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Substitution in First-Order Formulas. Part II. The Construction of First-Order Formulas[1]

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Summary. This article is part of a series of Mizar articles which constitute a formal proof (of a basic version) of Kurt Gödel's famous completeness theorem (K. Gödel, "Die Vollständigkeit der Axiome des logischen Funktionenkalküls", Monatshefte für Mathematik und Physik 37 (1930), 349-360). The completeness theorem provides the theoretical basis for a uniform formalization of mathematics as in the Mizar project. We formalize first-order logic up to the completeness theorem as in H. D. Ebbinghaus, J. Flum, and W. Thomas, Mathematical Logic, 1984, Springer Verlag New York Inc. The present article establishes that every substitution can be applied to every formula as in Chapter III par. 8, Definition 8.1, 8.2 of Ebbinghaus, Flum, Thomas. After that, it is observed that substitution doesn't change the number of quantifiers of a formula. Then further details about substitution and some results about the construction of formulas are proven.

MML Identifier: `SUBSTUT2`.

The papers [15], [10], [17], [3], [7], [13], [1], [11], [2], [6], [18], [9], [8], [12], [14], [16], [5], and [4] provide the terminology and notation for this paper.

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1. Further Properties of Substitution

For simplicity, we adopt the following convention: i, k, n denote natural numbers, p, q, r, s denote elements of CQC-WFF, x, y denote bound variables, P denotes a k-ary predicate symbol, l, l_1 denote variables lists of k, S_1 denotes a CQC-substitution, and S, S_2 denote elements of CQC-Sub-WFF.

Next we state several propositions:

(1) For every S_1 there exists S such that $S_{\mathbf{1}} = \mathrm{VERUM}$ and $S_{\mathbf{2}} = S_1$.

(2) For every S_1 there exists S such that $S_{\mathbf{1}} = P[l_1]$ and $S_{\mathbf{2}} = S_1$.

(3) Let k, l be natural numbers. Suppose P is a k-ary predicate symbol and a l-ary predicate symbol. Then $k = l$.

(4) If for every S_1 there exists S such that $S_{\mathbf{1}} = p$ and $S_{\mathbf{2}} = S_1$, then for every S_1 there exists S such that $S_{\mathbf{1}} = \neg p$ and $S_{\mathbf{2}} = S_1$.

(5) Suppose for every S_1 there exists S such that $S_{\mathbf{1}} = p$ and $S_{\mathbf{2}} = S_1$ and for every S_1 there exists S such that $S_{\mathbf{1}} = q$ and $S_{\mathbf{2}} = S_1$. Let given S_1. Then there exists S such that $S_{\mathbf{1}} = p \wedge q$ and $S_{\mathbf{2}} = S_1$.

Let us consider p, S_1. Then $\langle p, S_1 \rangle$ is an element of $[\!:\mathrm{WFF}, \mathrm{vSUB}:\!]$.

We now state several propositions:

(6) $\mathrm{dom}\,\mathrm{RestrictSub}(x, \forall_x p, S_1)$ misses $\{x\}$.

(7) If $x \in \mathrm{rng}\,\mathrm{RestrictSub}(x, \forall_x p, S_1)$, then $\text{S-Bound}(\langle \forall_x p, S_1 \rangle) = \mathrm{x}_{\mathrm{upVar}(\mathrm{RestrictSub}(x, \forall_x p, S_1), p)}$.

(8) If $x \notin \mathrm{rng}\,\mathrm{RestrictSub}(x, \forall_x p, S_1)$, then $\text{S-Bound}(\langle \forall_x p, S_1 \rangle) = x$.

(9) $\mathrm{ExpandSub}(x, p, \mathrm{RestrictSub}(x, \forall_x p, S_1)) = ({}^{@}\mathrm{RestrictSub}(x, \forall_x p, S_1)) {+\cdot}\, x \restriction \text{S-Bound}(\langle \forall_x p, S_1 \rangle)$.

(10) If $S_{\mathbf{2}} = ({}^{@}\mathrm{RestrictSub}(x, \forall_x p, S_1)) {+\cdot}\, x \restriction \text{S-Bound}(\langle \forall_x p, S_1 \rangle)$ and $S_{\mathbf{1}} = p$, then $\langle S, x \rangle$ is quantifiable and there exists S_2 such that $S_2 = \langle \forall_x p, S_1 \rangle$.

(11) If for every S_1 there exists S such that $S_{\mathbf{1}} = p$ and $S_{\mathbf{2}} = S_1$, then for every S_1 there exists S such that $S_{\mathbf{1}} = \forall_x p$ and $S_{\mathbf{2}} = S_1$.

(12) For all p, S_1 there exists S such that $S_{\mathbf{1}} = p$ and $S_{\mathbf{2}} = S_1$.

Let us consider p, S_1. Then $\langle p, S_1 \rangle$ is an element of CQC-Sub-WFF.

Let us consider x, y. The functor $\mathrm{Sbst}(x, y)$ yielding a CQC-substitution is defined by:

(Def. 1) $\mathrm{Sbst}(x, y) = x \dot{\longmapsto} y$.

2. Facts about Substitution and Quantifiers of a Formula

Let us consider p, x, y. The functor $p(x, y)$ yields an element of CQC-WFF and is defined as follows:

(Def. 2) $p(x, y) = \mathrm{CQCSub}(\langle p, \mathrm{Sbst}(x, y) \rangle)$.

In this article we present several logical schemes. The scheme *CQCInd1* concerns a unary predicate $\mathcal{P}$, and states that:

For every p holds $\mathcal{P}[p]$

provided the parameters meet the following conditions:

- For every p such that the number of quantifiers in $p = 0$ holds $\mathcal{P}[p]$, and
- Let given k. Suppose that for every p such that the number of quantifiers in $p = k$ holds $\mathcal{P}[p]$. Let given p. If the number of quantifiers in $p = k + 1$, then $\mathcal{P}[p]$.

The scheme *CQCInd2* concerns a unary predicate $\mathcal{P}$, and states that:

For every p holds $\mathcal{P}[p]$

provided the following conditions are met:

- For every p such that the number of quantifiers in $p \leq 0$ holds $\mathcal{P}[p]$, and
- Let given k. Suppose that for every p such that the number of quantifiers in $p \leq k$ holds $\mathcal{P}[p]$. Let given p. If the number of quantifiers in $p \leq k + 1$, then $\mathcal{P}[p]$.

We now state three propositions:

(13) VERUM(x, y) = VERUM.

(14) $P[l](x, y) = P[\text{CQC-Subst}(l, \text{Sbst}(x, y))]$ and the number of quantifiers in $P[l]$ = the number of quantifiers in $P[l](x, y)$.

(15) The number of quantifiers in $P[l]$ = the number of quantifiers in CQCSub($\langle P[l], S_1\rangle$).

Let S be an element of QC-Sub-WFF. Then $S_{\mathbf{2}}$ is a CQC-substitution.

Next we state several propositions:

(16) $\langle \neg p, S_1\rangle$ = SubNot($\langle p, S_1\rangle$).

(17)(i) $(\neg p)(x, y) = \neg p(x, y)$, and

(ii) if the number of quantifiers in p = the number of quantifiers in $p(x, y)$, then the number of quantifiers in $\neg p$ = the number of quantifiers in $(\neg p)(x, y)$.

(18) Suppose that for every S_1 holds the number of quantifiers in p = the number of quantifiers in CQCSub($\langle p, S_1\rangle$). Let given S_1. Then the number of quantifiers in $\neg p$ = the number of quantifiers in CQCSub($\langle \neg p, S_1\rangle$).

(19) $\langle p \wedge q, S_1\rangle$ = CQCSubAnd($\langle p, S_1\rangle, \langle q, S_1\rangle$).

(20)(i) $(p \wedge q)(x, y) = p(x, y) \wedge q(x, y)$, and

(ii) if the number of quantifiers in p = the number of quantifiers in $p(x, y)$ and the number of quantifiers in q = the number of quantifiers in $q(x, y)$, then the number of quantifiers in $p \wedge q$ = the number of quantifiers in $(p \wedge q)(x, y)$.

(21) Suppose that

(i) for every S_1 holds the number of quantifiers in p = the number of quantifiers in CQCSub($\langle p, S_1 \rangle$), and
(ii) for every S_1 holds the number of quantifiers in q = the number of quantifiers in CQCSub($\langle q, S_1 \rangle$).
Let given S_1. Then the number of quantifiers in $p \wedge q$ = the number of quantifiers in CQCSub($\langle p \wedge q, S_1 \rangle$).

The function CFQ from CQC-Sub-WFF into vSUB is defined as follows:

(Def. 3) CFQ = QSub $\restriction$ CQC-Sub-WFF .

Let us consider p, x, S_1. The functor QScope(p, x, S_1) yielding a CQC-WFF-like element of [: QC-Sub-WFF, BoundVar :] is defined by:

(Def. 4) QScope(p, x, S_1) = $\langle \langle p$, CFQ($\langle \forall_x p, S_1 \rangle$)$\rangle, x \rangle$.

Let us consider p, x, S_1. The functor Qsc(p, x, S_1) yielding a second q.-component of QScope(p, x, S_1) is defined by:

(Def. 5) Qsc(p, x, S_1) = S_1.

The following propositions are true:

(22) $\langle \forall_x p, S_1 \rangle$ = CQCSubAll(QScope(p, x, S_1), Qsc(p, x, S_1)) and QScope(p, x, S_1) is quantifiable.

(23) Suppose that for every S_1 holds the number of quantifiers in p = the number of quantifiers in CQCSub($\langle p, S_1 \rangle$). Let given S_1. Then the number of quantifiers in $\forall_x p$ = the number of quantifiers in CQCSub($\langle \forall_x p, S_1 \rangle$).

(24) The number of quantifiers in VERUM = the number of quantifiers in CQCSub($\langle$ VERUM, $S_1 \rangle$).

(25) For all p, S_1 holds the number of quantifiers in p = the number of quantifiers in CQCSub($\langle p, S_1 \rangle$).

(26) If p is atomic, then there exist k, P, l_1 such that $p = P[l_1]$.

The scheme *CQCInd3* concerns a unary predicate $\mathcal{P}$, and states that:

For every p such that the number of quantifiers in p = 0 holds $\mathcal{P}[p]$

provided the following condition is satisfied:

- Let given r, s, x, k, l be a variables list of k, and P be a k-ary predicate symbol. Then $\mathcal{P}$[VERUM] and $\mathcal{P}[P[l]]$ and if $\mathcal{P}[r]$, then $\mathcal{P}[\neg r]$ and if $\mathcal{P}[r]$ and $\mathcal{P}[s]$, then $\mathcal{P}[r \wedge s]$.

3. Results about the Construction of Formulas

In the sequel F_1, F_2, F_3 denote formulae and L denotes a finite sequence.

Let G, H be formulae. Let us assume that G is a subformula of H. A finite sequence is called a path from G to H if it satisfies the conditions (Def. 6).

(Def. 6)(i) $1 \leq \operatorname{len} \mathrm{it}$,

(ii) $\mathrm{it}(1) = G$,

(iii) $\mathrm{it}(\operatorname{len} \mathrm{it}) = H$, and

(iv) for every k such that $1 \leq k$ and $k < \operatorname{len} \mathrm{it}$ there exist elements G_1, H_1 of WFF such that $\mathrm{it}(k) = G_1$ and $\mathrm{it}(k+1) = H_1$ and G_1 is an immediate constituent of H_1.

The following propositions are true:

(27) Let L be a path from F_1 to F_2. Suppose F_1 is a subformula of F_2 and $1 \leq i$ and $i \leq \operatorname{len} L$. Then there exists F_3 such that $F_3 = L(i)$ and F_3 is a subformula of F_2.

(28) For every path L from F_1 to p such that F_1 is a subformula of p and $1 \leq i$ and $i \leq \operatorname{len} L$ holds $L(i)$ is an element of CQC-WFF.

(29) Let L be a path from q to p. Suppose the number of quantifiers in $p \leq n$ and q is a subformula of p and $1 \leq i$ and $i \leq \operatorname{len} L$. Then there exists r such that $r = L(i)$ and the number of quantifiers in $r \leq n$.

(30) If the number of quantifiers in $p = n$ and q is a subformula of p, then the number of quantifiers in $q \leq n$.

(31) For all n, p such that for every q such that q is a subformula of p holds the number of quantifiers in $q = n$ holds $n = 0$.

(32) Let given p. Suppose that for every q such that q is a subformula of p and for all x, r holds $q \neq \forall_x r$. Then the number of quantifiers in $p = 0$.

(33) Let given p. Suppose that for every q such that q is a subformula of p holds the number of quantifiers in $q \neq 1$. Then the number of quantifiers in $p = 0$.

(34) Suppose $1 \leq$ the number of quantifiers in p. Then there exists q such that q is a subformula of p and the number of quantifiers in $q = 1$.

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