théorie des ensembles* by BAIRE, BOREL, JACQUES HADAMARD and LEBESGUE [35] argue strongly against ERNST ZERMELO's proof [41] of the wellordering theorem under the assumption of the axiom of choice. In contrast to ZERMELO's choices *ad libitum* they stipulate that mathematical functions have to be *définie* (defined) or at least *décrite* (described). It is assumed that Borel sets and Baire functions satisfy this rather vague criterium. The word *décrite* lead to the terms *descriptive set theory* and also to the hardly used *descriptive function theory*.

Let us mention for the sake of correctness that LEBESGUE's above-mentioned claim about the sufficiency of Baire functions and Borel sets for all analytic purposes is actually at fault, as was discovered by MICHAIL SUSLIN in 1917. In any case, LEBESGUE's mistake was a central argument for the foundational position of the *Cinq lettres* and thus marks a historically important viewpoint.

2 Descriptive set theory in early set theoretic books

HAUSDORFF's published and known unpublished contributions to descriptive set theory begin approximately at the time of the publication of the *Grundzüge der Mengenlehre* [12] in 1914. To set his book into context, we briefly sketch the presentation of the field in early set theoretical treatises.

CANTOR's *Beiträge zur Begründung der transfiniten Mengenlehre* [7], although published as two articles in the *Mathematische Annalen*, can be considered to be the first textbook in set theory. CANTOR presents his general theory with arbitrarily large ordinals and cardinals and does not cover point set theory.

The emphasis shifted with BOREL's *Leçons sur la théorie des fonctions* [4]: after introducing cardinalities, BOREL treats countable sets and the cardinality of the continuum in detail. He writes on p.20:

> Ces notions nous suffiront pour les applications que nous avons en vue.[4]

BOREL then goes on to treat perfect sets and "les ensembles mesurables" in great detail. The *Leçons* were an influential introduction to set theory for the mathematical community in France. They were the first book in the famous series *Collection de monographies sur la théorie des fonctions*. The flourishing french analytic school took a special attitude towards set theory which fostered the emergence of *descriptive* set theory: to develop point set theory broadly, whilst opposing CANTOR's theory of arbitrary sets and arbitrary cardinalities.

In Germany, ARTHUR SCHOENFLIES published two "reports" [36], [37] on the state of set theory, surveying point set theory and the theory of real functions. His reports reached a wide mathematical audience in the *Jahresbericht der DMV*, the journal of the german mathematical society (DMV). In 1913, SCHOENFLIES published a thoroughly revised and extended version of his reports as a book: *Entwickelung der Mengenlehre und ihrer Anwendungen. Erste Hälfte: Allgemeine Theorie der unendlichen Mengen und Theorie der Punktmengen* [38]. The presentation was traditional, treating sets theory as an ordinary mathematical discipline without foundational aspects. This book was soon overshadowed by HAUSDORFF's much more advanced *Grundzüge*.

3. Descriptive set theory was founded by LEBESGUE in 1905 (following BAIRE's work on non-continuous functions).

4. These notions will suffice for the intended applications.

The first textbook explicitly devoted to point set theory was *The Theory of Sets of Points* by WILLIAM HENRY YOUNG and GRACE CHISHOLM YOUNG [8]. It is mainly a presentation of various results by the YOUNGs. Its principal achievement from the descriptive perspective is the generalisation of CANTOR's cardinality theorem from closed sets to G_δ-sets, i.e., to countable intersections of open sets. The book suffered from verbose formulations and inefficient notions and notations.

Other set theory books before 1914 were written by HESSENBERG, HOBSON, SHEGALKIN, and SIERPINSKI. None of the books mentioned gave a comprehensive treatment of the entire field of set theory of the time, which included general set theory, the theory of point sets, and measure theory. The theory of point sets was often presented in a simplified general framework which was just sufficient to treat *point* sets but not *arbitrary* sets. Energy was often expended to eliminate e.g. the use of ordinal numbers from point set arguments. These limitations were partly due to foundational concerns. The development of point set theory was motivated and hindered by scruples about the justification of general set theoretic methods.

3 HAUSDORFF's *Grundzüge der Mengenlehre*

Grundzüge der Mengenlehre, FELIX HAUSDORFFs *opus magnum*, appeared in 1914 and ranks among the classics of mathematics. *Grundzüge* had a profound impact on the development of modern mathematics in the 20th century with its axiomatic and set theoretic foundations. Generations of mathematicians learnt general set theory from HAUSDORFF's book and were influenced by the systematic approach and style of HAUSDORFF.

Grundzüge der Mengenlehre was a textbook covering all areas of set theory of the time. By its success and the enormeous development of topology in particular it also was the *last* book of its kind. The first six chapter of *Grundzüge* give a comprehensive presentation of general set theory including HAUSDORFF's own contributions to the theory of ordered sets, to cardinal arithmetic, and to the algebra of sets.

Historically, the greatest impact was made by the topological part of the *Grundzüge*. HAUSDORFF develops a systematic theory of topological spaces on the basis of his axiomatisation of topological spaces. This theory is one of the earliest examples of a strictly axiomatic theory which is differentiated and specialised systematically by various supplementary notions and is widely applicable throughout mathematics. Set theoretic topology begins with *Grundzüge* der Mengenlehre.

The final chapter is an introduction to the new theory of measure and integration and it contains the spectacular sphere paradox of HAUSDORFF.

The approach of *Grundzüge* is thoroughly "modern". The introductory chapter contains HAUSDORFF's foundational attitude in terse words, indicating a broad awareness of the issues but also the determination to rather operate with mathematical objects than to discuss existence questions. So he states on p.1:

> Die Mengenlehre ist das Fundament der gesamten Mathematik.[5]

In an appendix to the introduction, HAUSDORFF expresses the expectation that ZERMELO's axiomatic approach can safeguard set theory against the naive antinomies [12], p. 450:

> Hierher gehört auch die vielumstrittene Frage, unter welchen Bedingungen ein mathematisches Objekt, etwa eine Zahl, eine Menge, eine Funktion als „definiert" anzusehen sei (die Frage nach der Definition einer „Definition"). Wir folgen der freien Auffassung CANTORS [...] und verlangen nicht, daß die logische Disjunktion, ob ein Ding einer Menge angehört oder nicht, mit unseren aktuellen Mitteln wirklich entschieden werden könne. [...] Dieser Mengenbegriff und dieser (DIRICHLETsche) Funktionenbegriff bindet sich weder an „Kriterien, die nur eine endliche Anzahl von Versuchen erfordern", noch an „analytische Darstellungen" u. dgl.[6]

5. Set theory is the foundation of the whole of mathematics.

4 Descriptive set theory in *Grundzüge der Mengenlehre*

One of the central themes of HAUSDORFF's work in mathematics, but also in his philosophical writings, are the question of "space" and "space-forms". In *Grundzüge der Mengenlehre* HAUSDORFF gradually specialises general sets to structures with spatial character. The transitions between general set theory, topology, descriptive set theory and measure theory are continuous. Descriptive set theory is primarily located in chapter 8 (point sets in special spaces), chapter 9 (maps and functions), and the appendix. HAUSDORFF and other set theorists of his time would have classified descriptive set theory under the theory of general metric spaces. This becomes apparent on the first page of the preface of HAUSDORFF's book *Mengenlehre* [16]:

> [...] daß [...] ich [...] den *topologischen* Standpunkt [...] aufgegeben und mich auf die einfachere Theorie der *metrischen* Räume beschränkt habe.[7]

One of the remarkable characteristics of *Grundzüge* is how radical the set theoretic viewpoint is applied to a variety of mathematical objects and theories. Notions are defined in maximal generality without regard to actual existence in other fields of mathematics. Infinitary methods from ordinal and cardinal theory are applied throughout. We exemplify this by the introduction to Borel sets in *Grundzüge*.

The first chapter of the book develops notions for infinitary set operations. Given a family $\boldsymbol{X}$ of sets, $\boldsymbol{X}_\sigma$ and $\boldsymbol{X}_\delta$ denote the family of all countable unions resp. intersections of elements of $\boldsymbol{X}$. If $\boldsymbol{F}$ and $\boldsymbol{G}$ are the classes of closed resp. open sets of a given space then $\boldsymbol{F}_\sigma$ and $\boldsymbol{G}_\delta$ are the classes of all countable unions of closed sets resp. of all countable intersections of open sets. This notation can be continued. E.g., $\boldsymbol{F}_{\sigma\delta} = (\boldsymbol{F}_\sigma)_\delta$ is the class of all countable intersections of $\boldsymbol{F}_\sigma$-sets. These notations were introduced by HAUSDORFF and are still being used to denote small Borel complexities. By a transfinite recursion, HAUSDORFF defines the classes $\boldsymbol{F}_{(\sigma\delta)}$ and $\boldsymbol{G}_{(\delta\sigma)}$ which are closed with respect to the formation of countable unions and countable intersections. Both classes coincide in metric spaces and form the class of *Borel sets*. The term is due to HAUSDORFF ([12], p.466):

> [...] alle BORELschen Mengen [...], d.h. die aus den Gebieten oder abgeschlossenen Mengen durch Summen- und Durchschnittsbildung über Folgen hervorgehen.[8]

This extends an earlier usage of SCHOENFLIES [38], p.350 for $\boldsymbol{G}_\delta$-sets. HAUSDORFF applies general cardinality considerations to the family of Borel sets and emphasises that the Borel sets form a minute part of the family of all pointsets:

> Im euklidischen Raume bilden also die Mengen, die aus abgeschlossenen Mengen oder Gebieten durch Summen- und Durchschnittsbildung von Folgen entstehen, immer noch einen verschwindend kleinen Teil des Systems aller Punktmengen.[9] ([12], p.305)

Then HAUSDORFF goes on to consider Borel sets in connection with metric spaces, complex functions and irrational numbers. He demonstrates by "concrete" examples that $\boldsymbol{G}_\delta$- and $\boldsymbol{F}_\sigma$-sets occur naturally in various areas of mathematics.

6. Here also belongs the much-disputed question under which conditions we may take a mathematical object, be it a number, a set, or ar function as being "defined" (asking for a definition of "definition"). We follow the free opinion of CANTOR's [...] and do not require that the logical disjunction whether something belongs to a set or not can be really decided with our actual means. [...] This notion of set and this (DIRICHLET) notion of function is neither bound to "criteria which only require a finite number of attempts" nor to "analytic presentations" or something similar.

7. ... that I have abandoned the *topological* position and have restricted myself to the simpler theory of *metric* spaces.

8. [...] all BOREL sets [...], i.e., those which are generated from open sets or closed sets by the formation of unions and intersections of sequences.

9. In euclidean space, those sets which are generated from closed sets or open sets by the formation of unions and intersections of sequences form a vanishing part of the system of all point sets.

The *Grundzüge* contain a number of further expositions and results relevant to descriptive set theory. HAUSDORFF introduces the point class of *reducible sets*, which is defined by a variant of the CANTOR-BENDIXSON derivative. He studies various representations or normal forms of members of this class and shows that it is equal to $\boldsymbol{F}_\sigma \cap \boldsymbol{G}_\delta$. The theory of Baire functions is presented in the general setting of metric spaces. The results do not exceed what was known to BAIRE, BOREL and LEBESGUE but HAUSDORFF formulates them *set* theoretically rather than *function* theoretically. This was not the common approach at the time but has become widely accepted eventually: the subject is called descriptive *set* theory and not descriptive *function* theory.

The appendix of *Grundzüge* contains an important precursor of the cardinality result for Borel sets to be discussed below and a measure theoretic landmark result about sets of reals numbers: the HAUSDORFF paradoxical decomposition of the 2-dimensional sphere which yields a counterexample to the finitely additive measure problem is a radically *anti*-descriptive result. Under the axiom of choice, exotic sets of reals can be constructed which behave completely different from the ordinary descriptive sets. HAUSDORFF [12], p.469 expresses the paradox drastically as follows:

> Der Beweis beruht auf der merkwürdigen Tatsache, daß eine Kugelhälfte und ein Kugeldrittel kongruent sein können [...][10]

5 Cardinalities of Borel sets

HAUSDORFF's most significant single result in descriptive set theory is the determination of the cardinalities of Borel sets [13] in 1916:

> Jede Borelsche Menge ist entweder endlich oder abzählbar oder von der Mächtigkeit des Kontinuums.[11]

Actually, HAUSDORFF extends CANTOR's perfect subset theorem from closed sets to Borel sets. The same result was obtained by ALEXANDROFF [2] by an alternative method which foreshadowed the Suslin operation. The Alexandroff-Hausdorff theorem extends the above-mentioned result of YOUNG and the corresponding cardinality theorem for $\boldsymbol{G}_{\delta\sigma\delta}$-sets proved by HAUSDORFF in *Grundzüge*. The $\boldsymbol{G}_{\delta\sigma\delta}$-result is contained in the appendix of *Grundzüge* and was obviously proved shortly before its publication. The theorem is proved generally for spaces which are now being called *polish spaces* [12], p.465:

> In einem vollständigen Raume mit abzählbarer dichter Teilmenge ist eine Menge $G_{\delta\sigma\delta}$, wenn sie unabzählbar ist, von der Mächtigkeit des Kontinuums.[12]

Let us indicate HAUSDORFF's main construction in modern terms as it is a predecessor to the general proof. For an uncountable $\boldsymbol{G}_{\delta\sigma\delta}$-set

$$X = \bigcap_{a\in\omega} \bigcup_{i\in\omega} \bigcap_{n\in\omega} X_{a,i,n} \ (\text{all } X_{a,i,n} \text{ open intervals})$$

HAUSDORFF obtains a non-empty perfect set $U \subseteq X$ of the form $U = \bigcap_n \bigcup_{s\in 2^n} U_s$, where $\{U_s\}_{s\in 2^{<\omega}}$ is a dyadic system of closed intervals with the following properties:

i. for $s\in 2^n$: $U_{s^\frown i} \subseteq U_s$, $U_{s^\frown 0} \cap U_{s^\frown 1} = \emptyset$, and the interval U_s has lengths $\leqslant n^{-1}$; this ensures that U is homeomorphic to CANTOR's discontinuum;

ii. for $s \in 2^a$ is $U_s \cap S \cap X_{0,i_0} \cap X_{1,i_1} \cap \ldots \cap X_{a,i_a}$ uncountable, where the numbers $i_0, i_1, \ldots$, are defined simultaneously with the intervals U_s and where $X_{a,i} = \bigcap_{n\in\omega} X_{a,i,n}$;

10. The proof rests on the remarkable fact that a half sphere and a third of a sphere may be congruent.

11. Every Borel set is either finite or denumerable or has the cardinality of the real continuum.

12. In a complete space with a countable dense subset every set $G_{\delta\sigma\delta}$ which is uncountable is of the cardinality of the continuum.

iii. for $s \in s^n$ we have $U_s \subseteq \bigcap_{a \leqslant n, \nu \leqslant n} X_{a,i_a,\nu}$; this implies that $U \subseteq X$.

For the recursive definition of this system HAUSDORFF uses that every uncountable subset of the space has at least two condensation points.

Following this proof, HAUSDORFF remarks optimistically [12], p.466:

> Der Versuch scheint nicht aussichtslos, das gleiche für alle BORELschen Mengen zu beweisen.[13]

In HAUSDORFF's general proof the unfolding of a Borel set is carried out similarly along the paths of a well-founded tree which describes the definition of the Borel set; property (iii) is only stipulated for the end-points of the tree. HAUSDORFF is well aware that the cardinality result for Borel sets is far from proving CANTOR's continuum hypothesis in general [12], p.305:

> Wüßte man für *alle* Mengen eines euklidischen Raumes, was man [...] von den abgeschlossenen Mengen F oder den Mengen G_δ weiß, daß sie nämlich endlich, abzählbar oder von der Mächtigkeit $\aleph$ sind, so wäre $\aleph$ die nächste Mächtigkeit über $\aleph_0$ und damit die Kontinuumsfrage im Sinne der CANTORschen Vermutung $\aleph = \aleph_1$ entschieden. Um aber einzusehen, wie weit man noch von diesem Ziel entfernt ist, genügt es sich zu erinnern, daß das System der Mengen F oder G_δ nur einen verschwindend kleinen Teil des Systems aller Punktmengen bildet.[14]

6 The reception of Grundzüge in descriptive set theory

The dissemination of HAUSDORFF's *Grundzüge* which was published in April 1914 was much impaired by the outbreak of World War I. Only a small number of reviews appeared until 1920. After that it began to be widely used as a textbook and a general reference, and the many innovations and ideas contained in *Grundzüge* were taken up by young researchers.

In 1920, the journal *Fundamenta Mathematicae* was founded at Warsaw and it was specifically aimed at set theory and its applications. *Fundamenta* concentrated on the areas set theory, topology, theory of real functions, theory of measure and integration, functional analysis, logic and foundations of mathematics and *Grundzüge* was ideally suited to be a basis and a stimulation for research in most of those areas. Due to the dynamics of the field the main emphasis was on general topology, but a substantial part of the publications belonged to descriptive set theory. From the first volume, *Fundamenta Mathematicae* ranked under the leading mathematical journals.

From the beginning *Grundzüge* were used with remarkable frequency as a standard reference in articles published in *Fundamenta*; they were cited in 88 out of 558 articles in the first 20 Volumes of *Fundamenta* (1920 – 1933) (see [34], p.58). Among these were papers on descriptive set theory by prominent authors like PAUL ALEXANDROFF, STEFAN BANACH, KAZIMIERZ KURATOWSKI, ADOLF LINDENBAUM, NIKOLAI LUSIN, STEFAN MAZURKIEWICZ, JOHN V. NEUMANN, WACLAW SIERPINSKI, and ALFRED TARSKI. Moreover the notions introduced by HAUSDORFF, in particular in topology, quickly became general knowledge and were taken over without references.

Several authors of textbooks which appeared after the *Grundzüge* refered their readers to HAUSDORFF for further studies. ABRAHAM FRAENKEL [1], p.394 writes:

> Für eindringende Studien kommt allein das ausgezeichnete Lehrbuch von Hausdorff in Betracht.[15]

13. The attempt to prove the same for all BOREL sets does not appear to be hopeless.

14. If one knew about *all* sets of a euclidean space what one knows about the closed sets F or the sets G_δ , namely that they are finite, denumerable or of cardinality $\aleph$, then $\aleph$ would be the next cardinality above $\aleph_0$ and the continuum problem would be decided according to CANTORs conjecture $\aleph = \aleph_1$. But to realise how far one is from that goal it suffices to remember that the system of sets F or G_δ forms only a vanishingly small part of the system of all point sets.

15. Only the excellent textbook by Hausdorff can be considered for deep studies.

7 Hausdorff's further work in descriptive set theory

Descriptive set theory comprises a large part of Hausdorff's work. About seven published articles ([13], [14], [15], [17], [19], [20], [25]) can be classified under descriptive set theory, the papers [10], [11], [18], [22], [23], [24], [26] also contain descriptive parts. Hausdorff's book *Mengenlehre* [16] (officially declared to be the second edition of *Grundzüge*) is a completely rewritten book with an emphasis on descriptive set theory. Together with its reedition [21] *Mengenlehre* can be viewed as an updated textbook in descriptive set theory and set theoretic topology. The handwritten scientific *Nachlass* [9] of about 26000 pages which is kept in the University Library at Bonn contains about 2000 pages with material classifiable under descriptive set theory. A catalogue of the *Nachlass* [33] was compiled by Walter Purkert. It can also be accessed via the internet at:
`http://hss.ulb.uni-bonn.de:90/ulb_bonn/veroeffentlichungen/hausdorff_felix`

8 Discussion

The rôle of Felix Hausdorff within descriptive set theory is often underestimated, due to a variety of factors. After the Alexandroff–Hausdorff theorem and Suslin's discovery of analytic sets attention shifted from Borel sets to analytic and projective sets. Although Hausdorff continued to work in descriptive set theory he pursued a host of other mathematical interests with success and in particular his general topology. In terms of numbers of publications, Hausdorff was quickly overtaken by the younger generation.

In 1930 Lusin published the *Leçons sur les ensembles analytiques et leurs applications* [30] which was the first book solely devoted to descriptive set theory. Lusin elaborated foundational issues in line with Borel's standpoint. Kuratowski's *Topology I* [28] included a presentation of descriptive set theory which was rather more detailed than Hausdorff's books. Both monographs had a strong influence on the further development of descriptive set theory and became standard references. In the 1950, set theory and descriptive set theory began to evolve from their "naive" Cantorean era to a more sophisticated metamathematical phase with new techniques and paradigms.

Although the visibility of Hausdorff in descriptive set theory decreased over the years, his overall influence is still strong. His approach from a perspective of general set theory and some his his techniques and notions have been embraced by the folklore of descriptive set theory. On a wider scale, hausdorff was one of the pioneers of modern, structural mathematics without which present-day descriptive set theory cannot be carried out. Bearing in mind Hausdorff's original contributions to descriptive set theory and the historic proximity or even identification of descriptive set theory and the theory of metric spaces it appears justified to call Hausdorff one of the founders of descriptive set theory in line with Baire, Borel and Lebesgue. Alexandroff and Hopf write ([32], p.20):

> Ihre weitere Entwicklung [die weitere Entwicklung der deskriptiven Mengenlehre – P.K.] beginnt elf Jahre später mit dem Mächtigkeitssatz für Borelsche Mengen.[16]

And in the *Dictionary of Scientific Biography*, M. Katetov [31] describes Hausdorff's achievements with the words:

> His [Hausdorff's – P.K.] broad approach, his aesthetic feeling, and his sense of balance may have played a substantial part. He succeeded in creating a theory of topological and metric spaces into which the previous results fitted well, and he enriched it with many new notions and theorems. From the modern point of view, the *Grundzüge* contained, in addition to other special topics, the beginnings of the theories of topological and metric spaces, which are now included in all textbooks on the subject. In the *Grundzüge*, these theories were laid down in such a way that a strong impetus was provided for their further development. Thus, Hausdorff can rightly be considered the founder of general topology and of the general theory of metric spaces.

16. The further development of descriptive set theory begins eleven years later with the cardinality theorem for the Borel sets.

9 Biographical data of Felix Hausdorff

Felix Hausdorff was born on November 8, 1868 at Breslau (now Wroclaw, Poland). He spent his childhood at Leipzig. From 1887 to 1891 he studied mathematics and astronomy at Freiburg, Berlin and Leipzig. He obtained his doctoral degree in 1891 and his habilitation in 1895 from the University of Leipzig in the fields of mathematics and astronomy. Hausdorff began to work in set theory in 1901. Since 1903 he was a professor of mathematics at Leipzig, Bonn and Greifswald. From 1921 to 1935 he held a chair of mathematics at the University of Bonn. As a jew, Hausdorff and his family were subjected to grave threats and humiliations during the Nazi period. To avoid being deported to a concentration camp, Felix Hausdorff, his wife Charlotte, and his sister in law Edith Pappenheim jointly committed suicide on January 26, 1942.

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