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THE CONSISTENCY STRENGTH OF THE FREE-SUBSET PROPERTY FOR ω_ω

PETER KOEPKE

A subset X of a structure S is called *free* in S if $\forall x \in X\ x \notin S[X - \{x\}]$; here, $S[Y]$ is the substructure of S generated from Y by the functions of S. For κ, λ, μ cardinals, let $\mathrm{Fr}_\mu(\kappa, \lambda)$ be the assertion:

> for every structure S with $\kappa \subset S$ which has at most μ functions and relations there is a subset $X \subset \kappa$ free in S of cardinality $\geq \lambda$.

We show that $\mathrm{Fr}_\omega(\omega_\omega, \omega)$, the *free-subset property* for ω_ω, is equiconsistent with the existence of a measurable cardinal (2.2,4.4). This answers a question of Devlin [De].

In the first section of this paper we prove some combinatorial facts about $\mathrm{Fr}_\mu(\kappa, \lambda)$; in particular the first cardinal κ such that $\mathrm{Fr}_\omega(\kappa, \omega)$ is weakly inaccessible or of cofinality ω (1.2). The second section shows that, under $\mathrm{Fr}_\omega(\omega_\omega, \omega)$, ω_ω is measurable in an inner model. For the convenience of readers not acquainted with the core model K, we first deduce the existence of $0^\#$ (2.1) using the inner model L. Then we adapt the proof to the core model and obtain that ω_ω is measurable in an inner model. For the reverse direction, we essentially apply a construction of Shelah [Sh] who forced $\mathrm{Fr}_\omega(\omega_\omega, \omega)$ over a ground model which contains an ω-sequence of measurable cardinals. We show in §4 that indeed a *coherent sequence of Ramsey cardinals* suffices. In §3 we obtain such a sequence as an endsegment of a Prikry sequence.

The techniques of this paper can be applied to other situations. We may easily replace ω_ω by cardinals like $\omega_{\omega+\omega}, \omega_{\omega\cdot\omega}, \ldots$ in the above. Using "higher" core models I could show that $\mathrm{Fr}_\omega(\omega_{\omega_1}, \omega_1)$ is equiconsistent with the existence of ω_1 measurable cardinals. These results form part of my doctoral thesis.

§1. In the context of partition cardinals one often obtains better indiscernibility properties by choosing the homogeneous set in a minimal way. We use similar tricks to get "strongly free" sets.

1.1. LEMMA. *Let λ be an infinite cardinal and assume* $\mathrm{Fr}_\mu(\kappa, \lambda)$. *Let S be a structure with $\kappa \subset S$ which has at most μ functions and relations. Then there is a subset $X \subset \kappa$ free in S with monotone enumeration $(x_i : i < \lambda)$ such that*:

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(i) $i < \lambda \to [x_i, x_i^+) \cap S[x_i \cup \{x_j{:}i < j < \lambda\}] = \varnothing$ (x_i^+ *is the smallest cardinal* $> x_i$); *and, in particular,*

(ii) $i < \lambda \to x_i \notin S[x_i \cup \{x_j{:}\ i < j < \lambda\}]$.

PROOF. We assume that S contains the relation $< \bigcap \kappa^2$ and possesses a set of Skolem functions for itself. Let $Y \subset \kappa$ be free in S of cardinality λ. Construct a sequence $(x_i{:}\ i < \lambda)$ of ordinals $< \kappa$ and a sequence $(W_i{:}\ i < \lambda)$ of finite subsets of Y by recursion: Let $i < \lambda$ and let $(x_j{:} j < i)$, $(W_j{:} j < i)$ be defined. Then let $x_i =$ the smallest $\alpha < \kappa$ such that for some finite $W \subset Y$:

(*) if $j_1, \dots, j_m < i$ then
$\{x_{j_1}, \dots, x_{j_m}\}, \{\alpha\}, (Y - W_{j_1} - \cdots - W_{j_m} - W)$ are pairwise disjoint, and
$\{x_{j_1}, \dots, x_{j_m}\} \cup \{\alpha\} \cup (Y - W_{j_1} - \cdots - W_{j_m} - W)$ is a free subset in S.

Let W_i be such a W for x_i.

If $(x_j{:} j < i)$, $(W_j{:} j < i)$ are constructed consider $\alpha \in Y - \bigcup \{W_j{:} j < i\}$ and $W = \{\alpha\}$. α and W satisfy (*) at i. Hence the recursion does not break down. $X{:} = \{x_i{:}\ i < \lambda\}$ is free in S with monotone enumeration $(x_i{:}\ i < \lambda)$.

(ii) *holds for* X. Assume not. Let f be a function of S, $\vec{\alpha} < x_i$, $i < k_1 \le \cdots \le k_n < \lambda$, and $x_i = f(\vec{\alpha}, x_{k_1}, \dots, x_{k_n})$. For convenience assume that $\vec{\alpha} = \alpha$ has just one member. Assume also that α is minimal with $x_i = f(\alpha, x_{k_1}, \dots, x_{k_n})$. There is a function g of S such that $\alpha = g(x_i, x_{k_1}, \dots, x_{k_n})$. Let $W = W_i \cup W_{k_1} \cup \cdots \cup W_{k_n}$.

Claim. α, W *satisfy* (*) *at* i.

PROOF. Let $j_1, \dots, j_m < i$ be pairwise distinct.

(a) By construction,

$$x_{j_1} \notin S[\{x_{j_2}, \dots, x_{j_m}\} \cup \{x_i, x_{k_1}, \dots, x_{k_n}\} \cup (Y - W_{j_1} - W_{j_2} - \cdots - W_{j_m} - W_i - W_{k_1} - \cdots - W_{k_n})];$$

and since $\alpha = g(x_i, x_{k_1}, \dots, x_{k_n})$,

$$x_{j_1} \notin S[\{x_{j_2}, \dots, x_{j_m}\} \cup \{\alpha\} \cup (Y - W_{j_1} - W_{j_2} - \cdots - W_{j_m} - W)].$$

(b) By construction,

$$x_i \notin S[\{x_{j_1}, \dots, x_{j_m}\} \cup \{x_{k_1}, \dots, x_{k_n}\} \cup (Y - W_{j_1} - \cdots - W_{j_m} - W_i - W_{k_1} - \cdots - W_{k_n})].$$

Since $x_i = f(\alpha, x_{k_1}, \dots, x_{k_n})$,

$$\alpha \notin S[\{x_{j_1}, \dots, x_{j_m}\} \cup (Y - W_{j_1} - \cdots - W_{j_m} - W)].$$

(c) Let $y \in Y - W_{j_1} - \cdots - W_{j_m} - W$. By construction,

$$y \notin S[\{x_{j_1}, \dots, x_{j_m}\} \cup \{x_i, x_{k_1}, \dots, x_{k_n}\} \cup (Y - W_{j_1} - \cdots - W_{j_m} - W_i - W_{k_1} - \cdots - W_{k_n} - \{y\})].$$

Since $\alpha = g(x_i, x_{k_1}, \dots, x_{k_n})$,

$$y \notin S[\{x_{j_1}, \dots, x_{j_m}\} \cup \{\alpha\} \cup (Y - W_{j_1} - \cdots - W_{j_n} - W - \{y\})].$$

QED (claim).

But the claim contradicts the minimal choice of x_i. Hence (ii) holds for X.

To obtain an X satisfying (i) as well we assume further that S possesses "cardinality"-functions $r, s\colon \kappa \times \kappa \to \kappa$ with the property

$$\alpha < \beta < \kappa \to (r(\alpha, \beta) < \operatorname{card}(\beta) \,\&\, s(r(\alpha, \beta), \beta) = \alpha).$$

Then take $X \subset \kappa$ with monotone enumeration $(x_i\colon i < \lambda)$ which satisfies (ii).

(i) *holds for* X. Assume not; let $\beta \in [x_i, x_i^+) \cap S[x_i \cup \{x_j\colon i < j < \lambda\}]$. Of course $x_i < \beta$. Let $\gamma = r(x_i, \beta) < \operatorname{card}(\beta) \leq x_i$, and we have $x_i \in S[x_i \cup \{x_j\colon i < j < \lambda\}]$, since $x_i = s(\gamma, \beta)$. This contradicts (ii). QED

1.2. LEMMA. *Let λ be an infinite cardinal and let κ be the least cardinal such that* $\mathrm{Fr}_\omega(\kappa, \lambda)$. *Then*:

(i) *κ is a limit cardinal.*

(ii) *For all $\mu < \kappa$,* $\mathrm{Fr}_\mu(\kappa, \lambda)$.

(iii) *κ is weakly inaccessible or* $\operatorname{cof}(\kappa) = \operatorname{cof}(\lambda)$.

(iv) $\kappa \geq \omega_\lambda$.

PROOF. (i) is standard; see [Sh].

(ii) Let $\mu < \kappa$, and let $A = (\kappa, (f_\nu\colon \nu < \mu))$ be an algebra. For $n < \omega$ define $F_n\colon \kappa^{n+1} \to \kappa$ by $F_n(\nu, x_1, \ldots, x_n) = f_\nu(x_1, \ldots, x_n)$, if $\nu < \mu$ and f_ν is n-ary, and $F_n(\nu, x_1, \ldots, x_n) = 0$ else.

Since not $\mathrm{Fr}_\omega(\mu, \lambda)$, let $B = (\mu, (g_n\colon n < \omega))$ have no free subset of cardinality λ. Set $S = (\kappa, (F_n\colon n < \omega), (g_n\colon n < \omega))$, and let $X \subset \kappa$, $\operatorname{card}(X) = \lambda$, be free in S satisfying 1.1(ii). Let $Y = X - \mu$. By the choice of B, $\operatorname{card}(Y) = \lambda$. Y is free in A: if $y \in Y$ then

$$y \notin S[\mu \cup (Y - \{y\})] \supset A[Y - \{y\}].$$

(iii) Assume that κ is not weakly inaccessible and $\operatorname{cof}(\kappa) \neq \operatorname{cof}(\lambda)$. Let $\mu = \operatorname{cof}(\kappa)$. Let $(\theta_\nu\colon \nu < \mu)$ be a sequence of cardinals cofinal in κ; for $\nu < \mu$ let $S^\nu = (\theta_\nu, (f_i^\nu\colon i < \omega))$ be a structure with no free subset of size λ. Let $S = (\kappa, (f_i^\nu\colon i < \omega, \nu < \mu))$. By (ii), there is a free subset $X \subset \kappa$ in S, $\operatorname{card}(X) = \lambda$. Since $\mu \neq \operatorname{cof}(\lambda)$, $\operatorname{card}(X \cap \theta_\nu) = \lambda$ for some $\nu < \mu$. But $X \cap \theta_\nu$ is free in S^ν. Contradiction.

(iv) For λ regular this follows from (i) and (iii). If λ is a singular cardinal we have $\kappa \geq \omega_{\lambda'}$ for all regular $\lambda' < \lambda$; thus $\kappa \geq \omega_\lambda$. QED

§2.

2.1. THEOREM. *If* $\mathrm{Fr}_\omega(\omega_\omega, \omega)$ *then* $0^\#$ *exists.*

PROOF. Assume $\mathrm{Fr}_\omega(\omega_\omega, \omega)$ but that $0^\#$ does not exist. Set $\kappa = \omega_\omega$. By the Jensen covering theorem for L [De-Je], L "covers" V: for all $A \subset \mathrm{On}$ there is $B \in L$ such that $A \subset B$ and $\operatorname{card}(B) \leq \operatorname{card}(A) + \omega_1$. So there is $E \in L$ with $\{\omega_i\colon i < \omega\} \subset E \subset \kappa$ and $\operatorname{card}(E) = \omega_1$.

Let $S = (L_{\kappa^+}, E, (\alpha\colon \alpha < \omega_2))$ together with Skolem functions, where E and every $\alpha < \omega_2$ are constants of the structure. Take $X \subset \kappa$ with monotone enumeration $(x_i\colon i < \omega)$ free in S such that 1.1(i) holds.

For $i < \omega$, let $M_i = S[\{x_j\colon i \leq j < \omega\}]$ and let $\pi_i\colon M_i \simeq \bar{M}_i$, where $\bar{M}_i$ is transitive. For $i \leq j < \omega$, let $\pi_{ji} = \pi_i \circ \pi_j^{-1}\colon \bar{M}_j \to \bar{M}_i$.

(1) *π_{ji} is an elementary embedding and* $\pi_{ji} \restriction \omega_2 = \mathrm{id} \restriction \omega_2$.

For $i < \omega$ let $E_i = \pi_i(E)$; $\pi_{ji}(E_j) = E_i$. Let $\leq_L$ be the canonical wellordering of L. Since $\leq_L$ is absolute for transitive ZF^--models we get $E_j \leq_L E_i$ for $i \leq j < \omega$. There is $i < \omega$ such that $E_{i+1} = E_i$. Since "x is the αth element of E_i" is uniformly definable in M_i and in M_{i+1} and since $\pi_{i+1,i}$ is the identity on ω_2, we have:

(2) $\pi_{i+1,i} \restriction E_{i+1} = \mathrm{id} \restriction E_{i+1}$.

Let $\delta = \pi_{i+1}(x_i^+) \in E_{i+1}$. Then

$$\begin{aligned}\pi_{i+1,i}(\delta) &= \pi_i(x_i^+) > \pi_i(x_i) = \mathrm{otp}(M_i \cap x_i) \geq \mathrm{otp}(M_{i+1} \cap x_i) \\ &= \mathrm{otp}(M_{i+1} \cap x_i^+)[\text{by } 1.1(\mathrm{i})] = \pi_{i+1}(x_i^+) = \delta,\end{aligned}$$

contradicting (2). QED

2.2. THEOREM. *If* $\mathrm{Fr}_\omega(\omega_\omega, \omega)$ *then there is an inner model with a measurable cardinal* $\leq \omega_\omega$.

PROOF. Set $\kappa = \omega_\omega$. Assume $\mathrm{Fr}_\omega(\kappa, \omega)$ but that there is no inner model with a measurable cardinal $\leq \kappa$. Then $0^\dagger$ does not exist and by the Dodd-Jensen covering theorem [Do], V is covered *either* by the core model K, *or* by some $L[U] \models$ "U is a normal measure on some cardinal $> \kappa$", *or* by some $L[U, C]$ where C is a Prikry sequence for $L[U]$ and $L[U] \models$ "U is a normal measure on some cardinal $> \kappa$". In any case, since $P(\kappa) \cap K = P(\kappa) \cap L[U] = P(\kappa) \cap L[U, C]$, there is $E \in K$ such that $\{\omega_i : i < \omega\} \subset E \subset \kappa$ and $\mathrm{card}(E) = \omega_1$. Let $S = (H^K_{\kappa^+}, E, (\alpha : \alpha < \omega_2))$ together with Skolem functions as in the proof of 2.1. As above we get $E_j \leq_K E_i$ for $i \leq j < \omega$, where $\leq_K$ is the canonical wellordering of K. The remaining argument goes through unchanged. QED

§3. Assume κ is a measurable cardinal with normal ultrafilter U. Let

$$P = \{(a, X) : a \in [\kappa]^{<\omega}, X \in U, \max a < \min X\}$$

be the set of Prikry conditions for κ, U with the usual order. Let G be P-generic over V; let $(\kappa_i : i < \omega)$ be the Prikry sequence induced by G.

3.1. LEMMA. *In* $V[G]$, *the following principle holds: if* $f: [\kappa]^{<\omega} \to \kappa$ *is* regressive, *i.e.* $f(x) < \min x$ *for* $x \in [\kappa]^{<\omega}$, *then there are* $m < \omega$ *and* $(A_i : m \leq i < \omega)$ *such that*

(i) $A_i \subset \kappa_i$ *is cofinal in* κ_i, *and*

(ii) *if* $x, y \in [\kappa]^{<\omega}$, $x, y \subset \bigcup\{A_i : m \leq i < \omega\}$ *and if* $\mathrm{card}(x \cap A_i) = \mathrm{card}(y \cap A_i)$ *for* $m \leq i < \omega$, *then* $f(x) = f(y)$.

PROOF. Assume $(a, X) \Vdash$ "$\mathring{f} : [\kappa]^{<\omega} \to \kappa$ is regressive". It suffices to show that some extension of (a, X) forces the above property for $\mathring{f}$. Let H be a transitive structure containing $\kappa, U, P, (a, X)$, and $\mathring{f}$ as constants, $\kappa \subset H$, and which reflects enough of V to make the following go through. Moreover we assume that H possesses Skolem functions for itself. Since U is a normal measure on κ there is $Y \in U$ which is a set of *good indiscernibles* for H, i.e.,

(1) *for every H-formula* ψ, $x, y \in [Y]^{<\omega}$, $\vec{\alpha} < \min x \cup y$:

$$H \models \psi(\vec{\alpha}, x) \quad \textit{iff} \quad H \models \psi(\vec{\alpha}, y).$$

Let $Z = \{\nu \in Y : Y \cap \nu \text{ is cofinal in } \nu\} \cap X$. We show that $(a, Z) \leq (a, X)$ is as desired.

Let G' be P-generic over V, $(a, Z) \in G'$; let $(\kappa'_i : i < \omega)$ be the induced Prikry sequence. Let $a = \{\kappa'_0, \ldots, \kappa'_{m-1}\}$, and define $(A_i : m \leq i < \omega)$ by $A_i = Y \cap (\kappa'_i - \kappa'_{i-1})$.

Let $x, y \in [\kappa]^{<\omega}$, $x, y \subset \bigcup \{A_i : m \leq i < \omega\}$, and $\operatorname{card}(x \cap A_i) = \operatorname{card}(y \cap A_i)$ for $m \leq i < \omega$. Take n such that $x, y \subset \kappa'_n$. There are H-terms t, w such that

$$(\{\kappa'_0,,, \kappa'_{n-1}\}, t(\kappa'_0,,, \kappa'_{n-1}, x)) \Vdash \mathring{f}(x) = w(\kappa'_0,,, \kappa'_{n-1}, x) < \min x,$$
$$(\{\kappa'_0,,, \kappa'_{n-1}\}, t(\kappa'_0,,, \kappa'_{n-1}, y)) \Vdash \mathring{f}(y) = w(\kappa'_0,,, \kappa'_{n-1}, y) < \min y.$$

(1) implies that $w(\kappa'_0,,, \kappa'_{n-1}, x) = w(\kappa'_0,,, \kappa'_{n-1}, y)$, and so

$$(\{\kappa'_0,,, \kappa'_{n-1}\}, t(\kappa'_0,,, \kappa'_{n-1}, x) \cap t(\kappa'_0,,, \kappa'_{n-1}, y)) \Vdash \mathring{f}(x) = \mathring{f}(y).$$

There is $z \in Z - \kappa'_n$ such that $z \in t(\kappa'_0,,, \kappa'_{n-1}, x) \cap t(\kappa'_0,,, \kappa'_{n-1}, y)$, and, by (1),

$$Z - \kappa'_n \subset t(\kappa'_0,,, \kappa'_{n-1}, x) \cap t(\kappa'_0,,, \kappa'_{n-1}, y).$$

Hence,

$$(\{\kappa'_0,,, \kappa'_{n-1}\}, Z - \kappa'_n) \Vdash \mathring{f}(x) = \mathring{f}(y),$$

and thus $\mathring{f}(x) = \mathring{f}(y)$ is true in $V[G']$ since $(\{\kappa'_0,,, \kappa'_{n-1}\}\ Z - \kappa'_n) \in G'$. QED

3.2. THEOREM. *In $V[G]$ there is an ascending sequence $(\lambda_i : i < \omega)$ of cardinals cofinal in κ which forms a* coherent sequence of Ramsey cardinals, *i.e., for every regressive $f : [\kappa]^{<\omega} \to \kappa$ there are $(A_i : i < \omega)$ such that:*

(i) $A_i \subset \lambda_i$ is cofinal in λ_i, and

(ii) if $x, y \in [\kappa]^{<\omega}, x, y \subset \bigcup \{A_i : i < \omega\}$ and $\operatorname{card}(x \cap A_i) = \operatorname{card}(y \cap A_i)$ for $i < \omega$, then $f(x) = f(y)$.

PROOF. It is enough to see that 3.1 works with some fixed $m = m_0$ for all functions f. Assume not. For every $m < \omega$ take regressive $f^m : [\kappa]^{<\omega} \to \kappa$ with no homogeneous sequence $(A_i : m \leq i < \omega)$. Code the functions f^m into one regressive $f : [\kappa]^{<\omega} \to \kappa$ and apply 3.1. Then, if $(A_i : m \leq i < \omega)$ is homogeneous for f, it is also homogeneous for f^m. Contradiction. QED

§4. If $(\lambda_i : i < \omega)$ is a coherent sequence of Ramsey cardinals it satisfies a weak variant of property (3c) in Shelah [Sh]. We will employ the forcing technique of [Sh]. Since we assume a weaker indiscernibility property, we have to give more consideration to the organisation of the argument.

The following principle will hold in the generic extension:

4.1. DEFINITION. Let $(*)$ be the assertion: If $f : [\omega_\omega]^{<\omega} \to 2$ then there is $(C_i : i < \omega)$ such that:

(i) C_i is a cofinal subset of ω_{2i+2}, and

(ii) if $i_0 < \cdots < i_{n-1} < \omega$ and $\alpha_0, \beta_0 \in C_{i_0}, \ldots, \alpha_{n-1}, \beta_{n-1} \in C_{i_{n-1}}$ then

$$f(\alpha_0, \ldots, \alpha_{n-1}) = f(\beta_0, \ldots, \beta_{n-1}).$$

4.2. LEMMA. *$(*)$ implies* $\operatorname{Fr}_\omega(\omega_\omega, \omega)$.

PROOF. Easy.

Fix a coherent sequence $(\kappa_i : i < \omega)$ of Ramsey cardinals with supremum κ. Let $(P, \leq)$ be the following set of conditions:

$$P = \{(p_i : i < \omega) : p_0 \in \operatorname{Col}(\omega_1, \kappa_0), p_i \in \operatorname{Col}(\kappa^+_{i-1}, \kappa_i) \text{ for } 1 \leq i < \omega\},$$

where $\mathrm{Col}(\sigma,\rho)$ are the Levy conditions for collapsing the inaccessible ρ to σ^+; $(q_i\colon i<\omega)\le(p_i\colon i<\omega)$ iff $\forall i\, q_i\supset p_i$. Let G be P-generic over V. In $V[G]\colon \kappa_0=\omega_2$, $\kappa_1=\omega_4,\ldots,\kappa=\omega_\omega$.

4.3. THEOREM. *(*) holds in* $V[G]$.

PROOF. Let $p=(p_i\colon i<\omega)\in P$ and $p\Vdash$ "$\mathring{f}\colon[\kappa]^{<\omega}\to 2$". It suffices to show that some extension of p forces (*) for $\mathring{f}$. Let

$$R=\{(\alpha_0,\ldots,\alpha_{m-1})\colon m<\omega\;\&\;\forall i<m\,\alpha_i<\kappa_i\}$$

and wellorder R by putting $(\alpha_0,\ldots,\alpha_{m-1})<'(\beta_0,\ldots,\beta_{n-1})$ iff

(a) $m<n$, or

(b) $m=n$ and there is an $i<m$ with $\alpha_i<\beta_i$, $\alpha_{i+1}=\beta_{i+1},\ldots,\alpha_{m-1}=\beta_{m-1}$.

We construct by recursion on $<'$ a sequence $(p(r)\colon r\in R)$, $p(r)=(p_i(r)\colon i<\omega)\in P$, and a sequence $(w(r)\colon r\in R)$ such that:

(1) *If* $i<\omega$, $s=(\alpha_0,\ldots,\alpha_{m-1})<'r=(\beta_0,\ldots,\beta_{n-1})$ *and* $\alpha_i=\beta_i,\ldots,\alpha_{m-1}=\beta_{m-1}$, *then* $p_i\subset p_i(s)\subset p_i(r)$.

Assume that $r=(\beta_0,\ldots,\beta_{n-1})\in R$ and that for $s<'r$ $p(s)$ is constructed satisfying (1). Define $\tilde{p}(r)=(\tilde{p}_i(r)\colon i<\omega)$:

$$\tilde{p}_i(r)=p_i\cup\bigcup\{p_i(s)\colon s=(\alpha_0,\ldots,\alpha_{m-1})<'(\beta_0,\ldots,\beta_{n-1})\text{ and }\alpha_i=\beta_i,\ldots,\alpha_{m-1}=\beta_{m-1}\}.$$

$\tilde{p}(r)$ is a condition since $\mathrm{Col}(\kappa_{i-1}^+,\kappa_i)$ is closed under decreasing sequences of length κ_{i-1} (put $\kappa_{-1}=\omega$). Take $w(r)\in\{0,1\}$ and $p(r)=(p_i(r)\colon i<\omega)\le\tilde{p}(r)$ such that $p(r)\Vdash\mathring{f}(\beta_0,\ldots,\beta_{n-1})=w(r)$. The definition of $p(r)$ agrees with property (1), hence the recursion works.

For $i<n<\omega$ and $\beta_i<\kappa_i,\ldots,\beta_{n-1}<\kappa_{n-1}$ put

$$p_i'(\beta_i,\ldots,\beta_{n-1})=\bigcup\{p_i(\alpha_0,\ldots,\alpha_{m-1})\colon(\alpha_0,\ldots,\alpha_{m-1})\in R,\ m\le n,\text{ and }\alpha_i=\beta_i,\ldots,\alpha_{m-1}=\beta_{m-1}\}.$$

Since $(\kappa_i\colon i<\omega)$ is a coherent sequence of Ramsey cardinals there is $(A_i\colon i<\omega)$, each A_i is cofinal in κ_i, and the homogeneity properties (2) and (3) hold:

(2) *If* $\alpha_0,\beta_0\in A_0,\ldots,\alpha_{n-1},\beta_{n-1}\in A_{n-1}$ *then*

$$w(\alpha_0,\ldots,\alpha_{n-1})=w(\beta_0,\ldots,\beta_{n-1}).$$

(3) *If* $i<n<\omega$ *and* $\alpha_{i+1},\beta_{i+1}\in A_{i+1},\ldots,\alpha_{n-1},\beta_{n-1}\in A_{n-1}$ *then*

$$\forall\alpha_i<\kappa_i\,p_i'(\alpha_i,\alpha_{i+1},\ldots,\alpha_{n-1})=p_i'(\alpha_i,\beta_{i+1},\ldots,\beta_{n-1}).$$

By (3), we may put:

$$p_i''(\alpha_i)=\bigcup\{p_i'(\alpha_i,\alpha_{i+1},,,\alpha_{n-1})\colon i<n<\omega\text{ and }\alpha_{i+1}\in A_{i+1},,,\alpha_{n-1}\in A_{n-1}\};$$

$p_i''(\alpha_i)\in\mathrm{Col}(\kappa_{i-1}^+,\kappa_i)$. Let $i<\omega$. The standard proof of the κ_i-antichain condition for $\mathrm{Col}(\kappa_{i-1}^+,\kappa_i)$ yields $B_i\subset A_i$, B_i cofinal in κ_i such that $\{p_i''(\alpha_i)\colon\alpha_i\in B_i\}$ forms a Δ-system of functions; let $\bar{p}_i$ be its kernel. Put $\bar{p}=(\bar{p}_i\colon i<\omega)$; $\bar{p}\le p$.

We show that $\bar{p}$ forces (*) for $\mathring{f}$: Let $\bar{G}$ be P-generic over V, $\bar{p}\in\bar{G}$. Let $\bar{G}_i$ be the projection of $\bar{G}$ onto the ith coordinate; $\bar{G}_i$ is $\mathrm{Col}(\kappa_{i-1}^+,\kappa_i)$-generic over V. Put $C_i=\{\alpha_i\in B_i\colon p_i''(\alpha_i)\in\bar{G}_i\}$.

(4) C_i *is cofinal in* κ_i.

PROOF. It suffices to show that for $\nu < \kappa_i$, the set

$$D_\nu = \{q \in \mathrm{Col}(\kappa_{i-1}^+, \kappa_i) : q \leq p_i''(\alpha_i) \text{ for some } \alpha_i \in B_i - \nu\}$$

is dense below $\bar{p}_i$ in $\mathrm{Col}(\kappa_{i-1}^+, \kappa_i)$. Let $q \leq \bar{p}_i$. Since $\bar{p}_i$ is the kernel of $\{p_i''(\alpha_i) : \alpha_i \in B_i\}$ and $\mathrm{card}(q) \leq \kappa_{i-1}$, q is incompatible with at most κ_{i-1} of the functions $p_i''(\alpha_i)$, $\alpha_i \in B_i$. So there is $\alpha_i \in B_i - \nu$ such that q and $p_i''(\alpha_i)$ are compatible; and then $q \cup p_i''(\alpha_i) \in D_\nu$. QED(4)

(5) *Let $n < \omega$ and $\alpha_0, \beta_0 \in C_0, \ldots, \alpha_{n-1}, \beta_{n-1} \in C_{n-1}$. Then, in $V[\bar{G}]$,*

$$\mathring{f}^{V[\bar{G}]}(\alpha_0, \ldots, \alpha_{n-1}) = \mathring{f}^{V[\bar{G}]}(\beta_0, \ldots, \beta_{n-1}).$$

PROOF. By definition of the C_i and since the elements of the generic set are compatible, there is $\tilde{p} = (\tilde{p}_i : i < \omega) \in \bar{G}$ such that $\tilde{p} \leq \bar{p}$ and

$$\begin{aligned}
&p_0(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \subset p_0''(\alpha_0) \subset \tilde{p}_0,\\
&p_1(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \subset p_1'(\alpha_1, \ldots, \alpha_{n-1}) \subset p_1''(\alpha_1) \subset \tilde{p}_1,\\
&\quad\vdots\\
&p_{n-1}(\alpha_0, \ldots, \alpha_{n-1}) \subset p_{n-1}'(\alpha_{n-1}) \subset p_{n-1}''(\alpha_{n-1}) \subset \tilde{p}_{n-1},\\
&\quad p_n(\alpha_0, \ldots, \alpha_{n-1}) \subset \bar{p}_n = \tilde{p}_n,\\
&\quad\vdots\\
&\quad p_t(\alpha_0, \ldots, \alpha_{n-1}) \subset \bar{p}_t = \tilde{p}_t,\\
&\quad\vdots
\end{aligned}$$

We moreover assume that $\tilde{p}$ satisfies the same relations for the sequence $(\beta_0, \ldots, \beta_{n-1})$. Thus:

$$\begin{aligned}
&\tilde{p} \leq p(\alpha_0, \ldots, \alpha_{n-1}) \Vdash \mathring{f}(\alpha_0, \ldots, \alpha_{n-1}) = w(\alpha_0, \ldots, \alpha_{n-1}),\\
&\tilde{p} \leq p(\beta_0, \ldots, \beta_{n-1}) \Vdash \mathring{f}(\beta_0, \ldots, \beta_{n-1}) = w(\beta_0, \ldots, \beta_{n-1}).
\end{aligned}$$

By (2), $w(\alpha_0, \ldots, \alpha_{n-1}) = w(\beta_0, \ldots, \beta_{n-1})$. QED(5)

Using simple coding arguments we may assume that if (5) holds for $\mathring{f}$ then all instances of $(*)$ hold for $\mathring{f}$. Hence $\bar{p}$ forces $(*)$ for $\mathring{f}$. QED

Combining 3.2, 4.3 and 4.2 we obtain:

4.4. THEOREM. *If κ is a measurable cardinal then there is a two-stage generic extension of V in which $\mathrm{Fr}_\omega(\omega_\omega, \omega)$ holds.*

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UNIVERSITÄT FREIBURG
FREIBURG, WEST GERMANY

Current address: Mathematical Institute, St. Giles, Oxford, England.

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