SOME APPLICATIONS OF SHORT CORE MODELS*

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Communicated by A. Prestel
Received 29 October 1985

We survey the definition and fundamental properties of the family of short core models, which extend the core model $K$ of Dodd and Jensen to include $\alpha$-sequences of measurable cardinals ($\alpha \in \text{On}$). The theory is applied to various combinatorial principles to get lower bounds for their consistency strengths in terms of the existence of sequences of measurable cardinals. We consider instances of Chang's conjecture, 'accessible' Jónsson cardinals, the free subset property for small cardinals, a canonization property of $\omega_\alpha$, and a non-closure property of elementary embeddings of the universe. In some cases, equiconsistencies are obtained.

0. Introduction

A major theme in axiomatic set theory is the ranking of consistency strengths of combinatorial principles by the linear scale of large cardinal axioms. Typically, a forcing construction is employed to extend a model of a large cardinal property to a model of the combinatorial property considered, whereas, given the principle, one seeks for large cardinals within inner models. Constructible models of set theory—Gödel's model $L$ of constructible sets, Silver's $L^\alpha$ for a measurable cardinal [18], Mitchell's $L^F$ for a coherent sequence of measures [11, 12], the core model $K$ of Dodd and Jensen [5] and its generalizations—provide natural inner models for large cardinals up to high orders of measurability. If an ordinal is, say, measurable in some inner model, then it is measurable within some Silver model $L^\alpha$. Thus in applications of inner models it is often advantageous to restrict to well-structured 'L-like' models right away.

In my doctoral dissertation [9] I studied the family of short core models which, roughly speaking, approximate inner models with $\alpha$ measurable cardinals for some ordinal $\alpha$. The covering and condensation properties of short core models were applied to obtain information on the consistency strengths of certain instances of Chang's conjecture, the Jónsson property and the free subset property. The family of short core models which forms just a small subfamily of the general core models for sequences of measures as studied by Mitchell [13, 14] was chosen for several reasons: Since in these models every ordinal carries at most one measure the generalized fine structure theory of Dodd [4] can be

* This research was supported by a Feodor Lynen Research Fellowship of the Alexander von Humboldt Foundation, Bonn and by a Junior Research Fellowship at Wolfson College, Oxford.
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applied (if there were measurable cardinals of Mitchell order $\geq \omega$, a much more complicated fine structure is necessary [6]). Short core models have long stretches of ordinals without measures where the models behave very $L$-like. Short core models satisfy covering theorems with Prikry sequences which are known to fail for large core models. Apart from results and techniques which undoubtedly will be generalized to higher core models we also obtained some equiconsistencies of the strength "there are $\alpha$ measurable cardinals".

The aim of the present article is to give an overview of short core model theory and, taking fundamental properties of mice and short core models for granted, to prove our applications in detail. The theory of my thesis has been improved and I also include results on canonical form properties (Shelah [17]) and on a non-closure property of elementary embeddings as considered by Sureson [20]. Section 1 informally shows how core models canonically arise if one wants to prove the existence of inner models with (many) measurable cardinals. We sketch how such models might be obtained from strong combinatorial principles. Section 2 gives an outline of the coarse (=non-fine-structural) theory of mice. It is possible to a certain extent—and will be done in this paper—to define and use short core models without any explicit fine structure, the fine-structural details being nicely encapsulated within the fundamental theorems on short core models.

These theorems are presented in Section 3. The remaining sections contain the applications of our theory which we indicate here by typical instances (the combinatorial principles will be defined later):

**Theorem 1.** The Chang property $(\omega_3, \omega_2) \rightarrow (\omega_2, \omega_1)$ implies that for every $\alpha \in \text{On}$ there is an inner model with $\alpha$ measurable cardinals (4.3, 2.14).

**Theorem 2.** If $\lambda$ is a Jónsson cardinal such that $\lambda = \omega_\xi$, $\xi < \lambda$, or such that $\lambda$ is regular but not weakly hyper-Mahlo, then for every $\alpha \in \text{On}$ there is an inner model with $\alpha$ measurable cardinals (5.2, 5.4, 2.14).

**Theorem 3.** The theories "ZFC + there is a Jónsson cardinal of cofinality $\omega_1$" and "ZFC + there are $\omega_1$ measurable cardinals" are equiconsistent (5.7).

**Theorem 4.** The free subset property $\text{Fr}_\alpha(\omega_\omega; \omega_1)$ is equiconsistent to "There are $\omega_1$ measurable cardinals" (6.7).

**Theorem 5.** Assume that $\langle \omega_k(\alpha) \mid n < \omega \rangle$ has a $\langle 3 \mid n < \omega \rangle$-canonical form for $i$-times

$\{2, 2, \ldots, 2\}^{2^i}$; $i < \omega$}, where the sequence $\langle \omega_k(\alpha) \mid n < \omega \rangle$ has supremum $\omega_\alpha$. Then for every $\alpha \in \text{On}$ there is an inner model with $\alpha$ measurable cardinals (7.2, 2.14).

**Theorem 6.** If there is an elementary embedding $j : V \rightarrow M$, with critical point $\kappa$
such that $^\alpha M \subseteq M$ and $^\alpha M \not\subseteq M$, then for every $\alpha \in \text{On}$ there is an inner model with $\alpha$ measurable cardinals (8.1, 2.14).

The combinatorial principles considered could be weakened while still yielding the same consequences. In Theorem 1, e.g., a weak Chang property as defined in [7] would give the same conclusion.

This article requires some acquaintance with constructibility in terms of the relativized $J_\alpha$-hierarchy and of iterated ultrapowers. Standard set-theoretical notation will be used throughout.

1. Motivation

Assume we are to define an inner model with $\alpha$ measurable cardinals, that is, $\alpha$ should be the ordertype of the set of measurable cardinals. The Mitchell models $L^F$ are constructible models of this kind, and we have to find a sequence $F$ of filters such that:

\[(*) L^F \models \text{"}F \text{ is a sequence of measures on measurable cardinals and the ordertype of measurables is } \alpha\text{"}.\]

The subsequent informal argument will provide us with a 'local' criterion for $(*)$ to hold. It will allow the construction of $F$ by recursion on the ordinals in the domain of $F$ (we stipulate that $\text{dom}(F) \subseteq \text{On}$ and $F(\kappa)$ is a filter on $\kappa$ for every $\kappa \in \text{dom}(F)$).

Let us analyse the situation where $(*)$ fails. To facilitate our reasoning we assume that $F$ is countably complete (i.e., for every $\kappa \in \text{dom}(F)$ and $\{X_i \mid i < \omega\} \subseteq F(\kappa) : \bigcap \{X_i \mid i < \omega\} \neq \emptyset$, and that $\alpha < \min \text{dom}(F)$. Since $(*)$ is a $\Pi_1$-condition there will be a $\beta \in \text{On}$ such that:

- $J_\beta[F] \models \text{"}F \text{ is a sequence of measures"}$, but
- $J_{\beta+1}[F] \not\models \text{"}F \text{ is not a sequence of measures"}$.\]

In $J_{\beta+1}[F]$ there is a $\kappa \in \text{dom}(F)$ and a $c \subseteq \kappa$ which codes a counterexample to $F(\kappa)$ being a normal measure on $\kappa$ ($c$ could be a non-measured subset of $\kappa$, or it could code a regressive function which is not constant almost everywhere). In some weak sense, $c$ has to be definable over $M := J_\beta[F]$ (using $F$), but to avoid fine-structural arguments we assume here that $c$ is $\Sigma_1$ over $M$. We shall locate $c$ within a naturally defined constructible model.

Let $U := F^\kappa$. A $U$-mouse is a structure $N = J_\kappa[G, U]$ such that:

(i) $N \models \text{"}G \text{ is a sequence of measures on ordinals } > \text{sup}(U)".\]

(ii) $\text{opt dom}(U \cup G) \leq \min \text{dom}(U)$ if $\text{dom}(U) \neq \emptyset$ and $\text{opt}(\text{dom}(G) \cap \eta) < \min \text{dom}(G)$ for all $\eta < \omega \gamma$ if $\text{dom}(U) = \emptyset$.

(iii) $N$ is iterable by the measures in $G$ in terms of iterated ultrapowers.
The low part of the U-mouse \(N\) is defined to be \(\text{lp}(N) := H^N_{\lambda}\), if \(\lambda = \min(\text{dom}(G))\) exists, and \(\text{lp}(N) := N\) otherwise.

\(K[U] := \bigcup \{\text{lp}(N) \mid N\ is\ a\ U\-mouse\}\) is an inner model of ZFC. In case \(K[U] \models \text{"U is a sequence of measures"}, K[U] \) is called a core model.

Now, by the countable completeness of \(F\), \(M = \mathcal{J}_\beta[F \setminus U, U]\) is a U-mouse. Iterate \(M\) by all the measures in \(F \setminus U\ \mu\)-many times for \(\mu\) a sufficiently big regular cardinal; let \(N = \mathcal{J}_\gamma[G, U]\) be this iterate. Indiscernibility arguments show that \(N\) can be extended to a \(U\-mouse\ \(N^+ = \mathcal{J}_{\gamma+1}[G^+, U]\), \(G^+ \supseteq G\). Since the iteration maps are \(\Sigma_1\)-elementary, \(c \in \text{lp}(N^+) \subseteq K[U]\). (If the definition of \(c\) over \(M\) is more involved than \(\Sigma_1\), \(M\) has to be iterated in a more elaborate, fine-structure preserving way.) So if \(F\) is not a sequence of measures in \(L^F\), then \(F(\kappa)\) does not measure \(\mathcal{P}(\kappa) \cap K[F \upharpoonright \kappa]\) for some \(\kappa\). This implies the following criterion for a countably complete filter sequence to satisfy (*):

\((\dagger)\) Assume that for all \(\kappa \in \text{dom}(F)\), \(F(\kappa)\) is a normal measure on \(\mathcal{P}(\kappa) \cap K[F \upharpoonright \kappa]\). Then \(L^F \models \text{"F is a sequence of measures"}\).

This can be extended to:

\((\dagger\dagger)\) Assume that for all \(\kappa \in \text{dom}(F)\), \(F(\kappa)\) is a normal measure on \(\mathcal{P}(\kappa) \cap K[F \upharpoonright \kappa]\). Then \(K[F] \models \text{"F is a sequence of measures"}\). So \(K[F]\) is a core model, and all the models \(K[F \upharpoonright \kappa]\) are core models, too.

\((\dagger\dagger)\) is useful for constructing a good sequence \(F\) by recursion on \(\text{dom}(F)\).

We shall now briefly describe a strategy to show that a strong combinatorial principle \(P\) implies the existence of an inner model with \(\alpha\) measurable cardinals. If it is possible to apply \((\dagger\dagger)\) \(\alpha\) times we are done of course. Otherwise, application of \((\dagger\dagger)\) leads to a maximal core model \(K[U]\) where the measure sequence \(U\) cannot be properly end-extendend. Every \(x \in K[U]\) is in the low part of some \(U\-mouse\ M\), the low part consisting exactly of those elements of \(M\) which are not altered by iterations. \(M\) may be iterated up to a \(U\-mouse\ \(M^* = \mathcal{J}_\beta[C, U]\) where \(C\) is a sequence of closed unbounded filters on regular cardinals. So any set \(x \in K[U]\) and indeed any subset of \(K[U]\) may be considered within some \(M^*\) with a rather simple filter sequence. It then becomes conceivable that \(K[U]\) is an \(L\)-like inner model.

The maximality of \(K[U]\) implies that \(K[U]\) 'covers' the universe to a degree, which imposes traces of constructibility upon the set-theoretical universe. If our principle \(P\) now is sufficiently 'non-constructible' this leads to a contradiction, and the definition of filters according to \((\dagger\dagger)\) must be possible \(\alpha\) times as required.

This vague plan will be realized in various ways in the second half of this paper.

2. Iterable premise

We work with the relativized \(J\)-hierarchy \(J_\alpha[\bar{A}]\), \(\alpha \in \text{On}\), where the \(\bar{A}\) are used as predicates in the recursive definition of the hierarchy. \(J_\alpha[\bar{A}]\) will denote the set
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$[J_\alpha[\hat{A}]]$ as well as the $\in$-structure $\langle [J_\alpha[\hat{A}]], \in \upharpoonright [J_\alpha[\hat{A}]], \hat{A} \upharpoonright [J_\alpha[\hat{A}]] \rangle$. If $M = J_\alpha[\hat{A}]$, then $<_M$ denotes the canonical well-ordering of $M$. For details the reader is referred to Devlin [2].

A class $D$ is simple if (i) every $x \in D$ is of the form $x = \langle \kappa, a \rangle$ where $\kappa \in \text{On}$ and $a \subset \kappa$, and (ii) if $\langle \kappa, a \rangle \in D$ then $\langle \kappa, \kappa \rangle \in D$. A filter sequence as in Section 1 can be viewed as a simple predicate. For $D$ simple define: \( \text{dom}(D) := \{ \kappa \mid \langle \kappa, \kappa \rangle \in D \} \), \( D(\kappa) := \{ a \mid \langle \kappa, a \rangle \in D \} \) for $\kappa \in \text{dom}(D)$, and \( D \upharpoonright X := \{ \langle \kappa, a \rangle \in D \mid \kappa \in X \} \).

$U$ is a measure on $\kappa$ if $U$ is a non-principal, $\kappa$-complete, normal ultrafilter on $\kappa$. $F$ is a sequence of measures if $F$ is simple and $F(\kappa)$ is a measure on $\kappa$ for every $\kappa \in \text{dom}(F)$.

2.1. Definition. Let $D$ be simple. $M = J_\alpha[F, D]$ is a premouse over $D$ provided:

(i) $F$ is simple.

(ii) $\sup \text{dom}(D) < \min \text{dom}(F)$.

(iii) $M \triangleright \langle \text{"F is a sequence of measures"} \rangle$.

Then the set of measurables in $M$ is $\text{meas}(M) := \text{dom}(F \cap M)$.

The low part of $M$ is: $\text{lp}(M) := H^M_\kappa$, if $\kappa = \min \text{meas}(M)$ exists, and $\text{lp}(M) = M$ otherwise ($H_\kappa$ is the class of sets of hereditary cardinality $<\kappa$).

It is easy to see that every $\kappa \in \text{meas}(M)$ is regular within $M$.

2.2. Definition. Let $M = J_\alpha[F, D]$ be a premouse over $D$ and $\kappa \in \text{meas}(M)$. $\pi : M \rightarrow M'$ is called the ultrapower of $M$ at $\kappa$ if:

(i) $M'$ is transitive.

(ii) $\pi$ is $\Sigma_1$-elementary.

(iii) $M' = \{ \pi(f)(\kappa) \mid f \in M, f : \kappa \rightarrow M \}.$

(iv) For $x \in M$, $x \subset \kappa : x \in F(\kappa) \leftrightarrow \kappa \in \pi(x)$.

There is at most one ultrapower of $M$ at $\kappa$ and if it exists it can be obtained by the usual factoring of $(\ast M) \cap M$ modulo $F(\kappa)$. $M'$ will be a premouse over $D$ again, and so the operation may be iterated:

2.3. Definition. Let $M = J_\alpha[F, D]$ be a premouse over $D$. Let $I = \langle \kappa_i \mid i < \theta \rangle$ be an index, i.e., some sequence of ordinals. Then a system

\[ \text{It}(M, I) = \langle \langle M_i \mid i \leq \theta \rangle, \langle \pi_{ij} \mid i \leq j \leq \theta \rangle \rangle \]

is called the iterated ultrapower of $M$ by $I$ if:

(i) $M_0 = M$.

(ii) $\pi_{ij} : M_i \rightarrow M_j$, $\pi_{ii} = \text{id} \upharpoonright M_i$, $\pi_{ik} = \pi_{jk} \circ \pi_{ij}$ ($i \leq j \leq k \leq \theta$).

(iii) For each limit ordinal $k \leq \theta$, $\langle M_k, \langle \pi_{ik} \mid i \leq k \rangle \rangle$ is the transitive direct limit of $\langle \langle M_i \mid i < k \rangle, \langle \pi_{ij} \mid i \leq j < k \rangle \rangle$.

(iv) $M_i$ is a premouse over $D$ ($i \leq \theta$).
(v) If \( i < \theta \) and \( \kappa_i \in \text{meas}(M_i) \), then \( \pi_{i,i+1}:M_i \rightarrow M_{i+1} \) is the ultrapower of \( M_i \) at \( \kappa_i \).

(vi) If \( i < \theta \) and \( \kappa_i \in \text{meas}(M_i) \), then \( \pi_{i,i+1} = \text{id} \upharpoonright M_i \) and \( M_{i+1} = M_i \).

We shall write \( M_i \) for \( M_0 \) and \( \pi_i:M \rightarrow M_i \) for \( \pi_{00}:M \rightarrow M_0 \). \( \text{It}(M, I) \) is called an iteration \( \ni k \) if \( \kappa_i \ni k \) for all \( i < \theta \).

2.4. Lemma. Let \( \text{It}(M, I) \) be an iteration \( \ni k \) of \( M \) by \( I \). Then:

(i) \( \pi_i:M \rightarrow M_i \) is \( \Sigma_1 \)-elementary.

(ii) \( H^M_\kappa = H^{M_i}_\kappa \) and \( \pi_i \upharpoonright H^M_\kappa = \text{id} \upharpoonright H^{M_i}_\kappa \) (in particular, \( \pi_i \upharpoonright \kappa = \text{id} \upharpoonright \kappa \)).

(iii) \( \mathcal{P}(\kappa) \cap M = \mathcal{P}(\kappa) \cap M_i \).

2.5. Definition. A premouse \( M \) over \( D \) is iterable if \( \text{It}(M, I) \) exists for all indices \( I \).

We are now giving a few criteria for \( M \) to be iterable. Recall that a simple predicate \( F \) is countably complete if for every \( \kappa \in \text{dom}(F) \) and \( \{X_n \mid n < \omega \} \subset F(\kappa) \cap \{X_n \mid n < \omega \} \neq \emptyset \).

2.6. Theorem. Let \( M = J_a[F, D] \) be a premouse over \( D \).

(i) If \( F \) is countably complete, then \( M \) is iterable.

(ii) Assume that \( M \) is iterable and that \( \pi:M \rightarrow \tilde{M} \) is \( \Sigma_1 \)-elementary, \( \tilde{M} = J_a[F, D] \). Then \( \tilde{M} \) is an iterable premouse over \( D \).

For (ii), one shows that every iterate of \( \tilde{M} \) can be embedded canonically in some iterate of \( M \) so that \( \pi \), the iteration maps, and the embeddings commute.

2.7. Theorem. Let \( Q \) be a transitive model of a sufficiently large finite part of ZFC and assume \( \omega_1 \in Q \). Then iterable premice are absolute for \( Q \), i.e., if \( D, M \in Q \) then: \( M \) is an iterable premouse over \( D \) iff \( Q \models "M \) is an iterable premouse over \( D" \).

2.8. Theorem. Let \( Q \) be a transitive model of a sufficiently large part of ZFC, and let iterable premice be absolute for \( Q \). Let \( j:Q \rightarrow Q \) be elementary, and \( \bar{Q} \) transitive. Then iterable premice are absolute for \( \bar{Q} \), too.

The main tool in working with iterable premice is to 'compare' different ones by iterating them up to members of the same relativized \( J \)-hierarchy. This requires a coherence condition for the measure sequences to hold (Mitchell [11, 12]), and we shall ensure this condition by studying premice whose sequence of measurables is 'short':

2.9. Definition. Let \( M = J_a[F, D] \) be a premouse over \( D \). \( M \) is called short if
either (i) $D = \emptyset$ and $\text{opt}(\text{meas}(M) \cap \gamma) < \min \text{meas}(M)$ for all $\gamma < \omega\alpha$ or (ii) $D \neq \emptyset$ and $\text{opt} \text{meas}(M) \leq \min \text{dom}(D)$.

A $D$-premouse is a short premouse over $D$, and a $D$-mouse is an iterable short premouse over $D$.

The shortness of a $D$-mouse $M$ implies that for any iteration $\pi_I : M \rightarrow M_I$ we have $\text{meas}(M_I) = \pi_I^\#\text{meas}(M)$; no 'new' measurables come up in the iteration which greatly simplifies the structural analysis of iterates and many 'iterate and compare'-arguments.

In the proof of the following theorem, both $D$-mice are iterated up to a simple predicate $C$ consisting of closed unbounded filters.

2.10. Theorem. Let $M, N$ be $D$-mice. Then there are iterations $\pi_I : M \rightarrow M_I$, $\pi_J : N \rightarrow N_J$ such that $M_I, N_J$ are of the form $M_I = J_{\gamma}[C, D], N_J = J_{\delta}[C, D]$ for some simple predicate $C$. Assume further that $\pi_{I*} : M \rightarrow M_{I*}$, $\pi_{J*} : N \rightarrow N_{J*}$ is another pair of iterations to a common predicate $C^* : M_{I*} = J_{\gamma*}[C^*, D], N_{J*} = J_{\delta*}[C^*, D]$. Then: $\gamma \leq \delta$ iff $\gamma^* \leq \delta^*$.

2.11. Definition. For $D$-mice $M, N$ set $M \leq^*_D N$ iff there are iterations $\pi_I : M \rightarrow M_I$, $\pi_J : N \rightarrow N_J$ such that $M_I = J_{\gamma}[C, D], N_J = J_{\delta}[C, D]$, $C$ is simple, and $\gamma \leq \delta$. Define $M \sim_D N$ iff $M \leq^*_D N$ and $N \leq^*_D M$.

Using 2.10, we get:

2.12. Theorem. (i) $\sim_D$ is an equivalence relation on the class of $D$-mice.
(ii) $\leq^*_D$ induces a well-ordering of the $\sim_D$-equivalence classes.
(iii) If $M, N$ are $D$-mice and $\sigma : M \rightarrow N$ is $\Sigma_1$-elementary then $M \leq^*_D N$.
(iv) If $M \sim_D N$ are $D$-mice, $M = J_{\alpha}[F, D], N = J_{\beta}[G, D]$, and if $(F \upharpoonright \gamma) \cap M \cap N = (G \upharpoonright \gamma) \cap M \cap N$, then $\mathcal{P}(\gamma) \cap M = \mathcal{P}(\gamma) \cap N (\gamma < \omega\alpha, \omega\beta)$.

The set $0^\# \in \omega$ defined by Solovay [19] can be viewed as the smallest mouse transcending all the $J_{\alpha}$, $\alpha \in \text{On}$ (the existence of $0^\#$ is equivalent to the existence of an iterable premouse $J_{\kappa+1}[U]= "U$ is a measure on $\kappa"$). Likewise, one can define a real number $0^{\text{long}} \in \omega$ coding the smallest iterable premouse over $\emptyset$ which is not short. Analogous to the Jensen [3] covering theorem for $L$ which holds under the assumption "$0^\#$ does not exist", we will obtain a covering result assuming that "$0^{\text{long}}$ does not exist". This is no restriction for the intended applications as demonstrated by Theorem 2.14. There is no need here to exhibit $0^{\text{long}}$ explicitly and we define:

2.13. Definition. "$0^{\text{long}}$ exists" means that there is an iterable premouse over $\emptyset$ which is not short. "$0^{\text{long}}$" stands for "$0^{\text{long}}$ does not exist".
2.14. Theorem. If $0^{\text{long}}$ exists, then for every $\alpha \in \text{On}$ there is an inner model with $\alpha$ measurable cardinals $< \alpha^+ + \omega_2$.

To get this one iterates the non-short iterable premouse at its smallest measurable until the iterate has $\geq \alpha + 1$ measurables. The first $\alpha$ measures will then be measures in the associated Mitchell model.

3. Short core models

3.1. Definition. Let $D$ be simple such that $D = \emptyset$ or $\text{opt dom}(D) \leq \min(D)$. Define the class $K[D]$ as:

$$K[D] = \bigcup \{ \text{Ip}(M) \mid M \text{ is a } D\text{-mouse} \}.$$  

For $\alpha \in \text{On}$ set $K_\alpha[D] = H^K[D]_{\alpha}$. Fix $K[D]$ for the moment.

3.2. Theorem. $K[D]$ is a transitive inner model of ZFC.

Let $\bar{D} := D \cap K[D] \in K[D]$ and $K[D] \vdash \text{"}V = K[\bar{D}]\text{"}$. If $\alpha > \sup \text{dom}(D)$ is an uncountable cardinal in $K[D]$, then $K_\alpha[D] \vdash \text{"}V = K[\bar{D}]\text{"}$.  

3.3. Lemma. Let $Q$ be a transitive model of a sufficiently large finite part of ZFC + $\text{"}V = K[\bar{D}]\text{"}$, where $\bar{D} := D \cap Q \in Q$. Assume $\text{dom} (\bar{D}) = \text{dom}(D)$ and that $\bar{D}$-mice are absolute for $Q$. Then $Q \subset K_\alpha[D]$ where $\alpha = \text{On} \cap \text{Q} \leq \omega$.

3.4. Theorem. For $x, y \in K[D]$ set $x \leq_D y$ iff $x \leq_M y$ for every $D$-mouse $M$ where $x, y \in \text{Ip}(M)$. Then $\leq_D$ is a well-ordering of $K[D]$.  

This is due to the fact that $\leq_M$ is uniformly $\Sigma_1(M)$ and hence preserved under iterations of premice. The following technical result will be used in Section 6:

3.5. Lemma. Let $\sigma : P \rightarrow Q$ be an elementary map between transitive models of a sufficiently large finite part of ZFC. Let $D_Q := D \cap Q \in Q$. Let $Q \vdash \text{"}V = K[D_Q]\text{"}$ and let $D_Q$-mice be absolute for $Q$. Assume $D_P := D \cap P \in P$, $D_Q = \sigma(D_P)$, and $\text{dom}(D_P) = \text{dom}(D) = \text{dom}(D_Q)$. Then $x \leq_D \sigma(x)$ for all $x \in P$.

3.6. Definition. A model $K[U]$ is called a (short) core model if $K[U] \vdash \text{"}U \text{ is a sequence of measures}\text{"}$. A set $U$ such that $K[U]$ is a core model is called strong.

Most properties of short core models we have proved need the assumption "$0^{\text{long}}$ does not exist"; this condition will be included as ($\neg 0^{\text{long}}$) in the statement of theorems.
3.7. **Theorem** ($\neg 0^{\text{long}}$). Let $K[U]$ be a core model. Then $K[U] \models \text{GCH}$. Other $L$-like combinatorial properties also hold in $K[U]$.

3.8. **Definition.** For strong $U$, $U'$ set $U \leq^e U'$ if $U = U' \upharpoonright \eta$ for some $\eta \in \text{On}$ ("$U'$ is an end-extension of $U"$). Set $U <^e U'$ if $U \leq^e U'$ and $U \neq U'$. A strong $U$ is maximal if there is no $U' >^e U$.

3.9. **Theorem** ($\neg 0^{\text{long}}$). Let $U$ be strong and $\gamma \in \text{On}$. Then:
   
   (i) $U \upharpoonright \gamma$ is strong.
   
   (ii) $K[U \upharpoonright \gamma] \subseteq K[U]$, indeed $K[U \upharpoonright \gamma] = (K[U] \upharpoonright \gamma)^{K[U]}$ where $\bar{U} = U \cap K[U]$.
   
   (iii) $\mathcal{P}(\gamma) \cap K[U \upharpoonright \gamma] = \mathcal{P}(\gamma) \cap K[U]$.

3.10. **Theorem** ($\neg 0^{\text{long}}$). Let $\langle U_\beta \mid \beta < \eta \rangle$ be an $\leq^e$-ascending chain of strong sets. Then $\bigcup \{U_\beta \mid \beta < \eta\}$ is strong.

Theorem 3.10 readily implies:

3.11. **Theorem** ($\neg 0^{\text{long}}$). Let $U$ be strong. Then there is a maximal strong $U' \geq^e U$.

3.12. **Lemma** ($\neg 0^{\text{long}}$). Let $U$ be strong. Then for every regular cardinal $\eta$: $\sup(\text{dom}(U) \cap \eta) < \eta$.

3.13. **Theorem** ($\neg 0^{\text{long}}$). Let $U$ be strong and let $j: K[U] \rightarrow W$ be elementary, $W$ transitive. Then $W = K[U']$ where $U' = j(U \cap K[U])$.

The following results show that under $\neg 0^{\text{long}}$ the family of short core models is generated from a unique canonical core model by iterated ultrapowers. Iterated ultrapowers of inner models are very similar to iterated ultrapowers of premice (2.3).

3.14. **Theorem** ($\neg 0^{\text{long}}$). Let $K[U], K[U']$ be core models with $\text{dom}(U) = \text{dom}(U')$. Then $|K[U]| = |K[U']|$ and $U \cap K[U] = U' \cap K[U]$.

3.15. **Definition.** Let $U_{\text{can}}$ be the unique maximal strong sequence satisfying (i) $U_{\text{can}} \subseteq K[U_{\text{can}}]$, and (ii) if $\kappa \in \text{dom}(U_{\text{can}})$, then $\kappa$ is the minimal ordinal $\xi$ such that there is some $U' >^e U_{\text{can}} \upharpoonright \kappa$ with $\xi = \min(\text{dom}(U' \setminus (U_{\text{can}} \upharpoonright \kappa)))$. $U_{\text{can}}$ is called the canonical sequence and $K[U_{\text{can}}]$ is the canonical core model.

3.16. **Theorem** ($\neg 0^{\text{long}}$). Let $K[U]$ be a core model. Then there is an iteration $\pi_1: K[U_{\text{can}}] \rightarrow K[U']$ such that $U' \geq^e U$.

3.17. **Theorem** ($\neg 0^{\text{long}}$). Let $j: K[U_{\text{can}}] \rightarrow K[U]$ be elementary. Then $j$ is the
iteration map of a normal iterated ultrapower of $K[U_{can}]$, normal meaning that its index is a strictly increasing sequence of ordinals.

3.18. Theorem ($\neg 0^{long}$) (Embedding Theorem). Let $K[U]$ be a core model and let $j: K[U] \rightarrow K[U]$ be elementary with critical point $\kappa > \text{sup dom}(U)$. Assume $\delta$ is a regular cardinal $> \kappa$ which is a limit cardinal in $K[U]$. Then there exists a strong $U' >_e U$ with $\tau := \text{min dom}(U' \setminus U)$ satisfying (i) $\tau \geq \kappa$, and (ii) $\tau = \delta$ if $\delta = \omega_1$ and $\tau < \delta$ if $\delta > \omega_2$.

The following is a covering theorem for situations away from the measurables of the canonical sequence:

3.19. Theorem ($\neg 0^{long}$) (Covering Theorem). Let $\tau \geq \omega_2$ such that $\text{sup dom}(U_{can} \upharpoonright \tau + 1) < \tau$. Let $X \subset \tau$, card$(X)$ regular, and card$(X) < \text{card}(\tau)$. Then there exists $Z \in K[U_{can}]$, $X \subset Z$ such that card$_{K[U_{can}]}(Z) < \tau$.

3.20. Theorem ($\neg 0^{long}$). Let $\tau$ be an ordinal with $\text{sup dom}(U_{can} \upharpoonright \tau + 1) < \tau$. Then:

(i) If $\tau \geq \omega_2$ is a limit ordinal and $\text{cof}(\tau) < \text{card}(\tau)$, then $\tau$ is singular in $K[U_{can}]$.

(ii) If $\tau$ is a singular cardinal in $V$, then $\tau$ is singular in $K[U_{can}]$ and $\tau^+ = \tau^{+K[U_{can}]}$.

Proof. (i) Take $X \subset \tau$ cofinally such that card$(X) = \text{cof}(\tau) < \text{card}(\tau)$. By 3.19 there is $Z \in K[U_{can}]$ such that $X \subset Z$ and card$_{K[U_{can}]}(Z) < \tau$. So $\tau$ is singular in $K[U_{can}]$.

(ii) $\tau$ is singular in $K[U_{can}]$ by (i). Assume $\tau' := \tau^{+K[U_{can}]} < \tau^+$. Then $\text{cof}(\tau') < \tau = \text{card}(\tau')$. We can apply (i) to $\tau'$ and get $\tau'$ being singular in $K[U_{can}]$. Contradiction. □

3.21. Theorem ($\neg 0^{long}$). For all $X \subset \tau \in \text{On}$, card$(X)$ regular there exists $Y \in K[U_{can}]$ such that $X \subset Y$, and card$(Y) \leq \text{card}(X) + \text{card}(\gamma) + \omega_1$, where $\gamma = \text{sup dom}(U_{can} \upharpoonright \tau + 1)$.

Proof. Set $U := U_{can}$. Assume that $X \subset \tau$ is a counterexample to the theorem with $\tau$ minimal. Set $\gamma := \text{sup dom}(U_{can} \upharpoonright \tau + 1)$. Obviously $\tau > \gamma$, $\tau \geq \omega_2$, and card$(X) < \text{card}(\tau)$. By 3.19, there is $Z \in K[U]$ such that $X \subset Z$ and card$_{K[U]}(Z) < \tau$. Take $\tilde{\tau} \subset \tau$, $f \in K[U]$ so that $f: \tilde{\tau} \leftrightarrow Z$. Set $\tilde{X} := f^{-1}X$. By the minimality of $\tau$ there is $\tilde{Y} \in K[U]$, $\tilde{X} \subset \tilde{Y} \subset \tilde{\tau}$ and card$(\tilde{Y}) \leq \text{card}(X) + \text{card}(\gamma) + \omega_1$. But then $Y := f^*\tilde{Y} \in K[U]$, $X \subset Y$, and card$(Y) \leq \text{card}(X) + \text{card}(\gamma) + \omega_1$. Contradiction. □

To obtain covering properties close to the measure sequence we have to extend $K[U_{can}]$ by a Prikry system (note that the universe could be a Prikry extension of some core model).
3.22. Definition. Let $K[U]$ be a core model. A function $C: \text{dom}(U) \to V$ is called a Prikry system for $K[U]$ if:

(i) $C(\kappa) \subset \kappa$ and $\text{otp}(C(\kappa)) \leq \omega$ for $\kappa \in \text{dom}(U)$.

(ii) If $\{x_\kappa | \kappa \in \text{dom}(U)\} \in K[U]$ there is a finite set $p \subset \text{On}$ such that

$$\forall \kappa \in \text{dom}(U) \ (C(\kappa) \setminus p \neq \emptyset \to (x_\kappa \in U(\kappa) \leftrightarrow C(\kappa) \setminus p \subset x \subset \kappa)).$$

The extension of $K[U]$ by a Prikry system $C$ is denoted by $K[U, C]$.

3.23. Theorem (-0 long) (Covering Theorem with Prikry system). There is a Prikry system $C$ for $K[U_{can}]$ such that the following holds: Let $\tau \geq (\text{card} \ \text{ otp dom}(U_{can}))^{++}$. Let $X \subset \tau$, $\text{card}(X)$ regular, and $\text{card}(X) < \text{card}(\tau)$. Then there is a function $f: (\tau)^{<\omega} \to \tau$, $f \in K[U_{can}]$ and a $\gamma < \tau$ such that

$$X \subset Z := \{f(\vec{v}, \vec{\mu}) | \vec{v} < \gamma, \vec{\mu} \in \vec{C} \cap \tau\},$$

where $\vec{C} := \bigcup \{C(\kappa) | \kappa \in \text{dom}(U)\}$. In particular, $\text{card}^{K[U_{can}, C]}(Z) < \tau$.

The next result will be used as a condensation criterion in the two following sections:

3.24. Theorem. Let $D$ be a simple predicate with $\text{otp} \ \text{dom}(D) \leq \text{min}(D)$ or $D = \emptyset$. Let $Q$ be a transitive model of a sufficiently large finite part of $\text{ZFC} + \text{"}V = K[\check{D}]\text{"}$, $\check{D} = D \cap Q \in Q$. Let $\omega_1 \subset Q$ and $\text{dom}(D) = \text{dom}(\check{D})$. Then:

(i) Let $M$ be a $D$-mouse, $\text{meas}(M) \neq \emptyset$ and let $\kappa = \text{min} \ \text{meas}(M)$ be singular in $Q$. Then $\text{lp}(M) \subset Q$.

(ii) Let $\lambda \subset Q$ be a cardinal $> \sup(D)$ and assume the following condition is satisfied:

If $C \subset Q$ be closed unbounded in $\lambda$ then there exists a $\kappa \in C$ which is singular in $Q$.

Then $K_\lambda[D] \subset Q$.

Proof. (i) Let $f \in Q$, $f: v \to \kappa$ cofinal, $v < \kappa$. $\check{D}$-mice are absolute for $Q$ (2.7), and we can take a $D$-mouse $N \in Q$ such that $f \in \text{lp}(N)$. Let $M^*, N^*$ be iterates of $M, N$ respectively such that $N^* \subset M^*$ or $M^* \in N^*$ (2.10). If $N^* \subset M^*$ then $f \in \mathcal{P}(\kappa) \cap N = \mathcal{P}(\kappa) \cap N^* \subset \mathcal{P}(\kappa) \cap M^* = \mathcal{P}(\kappa) \cap M$, contradicting the regularity of $\kappa$ inside $M$. Thus $M^* \in N^*$, and $\text{lp}(M) = H^M_x = H^M_{x^*} \subset H^N_x \subset Q$.

(ii) Let $x \in K_\lambda[D]$. Take a $D$-mouse $M$ such that $x \in \text{lp}(M)$ and $\text{card}(M) < \lambda$ (3.2). If $\text{meas}(M) = \emptyset$, then $M \in L_\lambda[D] \subset Q$. So assume $\text{meas}(M) \neq \emptyset$. Taking the ultrapower by the smallest measurable $\lambda$-times we obtain an iteration $\text{It}(M, I) = \{M_i | i < \lambda\}$ of $M$ by $I = \{\kappa_i | i < \lambda\}$ so that $C := \{\kappa_i | i < \lambda\}$ is closed unbounded in $\lambda$. By the condition, some $\kappa_i, i < \lambda$, is singular in $Q$. By (i), $x \in \text{lp}(M) \subset \text{lp}(M_i) \subset Q$. □

In subsequent proofs, condensation arguments will yield embeddings of some
initial segment of a core model. The final result in this section shows how to get an embedding of the entire core model from this.

3.25. Theorem (\(-0^{\text{long}}\)). Let \(K[U]\) be a core model. Let \(\lambda\) be a cardinal \(>\gamma := \sup \text{dom}(U), \lambda \geq \omega_1\). Assume \(\pi : K_1[U] \rightarrow W\) is elementary, \(W\) transitive, and \(\pi\) has critical point \(\alpha > \gamma\). Then there is an elementary embedding \(\tilde{\pi} : K[U] \rightarrow K[U]\) with critical point \(\alpha\).

**Proof.** Set \(G := \{ x < \alpha | x \in K_1[U] \land \alpha \in \pi(x)\}\). By 3.13, it suffices to show that the ultrapower \(\text{Ult}(K_1[U], G)\) is well-founded. So assume not and let \(f_0, f_1, \ldots \in K[U]\) be functions such that \(\{ v < \alpha | f_{n+1}(v) \in f_n(v)\} \in G\) for \(n < \omega\). Let \(\{ f_n | n < \omega \} \subset K_n[U]\) where \(K_n[U]\) reflects enough properties of \(K[U]\). There is an elementary map \(\sigma : Q \rightarrow K_n[U]\) so that \(Q\) is transitive, \(\alpha \cup \{ f_n | n < \omega \} \subset \text{range}(\sigma)\), and \(\text{card}(Q) < \lambda\). Let \(\hat{U} := U \cap Q; \hat{U} \in Q\). By 2.8, \(\hat{U}\)-mice are absolute for \(Q\), and by 3.3, \(Q \subset K_1[U]\). Set \(\tilde{f}_n := \sigma^{-1}(f_n)\), \((n < \omega)\). Then, for \(n < \omega\):

\[\{ v < \alpha | f_{n+1}(v) \in f_n(v)\} = \{ v < \alpha | f_{n+1}(v) \in f_n(v)\} \in G,\]

hence

\[\alpha \in \pi(\{ v < \alpha | f_{n+1}(v) \in f_n(v)\}) = \{ v < \pi(\alpha) | \pi(f_{n+1})(v) \in \pi(f_n)(v)\},\]

and \(\pi(f_{n+1})(\alpha) \in \pi(f_n)(\alpha)\). Contradiction.  

4. Chang's conjecture

4.1. Definition. For \(k, \lambda, \mu, \nu, \theta\) cardinals let \((k, \lambda) \Rightarrow_\theta (\mu, \nu)\) be the assertion:

Every structure \((A, B, \ldots)\) where \(\text{card}(A) = k, \text{card}(B) = \lambda\) and whose type has cardinality \(\leq \theta\) possesses an elementary substructure \((\hat{A}, \hat{B}, \ldots)\) such that \(\text{card}(\hat{A}) = \mu, \text{card}(\hat{B}) = \nu\).

\((k, \lambda) \Rightarrow_\omega (\mu, \nu)\) is called Chang's conjecture for the pairs \((k, \lambda), (\mu, \nu)\).

Chang's conjecture is presented in Chang–Keisler [1, 7.3.1], and various relations between instances of Chang's conjecture are discussed there.

4.2. Lemma. \((\mu^+, \mu^+) \Rightarrow_\omega (\mu^+, \mu)\) implies \((\mu^+, \mu^+) \Rightarrow_\mu (\mu^+, \mu)\).

**Proof.** Assume \((\mu^+, \mu^+) \Rightarrow_\omega (\mu^+, \mu)\). Consider a structure \(S = (\mu^+, \mu^+, \langle R_v | v < \mu \rangle, \langle f_v | v < \mu \rangle)\) with relations \(R_v\) and functions \(f_v\). We suppose that the family of \(f_v\)s contains a set of Skolem functions for \(S\). Define a function \(F:\)

\[F(\bar{x}) = \sup(\{ f_v(\bar{x}) : v < \mu \cap \mu^+ \} < \mu^+).\]
Some applications of short core models

By $(\mu^{++}, \mu^{+}) \Rightarrow_\omega (\mu^{+}, \mu)$ there is $X < \langle \mu^{++}, \mu^{+}, F \rangle$ such that $\text{card}(X) = \mu^{+}$ and $\text{card}(X \cap \mu^{+}) = \mu$. Set $Y := \bigcup \{ f(\bar{x}) \mid v < \mu, \bar{x} \in X \}$. $Y < S$, $\text{card}(Y) = \mu^{+}$, and $Y \cap \mu^{+} \in \text{sup}(X \cap \mu^{+}) < \mu^{+}$. □

4.3. Theorem. Assume $(\mu^{++}, \mu^{+}) \Rightarrow_\omega (\mu^{+}, \mu)$, $\mu \geq \omega_{1}$. Then $0^{\text{long}}$ exists.

Proof. Assume $\neg 0^{\text{long}}$. Let $U := U_{\text{can}} \upharpoonright \mu^{++}$. By 3.12,

(1) $\text{sup dom}(U \upharpoonright \mu^{+}) < \mu^{+}$ and $\text{sup dom}(U) < \mu^{++}$.

Let $H$ be a transitive structure reflecting enough properties of $V$ with $\mu^{++}, U \in H$. By $(\mu^{++}, \mu^{+}) \Rightarrow_{\mu} (\mu^{+}, \mu)$ there is an $X < H$ such that $\mu^{+}, \mu^{++}, U \in X$, $(\mu + 1) \in X$, $\text{card}(X \cap \mu^{++}) = \mu^{+}$, and $\text{card}(X \cap \mu^{+}) = \mu$. Let $\pi: \bar{H} \cong X < H$, $\bar{H}$ transitive. Set $\alpha := \pi^{-1}(\mu^{+}), \beta := \pi^{-1}(\mu^{++}), \bar{U} := \pi^{-1}(U)$.

(2) $X \cap \mu^{+}$ is transitive.

Proof. Let $\gamma \in X \cap \mu^{+}$. There is $f \in X$, $f: \mu \to \gamma$ onto. Since $\mu \subseteq X$, $\gamma = f^{\gamma} \mu \subseteq X$. □(2)

Thus:

(3) $\alpha = X \cap \mu^{+}$ is the critical point of $\pi$.

(4) $\text{sup dom}(U \upharpoonright \mu^{+}) < \alpha$, since $\text{sup dom}(U \upharpoonright \mu^{+}) \in X$.

(5) $\beta = \mu^{+}$.

Proof. $\beta \geq \mu^{+}$ since $\text{card}(X \cap \mu^{++}) = \mu^{+}$. If $\beta > \mu^{+}$, then $\pi(\mu^{+})$ is a cardinal in $V$, and $\mu^{+} = \pi(\alpha) < \pi(\mu^{+}) < \pi(\beta) = \mu^{++}$. Contradiction. □(5)

Set $Q := (K[\bar{U}])^{\bar{H}}$. By (1), $\text{sup dom}(\bar{U}) < \beta$.

(6) $K_{\beta}[\bar{U}] \subseteq Q$.

Proof. We apply 3.24(ii). Let $C \subseteq \beta$ be closed unbounded in $\beta$. Choose $\kappa \in C$ such that $\kappa > \text{sup dom}(\bar{U})$, $\alpha$, and $\text{cof}(\kappa) \neq \text{cof}(\alpha)$; this is possible since there are at least two different cofinalities below $\mu^{+} \geq \omega_{2}$. $\bar{H} \vdash \text{cof}(\kappa) \neq \text{cof}(\alpha)$, hence $\text{cof}(\pi(\kappa)) \neq \text{cof}(\pi(\alpha)) = \mu^{+}$. $\text{cof}(\pi(\kappa)) < \mu^{+} = \text{card}(\pi(\kappa))$ and $\pi(\kappa) > \text{sup dom}(U)$. By 3.20(i), $K[U] \models " \pi(\kappa) \text{ is singular}"$. Thus $Q \models " \kappa \text{ is singular}"$, as required. □(6)

(7) $K[\bar{U}]$ is a core model.

Proof. $K[\bar{U}] \models " \bar{U} \text{ is a sequence of measures}"$, since $K_{\beta}[\bar{U}] \subseteq Q$ and $Q \models " \bar{U} \text{ is a sequence of measures}"$. □(7)

$\text{dom}(\bar{U} \upharpoonright \alpha) = \text{dom}(U \upharpoonright \alpha)$, and so by 3.14, $\bar{U} \upharpoonright \alpha = U \upharpoonright \alpha$. By (4), there is no end-extension of $U \upharpoonright \alpha$ with new measurables below $\mu^{+} = \beta$, and so $\bar{U} = \bar{U} \upharpoonright \alpha = U \upharpoonright \beta$. (6) and some absoluteness considerations imply:
So $\pi \models K_\beta[U] \to K_\mu^+[U]$ is elementary with critical point $\alpha$. By 3.25 there is $\mathfrak{A}$ such that

(9) $\mathfrak{A} : K[U \upharpoonright \mu^+] \to K[U \upharpoonright \mu^+]$ is elementary with critical point $\alpha$.

(10) $\mu^+$ is a limit cardinal in $K[U \upharpoonright \mu^+]$.

Proof. Assume instead $\mu^+ = \eta^+ K[U \upharpoonright \mu^+]$, $\eta \in \text{card}(K[U \upharpoonright \mu^+])$. $\eta \in X \cap \mu^+ = \alpha < \mu^+$ implies $K[U \upharpoonright \mu^+] \models \text{"$\alpha$ is singular"}$, which contradicts (9). $\square$(10)

By the embedding Theorem 3.18 there exists $U' \succ U \upharpoonright \mu^+$ with $\min \text{dom}(U' \setminus (U \upharpoonright \mu^+)) < \mu^+$. But this contradicts the definition of $U_{\text{can}}$. $\square$

5. 'Accessible' Jónsson cardinals

5.1. Definition. A cardinal $\lambda$ is called a Jónsson cardinal provided every structure of cardinality $\lambda$ whose type is countable possesses a proper elementary substructure of cardinality $\lambda$.

For details on Jónsson cardinals see Chang–Keisler [1, 7.3.2]. Every Jónsson cardinal is $\geq \omega_\alpha$ [1, exercise 7.3.15].

5.2. Theorem. Let $\lambda$ be a Jónsson cardinal and $\lambda = \omega_\xi$, $\xi < \lambda$. Then $0^{\text{long}}$ exists.

Proof. We assume $\neg 0^{\text{long}}$ and work for a contradiction.

Let $U := U_{\text{can}}$. Let $H$ be a transitive structure reflecting enough properties of the universe with $\lambda$, $U \in H$. Since $\lambda$ is Jónsson there is $X < H$ such that $\lambda$, $U \in X$, $\text{card}(X \cap \lambda) = \lambda$ and $X \cap \lambda \neq \lambda$. Let $\pi : H \equiv X < H$, $\bar{H}$ transitive. Set $\bar{U} := \pi^{-1}(U)$. Let $\alpha < \lambda$ be the critical point of $\pi$; let $\alpha' := \pi(\alpha)$.

(1) $[\alpha, \alpha') \cap X = \emptyset$.

(2) $\alpha'$ is regular in $V$, $\alpha' \geq \omega_1$.

Proof. If $\text{cof}(\alpha') < \alpha$, there is a sequence in $X$ converging to $\alpha'$, contradicting (1). If $\alpha \leq \text{cof}(\alpha') < \alpha'$, then $\text{cof}(\alpha') \in X$ which again contradicts (1). $\square$(2)

(3) $\sup \text{dom}(U \upharpoonright \alpha') < \alpha$ and $\text{dom}(U \upharpoonright \alpha') = \text{dom}(\bar{U} \upharpoonright \alpha)$.

$Z := \{ \beta < \lambda : \bar{H} \models \text{"$\beta$ is a cardinal"} \}$ is a closed set of ordinals of ordertype $\leq \xi < \lambda$. Choose a regular cardinal $\mu$ such that $\omega_\xi$, $\alpha$, $\xi < \mu < \lambda$. Set $\mu' = \pi(\mu)$. Let $\theta := \max(Z \cap \mu) < \mu$. Set $Q := (K[\bar{U} \upharpoonright \mu])^{\bar{H}}$. $\sup \text{dom}(\bar{U} \upharpoonright \mu) < \mu$, by 3.12.

(4) $K_\mu(\bar{U} \upharpoonright \mu) \subset Q$.

Proof. We apply 3.24(ii). Let $C \subset \mu$ be closed unbounded in $\mu$. Choose $\kappa \in C$
such that $\kappa > \sup \dom(\bar{U} \upharpoonright \mu)$, $\theta$, and $\cof(\kappa) \neq \cof(\theta)$. $\bar{H} \models \cof(\kappa) \neq \cof(\theta)$, and $\cof(\pi(\kappa)) \neq \cof(\pi(\theta))$. So $\cof(\pi(\kappa)) < \pi(\theta) = \card(\pi(\kappa))$, and $\pi(\kappa) > \sup \dom(U \upharpoonright \mu')$. By 3.20(i), $K[U \upharpoonright \mu'] \models \text{"\pi(\kappa) is singular"}$, and thus $Q \models \text{"\kappa is singular"}$, as required. \(\Box(4)\)

(5) $\bar{U} \upharpoonright \mu$ is strong.

Proof. $K[\bar{U} \upharpoonright \mu] \models \text{"\bar{U} \upharpoonright \mu is a sequence of measures"}$, since $K_\mu[\bar{U} \upharpoonright \mu] = Q$, and $Q \models \text{"\bar{U} \upharpoonright \mu is a sequence of measures"}$. \(\Box(5)\)

Since $\alpha$ is the critical point of $\pi$, $\dom(\bar{U} \upharpoonright \alpha) = \dom(U \upharpoonright \alpha)$. So by 3.14,

(6) $\bar{U} \upharpoonright \alpha = U \upharpoonright \alpha$.

By (3), $U \upharpoonright \alpha = U \upharpoonright \alpha'$. Since $U$ is ‘canonical’:

(7) $[\alpha, \alpha') \cap \dom(\bar{U}) = \emptyset$.

(8) $\alpha' \notin \dom(U)$, since $\alpha \notin \dom(\bar{U})$.

Absoluteness considerations and (4) imply:

(9) $K_\mu[U \upharpoonright \alpha] = (K_\mu[\bar{U} \upharpoonright \alpha])^\mu$.

The map $\pi \upharpoonright K_\mu[U \upharpoonright \alpha]: K_\mu[U \upharpoonright \alpha] \to K_\mu[U \upharpoonright \alpha']$ is elementary with critical point $\alpha$. By 3.25 there is $\bar{\pi}$ such that:

(10) $\bar{\pi}: K[U \upharpoonright \alpha] \to K[U \upharpoonright \alpha]$ is elementary with critical point $\alpha$.

As in 4.3(10),

(11) $\alpha' = \pi(\alpha)$ is a limit cardinal in $K[U \upharpoonright \alpha]$.

By the embedding Theorem 3.18 there exists a strong predicate $U' \succ U \upharpoonright \alpha$ with $\min \dom(U' \setminus (U \upharpoonright \alpha')) \leq \alpha'$. But by (3) and (8) this contradicts the fact that $U$ is canonical. \(\Box\)

5.3. Theorem (¬0\text{\text{Jónss})}. Let $\lambda$ be a Jónsson cardinal such that one of the following holds:

(i) $\omega < \cof(\lambda) < \lambda$.

(ii) $\lambda$ is regular but not weakly hyper-Mahlo.

Then $\dom(U_{\text{can}} \upharpoonright \lambda)$ is cofinal in $\lambda$.

A definition of weakly hyper-Mahlo can be found in Drake [8, Ch. 4, §3.6]. The subsequent proof also contains a definition of this notion.

Proof. Set $U := U_{\text{can}}$, and assume $\sup \dom(U \upharpoonright \lambda) < \lambda$. By the previous theorem, $\lambda = \omega_\alpha$. Hence $\lambda$ is a limit cardinal. Let $H$ be a transitive structure reflecting enough properties of the universe with $\lambda$, $U \in H$. Since $\lambda$ is Jónsson, there is $X < H$ such that $\lambda$, $U \in X$, $\card(X \cap \lambda) = \lambda$, and $X \cap \lambda \neq \lambda$. Let $\pi : H \equiv X < H$, $\bar{R}$.
transitive. Set $\bar{U} := \pi^{-1}(U)$. Let $\alpha < \lambda$ be the critical point of $\pi$; $\alpha' := \pi(\alpha)$. As in the proof of 5.2 we get:

1. $\alpha'$ is regular in $V$, $\alpha' \geq \omega_1$.
2. $\sup \dom(U \upharpoonright \alpha') < \alpha$.

Set $Q := (K[\bar{U} \upharpoonright \lambda])^\bar{U}$. $\sup \dom(\bar{U} \upharpoonright \lambda) < \lambda$. 

3. $K_1[\bar{U} \upharpoonright \lambda] \subseteq Q$.

**Proof.** We apply 3.24(ii). Let $C \subset \lambda$ be closed unbounded in $\lambda$.

Case (i): $\nu < \operatorname{cof}(\lambda) < \lambda$. Then there is a closed unbounded set $D \subset \lambda$ consisting of singular cardinals. We may assume that $D \in X$. $\bar{D} := \pi^{-1}(D)$ is closed unbounded in $\lambda$. Take $\kappa \in C \cap \bar{D}$ such that $\kappa > \sup \dom(\bar{U} \upharpoonright \lambda)$. $\pi(\kappa)$ is a singular cardinal $> \sup \dom(U \upharpoonright \lambda)$, and by 3.20(ii), $K[U] \vdash \text{"}\pi(\kappa)\text{ is singular"}$. Thus $Q \vdash \kappa$ is singular", as required.

Case (ii): $\lambda$ is regular but not weakly hyper-Mahlo. $\lambda$ is weakly inaccessible since $\lambda = \omega_1$. Adjoin a suitable set ‘$-1$’ as a new least element to the ordinals.

For $\beta \in \On$ define its (weak) Mahlo degree $M(\beta) \in [-1, \beta]$ by:

- $M(\beta) \geq 0$ iff $\beta$ is weakly inaccessible;
- $M(\beta) = \gamma$ iff for all $\delta < \gamma$ the set $\{ \eta < \beta \mid M(\eta) \geq \delta \}$ is stationary in $\beta$, $(\gamma > 0)$.

$\beta$ is weakly Mahlo if $M(\beta) \geq 1$. $\beta$ is weakly Mahlo if $M(\beta) > 0$. $\beta$ is weakly hyper-Mahlo if $M(\beta) > 0$. Thus $0 \leq M(\lambda) < \lambda$.

To every $\beta \leq \lambda$ with $0 \leq M(\beta) < \beta$ assign a closed unbounded set $D_\beta \subset \beta$ such that $\delta \in D_\beta$ implies $\delta$ is a limit cardinal and $M(\delta) < M(\beta)$. We may assume that the function $\langle D_\beta \mid \beta \leq \lambda, \, 0 \leq M(\beta) < \beta \rangle$ is an element of $X$.

Let $\bar{M}$ be defined in $\bar{H}$ as $M$ is in $V: \bar{M} = (M)^{\bar{H}}$. For $\beta \leq \lambda$, $0 \leq \bar{M}(\beta) < \beta$ set $\bar{D}_\beta := \pi^{-1}(D_{\pi(\beta)})$. $\bar{D}_\beta$ is closed unbounded in $\beta$; $\delta \in \bar{D}_\beta \rightarrow \pi(\delta)$ is a limit cardinal and $\bar{M}(\delta) < \bar{M}(\beta)$.

We assume that the closed unbounded set $C \subset \lambda$ has min($C$) $> \sup \dom(\bar{U} \upharpoonright \lambda)$.

Do the following construction until the recursion breaks down:

Set $\beta_0 := \lambda$, $\gamma_0 := \bar{M}(\lambda) < \lambda$. If $\beta_n$, $\gamma_n$ are defined, put $\beta_{n+1} := \omega_{\gamma_{n+1}}$-st element of $C \cap \bar{D}_{\beta_n}$, and $\gamma_{n+1} := \bar{M}(\beta_{n+1})$.

**Claim.** Let $\beta_n$, $\gamma_n$ be constructed and assume $\operatorname{cof}(\beta_n) > \omega_{\gamma_{n+1}}$, $\bar{M}(\beta_n) > 0$. Then $\beta_{n+1}$, $\gamma_{n+1}$ exist and $\operatorname{cof}(\beta_{n+1}) > \omega_{\gamma_{n+1}}$. Also $\gamma_{n+1} < \gamma_n$.

**Proof.** Because $\gamma_n = \bar{M}(\beta_n) \geq 0$, $\bar{D}_{\beta_n}$ is closed unbounded in $\beta_n$. $\operatorname{cof}(\beta_n) > \omega_{\gamma_{n+1}}$, and $\beta_{n+1}$ is the $\omega_{\gamma_{n+1}}$-st element of $C \cap \bar{D}_{\beta_n}$ exists. $\gamma_{n+1} = \bar{M}(\beta_{n+1}) < \bar{M}(\beta_n) < \gamma_n$, and so $\operatorname{cof}(\beta_{n+1}) = \omega_{\gamma_{n+1}} > \omega_{\gamma_{n+1}}$. □(Claim)

So $\beta_1$, $\gamma_1$ exist. Since $\gamma_n > \gamma_{n+1}$, the construction breaks down, and by the claim there must be $n \geq 1$ such that $\gamma_n = \bar{M}(\beta_n) = -1$. Set $\kappa := \beta_n$. Then $M(\pi(\kappa)) = -1$, and $\pi(\kappa)$ is a singular cardinal $> \sup \dom(U \upharpoonright \lambda)$. By 3.20(ii), $K[U \upharpoonright \lambda] \vdash \text{"}\pi(\kappa)\text{ is singular"}$. So $Q \vdash \kappa$ is singular", as required. □(3)
The proof can now be finished exactly as the proof of 5.2 from 5.2(5) onwards, taking $\mu := \mu' := \lambda$. □

As corollaries we get the following results:

5.4. Theorem. If there exists a regular Jónsson cardinal $\lambda$ which is not weakly hyper-Mahlo, then $\mathsf{0}^\text{long}$ exists.

Proof. Assume $\neg\mathsf{0}^\text{long}$. By 5.3, dom$(U_{\text{can}} | \lambda)$ is cofinal in $\lambda$, which contradicts 3.12. □

5.5. Theorem. Let $\lambda$ be a Jónsson cardinal with $\omega < \delta := \text{cof}(\lambda) < \lambda$. Then there is an inner model in which the set of measurables $< \lambda$ has ordertype $\geq \delta$.

Proof. If $\mathsf{0}^\text{long}$ exists this follows from 2.14. Assume that $\mathsf{0}^\text{long}$ does not exist, on the other hand. By 5.3, dom$(U_{\text{can}} | \lambda)$ is cofinal in $\lambda$, thus $\text{otp dom}(U_{\text{can}} | \lambda) \geq \delta$. □

Conversely, we have the well-known result of Prikry [15]:

5.6. Theorem. If $\langle \kappa_i | i < \delta \rangle$ is a strictly increasing sequence of measurable cardinals where $\delta$ is a limit ordinal $< \kappa_0$, then $\lambda := \sup \{ \kappa_i | i < \delta \}$ is a Jónsson cardinal.

Theorems 5.5 and 5.6 yield a family of equiconsistency results of which we present just one typical example:

5.7. Theorem. The theories “ZFC + there is a Jónsson cardinal of cofinality $\omega_1$” and “ZFC + there are $\omega_1$ measurable cardinals” are equiconsistent.

6. Free subsets

6.1. Definition. A subset $X$ of a structure $S$ is free in $S$ if $\forall x \in X \times \notin S[X \setminus \{x\}]$; here $S[Y]$ denotes the substructure of $S$ generated from $Y$ by the functions and constants of $S$. For cardinals $\kappa, \lambda, \mu$ let $\text{Fr}_\mu(\kappa, \lambda)$ be the assertion: Every structure $S$ with $\kappa \subseteq S$ whose type has cardinality $\leq \mu$ possesses a free subset $X \subseteq \kappa$ of cardinality $\geq \lambda$.

$\text{Fr}_{\leq \kappa}(\kappa, \lambda)$ stands for: $\forall \mu < \kappa \text{ Fr}_\mu(\kappa, \lambda)$.

This section extends techniques of [10] where we proved that $\text{Fr}_\omega(\omega_\omega, \omega)$ is equiconsistent to “there exists a measurable cardinal”. The following lemma is [10, 1.1]:
Let $A$ be an finite set and assume $F^+(K,A)$. Let $S$ be a $K$-subset whose type has cardinality $\leq \mu$. Then there is a subset $X \subset K$ free in $S$ with monotone enumeration $\langle x_i \mid i < \lambda \rangle$ such that:

(i) $i < \lambda \to [x_i, x_i^+] \cap S[x_i \cup \{x_j \mid i < j < \lambda\}] = \emptyset$

($x_i^+$ is the smallest cardinal $>x_i$). In particular:

(ii) $i < \lambda \to x_i \notin S[x_i \cup \{x_j \mid i < j < \lambda\}]$.

6.3. Lemma. Let $\lambda, \mu$ be infinite cardinals and let $\kappa$ be the least cardinal such that $\text{Fr}_\mu(x, \lambda)$. Then:

(i) $\kappa$ is a limit cardinal.

(ii) $\text{Fr}_{\leq \kappa}(\kappa, \lambda)$.

(iii) $\kappa$ is weakly inaccessible or $\text{cof}(\kappa) = \text{cof}(\lambda)$.

(iv) If $\mu = \omega_\xi$, then $\kappa \geq \omega_\xi + \lambda$.

Proof. Just as Lemma 1.2 of [10].

6.4. Theorem ($\omega_1^{\text{long}}$). Let $\kappa = \omega_\xi$, $\xi < \kappa$ and $\text{Fr}_{< \kappa}(\kappa, \lambda)$, where $\lambda$ is an uncountable cardinal. Then $\text{dom}(U_{\text{can}} \upharpoonright k)$ is cofinal in $k$.

Proof. Set $U := U_{\text{can}} \upharpoonright k$. Assume that $\nu := \sup \text{dom}(U) < k$. By 6.3(iv), $\kappa$ is a limit cardinal and so $\mu := \max(\nu^+, \xi^+, \omega_2) < k$.

(1) There is $\bar{k} < \kappa$ so that $\text{Fr}_\mu(\bar{k}, \omega)$.

Proof. If there is $\bar{k} < \kappa$ such that $\text{Fr}_\mu(\bar{k}, \lambda)$ we are done. So assume $\lambda$ is minimal with $\text{Fr}_\mu(\kappa, \lambda)$. By 6.3(iii), $\kappa$ is singular and $\text{cof}(\kappa) = \text{cof}(\lambda)$. Take a cardinal $\bar{k} < \lambda$ such that $\text{cof}(\bar{k}) \neq \text{cof}(\lambda)$. By 6.3(iii), there has to be $\bar{k} < \kappa$ such that $\text{Fr}_\mu(\bar{k}, \lambda)$. Then $\text{Fr}_\mu(\bar{k}, \omega)$. □

Fix $\bar{k} = \omega_\xi < \kappa$ such that $\text{Fr}_\mu(\bar{k}, \omega)$. By the covering theorem 3.21 there exists $E \in K[U]$ such that $\{\omega_i \mid i < \xi\} \subset E \subset \bar{k}$ and card($E$) $< \mu$. Let $K_0[U]$, $\theta > \bar{k}$ reflect sufficiently many properties of $K[U]$. Let $S := \langle K_0[U], E, \langle \alpha \mid \alpha \leq \mu \rangle \rangle$ augmented by a countable set of Skolem functions for $K_0[U]$; $E$ and every $\alpha \leq \mu$ are understood to be constants of $S$. By 6.2(i), there is a set $X \subset \bar{k}$ free in $S$ with monotone enumeration $\langle x_i \mid i < \omega \rangle$ so that:

(2) $[x_i, x_i^+ \cap S[x_i \cup \{x_j \mid i < j < \omega\}] = \emptyset$ for all $i < \omega$.

For $i < \omega$ set $M_i := S[\{x_i \mid i \leq j < \omega\}]$ and let $\pi_i : M_i \equiv \bar{M}_i$, $\bar{M}_i$ transitive. Set $U_i := \pi_i(U)$. For $i \leq j < \omega$ let $\pi_{ji} := \pi_i \circ \pi_j^{-1} : \bar{M}_j \to \bar{M}_i$.

(3) $\mu \subset M_i$ and $\pi_i \upharpoonright \mu = \text{id}$.

(4) $\bar{M}_i \vDash \"V = K[U_i]\"$, $U_i \cap \bar{M}_i \in \bar{M}_i$, and $U_i$-mice are absolute for $\bar{M}_i$ (the latter follows from 2.8).
Some applications of short core models

(5) \( \pi_\mu \) is elementary, \( \pi_\mu \upharpoonright \mu = \text{id} \), and \( \pi_\mu(U_i) = U_i \).

For \( i < \omega \) set \( E_i := \pi_i(E) \); \( \pi_j(E_i) = E_i \). Then 3.5 implies:

(6) \( E_i \preceq_U E_j \), for \( i \leq j < \omega \).

Since \( \preceq_U \) well-orders \( K[U] \) (3.4) there is \( i < \omega \) such that \( E_{i+1} = E_i \). "\( x \) is the \( \alpha \)-th element of \( E_i \)" , for \( \alpha < \text{otp}(E_i) < \mu \), is uniformly definable in \( M_i \) and \( M_{i+1} \), and since \( \pi_{i+1,i} \upharpoonright \mu = \text{id} \):

(7) \( \pi_{i+1,i} \upharpoonright E_{i+1} = \text{id} \).

Let \( \delta := \pi_{i+1}(x_i^+)) \in E_{i+1} \). Then:

\[
\pi_{i+1,i}(\delta) = \pi_i(x_i^+)) \upharpoonright \pi_i(x_i) = \text{otp}(M_i \cap x_i) \supseteq \text{otp}(M_{i+1} \cap x_i)
\]

\[= \text{otp}(M_{i+1} \cap x_i^+) \text{ (by (2))} = \pi_{i+1}(x_i^+) = \delta,\]

which contradicts (7). \( \square \)

6.5. Theorem. Assume \( \text{Fr}_{\omega_1}(\omega_1, \lambda) \), where \( \lambda \) is a cardinal with \( \omega_1 \leq \lambda < \omega_\lambda \). Then there is an inner model in which the set of measurables \( < \omega_\lambda \) has ordertype \( \geq \lambda \).

Proof. If \( 0^\text{long} \) exists the theorem follows by 2.14. So assume \( \neg 0^\text{long} \). Set \( \kappa := \omega_\lambda \).

By 6.3, \( \text{Fr}_{<\kappa}(\kappa, \lambda) \) holds. We shall show that \( \text{otp} \text{dom}(U_{\text{can}} \upharpoonright \kappa) \supseteq \lambda \). By 6.4, \( \text{dom}(U_{\text{can}} \upharpoonright \kappa) \) is cofinal in \( \kappa \), and for regular \( \lambda \) this implies \( \text{otp} \text{dom}(U_{\text{can}} \upharpoonright \kappa) \supseteq \lambda \).

Now consider the case that \( \lambda \) is a singular cardinal. It suffices to show that \( \text{otp}(U_{\text{can}} \upharpoonright \kappa) \supseteq \lambda' \) for every regular cardinal \( \lambda' < \lambda \). Let \( \lambda' < \lambda \) be regular, \( \lambda' \supseteq \omega_1 \). Let \( \kappa' < \kappa \) be minimal such that \( \text{Fr}_\kappa(\kappa', \lambda') \); let \( \kappa' = \omega_{\xi} \). Then:

(1) \( \xi < \kappa < \kappa' = \omega_{\xi} \).

(2) \( \text{Fr}_{<\kappa'}(\kappa', \lambda'), \) by 6.3(ii).

(3) \( \text{cof}(\kappa') \supseteq \lambda', \) by 6.3(iii).

By 6.4, \( \text{dom}(U_{\text{can}} \upharpoonright \kappa') \) is cofinal in \( \kappa' \). Then

\( \text{otp} \text{dom}(U_{\text{can}} \upharpoonright \kappa) \supseteq \text{otp} \text{dom}(U_{\text{can}} \upharpoonright \kappa') \supseteq \text{cof}(\kappa') \supseteq \lambda'. \) \( \square \)

Conversely, Shelah [17] proves:

6.6. Theorem. Assume GCH. Let \( \langle \kappa_i \mid i < \lambda \rangle \) be a strictly increasing sequence of measurable cardinals, \( \lambda = \omega_\xi, \xi < \lambda < \kappa_0 \). Let \( \kappa := \sup(\kappa_i \mid i < \lambda) \). Then there is a generic extension \( V[G] \) of \( V \) satisfying:

\( V[G] \models "\kappa = \omega_\lambda \text{ and } \text{Fr}_{\omega_1}(\omega_\lambda, \lambda)". \)

Theorem 6.5 and 6.6 imply a series of equiconsistency results of which we present one typical example:
6.7. Theorem. The theories "ZFC + Fr_\omega(\omega_\omega, \omega_1)" and "ZFC + there are \omega_1 measurable cardinals" are equiconsistent.

7. Canonical forms

7.1. Definition. Let the sequence \langle \omega_{k(n)} \mid n < \omega \rangle be strictly increasing with supremum \omega_\omega. By CF(k) we denote the following combinatorial property:

For every sequence \langle f_i \mid i < \omega \rangle of functions, f_i: (\omega_\omega)^2 \to 2 there exist sets \mathcal{B}_n = \omega_{k(n)} \setminus \omega_{k(n-1)}, \text{ card}(\mathcal{B}_n) \geq 3 such that for every i < \omega, n_1 < \cdots < n_i < \omega, and \( x_1, y_1, z_1, y_1' \in \mathcal{B}_n, \ldots, x_i, y_i, z_i, y_i' \in \mathcal{B}_n \), with \( x_1 < y_1 < x_2 < y_2 < \cdots < x_i < y_i, x'_1 < y'_1 < x'_2 < y'_2 < \cdots < x'_i < y'_i \) we have:

\[ f(x_1, y_1, x_2, y_2, \ldots, z_i, y_i) = f(x'_1, y'_1, x'_2, y'_2, \ldots, z'_i, y'_i). \]

Such a sequence \langle \mathcal{B}_n \mid n < \omega \rangle will be called homogeneous for \langle f_i \mid i < \omega \rangle. The \mathcal{B}_n's can be viewed as segments of a system of indiscernibles for the functions \langle f_i \mid i < \omega \rangle, so that the value of the functions is independent of which pair from a segment is chosen.

In the language of Shelah [16], who studies strong partition properties \( \forall \omega_\omega \), CF(k) is the same as: \langle \omega_{k(n)} \mid n < \omega \rangle has a \langle 3 \mid n < \omega \rangle-canonical form for \( \langle 2, 2, \ldots, 2 \rangle^{\omega_i} \mid i < \omega \rangle \).

The consistency strengths of (reasonable) canonical form properties for \omega_\omega where just singletons from each segment of indiscernibles are considered correspond to the existence of one measurable cardinal (see [17] and [10]). So CF(k) is basically the next stage as far as the strengths of canonical forms are concerned. A model for CF(k)—indeed for a much stronger canonical form property—was constructed by Shelah [16] from a ground model with \omega strongly compact cardinals \( \forall \omega \) with \( k(n) = (n + 5) \cdot n/2 + n + 1 \). The following theorem shows that high levels of measurability are indeed necessary for such a construction.

7.2. Theorem. Assume CF(k) for some function k. Then \( 0^{\text{long}} \) exists.

Proof. We assume \( \neg 0^{\text{long}} \) and work for a contradiction.

Set \( \kappa := \omega_\omega \) and \( \mathcal{U} := U_\text{can} \). Let \( H \) be a transitive structure reflecting enough properties of the universe and let \( \kappa, \mathcal{U} \in H \). Let \( H \) also possess a countable collection of Skolem functions for itself. We assume that the Skolem functions are suitably encoded into a sequence \langle f_i \mid i < \omega \rangle of functions \( f_i: (\kappa)^2 \to 2 \) to which we will apply the principle CF(k). Subsequently a homogeneous sequence for \( H \) will mean a homogeneous sequence for \( \langle f_i \mid i < \omega \rangle \).

We obtain a natural homogeneous sequence \langle W^*_n \mid n < \omega \rangle, W^*_n = \{x_n, y_n, z_n\}
for $H$ by successive minimal choices of the $x_n$: Let $\langle W_i^* | i < n \rangle$ be chosen. Let $x_n < \kappa$ be minimal such that there exists a homogeneous sequence $\langle W_i^* | i < n \rangle \cup \langle W_i | n \leq i < \omega \rangle$ for $H$ with $x_n = \min(W_n^*)$. Then let $W_n^* = \{x_n, y_n, z_n\}$ be such a $W_n^*$. $\langle W_n^* | n < \omega \rangle$ is clearly homogeneous for $H$.

Set $W_n := \{x_n, y_n\}$, $(n < \omega)$. It is readily seen that $\bigcup \{W_n | n < \omega\}$ is a free subset of $H$.

(1) $[x_n, x_n^+] \cap H[x_n \cup W_{n+1} \cup W_{n+2} \cup \cdots] = \emptyset$.

**Proof.** Assume not; let $\gamma \in H[x_n \cup W_{n+1} \cup W_{n+2} \cup \cdots]$, $x_n < \gamma < x_n^+$. There is $f \in H[x_n \cup W_{n+1} \cup W_{n+2} \cup \cdots]$, $f: \omega \leftrightarrow \gamma + 1$, where $\omega_r = \text{card}(x_n) < x_n$, and hence $x_n \in f^* \omega_r \subset H[x_n \cup W_{n+1} \cup W_{n+2} \cup \cdots]$. Let $x_n = t(u_0, \ldots, u_{k-1}, x_{n+1}, y_{n+1}, \ldots, x_i, y_i)$ for some $H$-term $t$, $u_0, \ldots, u_{k-1} < x_n$, $i \geq n + 1$. We may assume that $u_0, \ldots, u_{k-1}$ are successively chosen minimal for this equality to hold. Hence there are $H$-terms $t_0, \ldots, t_{k-1}$ such that

$$u_j = _i t(x_n, x_{n+1}, y_{n+1}, \ldots, x_i, y_i).$$

Set $u_j' := t(y_n, x_{n+1}, y_{n+1}, \ldots, x_i, y_i)$, for $j = 0, \ldots, k - 1$. Then

$$y_n = t(u_0', \ldots, u_{k-1}, x_{n+1}, y_{n+1}, \ldots, x_i, y_i),$$

and there must exist some $j < k$ such that $u_j \neq u_j'$.

Set $x_n' := u_j$, $y_n' := u_j'$, and $z_n' := t(z_n, x_{n+1}, y_{n+1}, \ldots, x_i, y_i)$. Then standard indiscernibility arguments show that $x_n', y_n', z_n'$ are pairwise distinct and that $\langle W_0^*, \ldots, W_{n-1}^*, \{x_n', y_n', z_n'\}, W_{n+1}^*, W_{n+2}^*, \ldots \rangle$ is a homogeneous sequence for $H$ with $\min(\{x_n', y_n', z_n'\}) \approx x_n' < x_n = \min(W_n)$. Contradiction. $\square(1)$

(2) Let $\omega_r \leq x_n < \omega_{r+1}$. Then $\omega_r < x_n$ and $\sup \text{dom}(U \upharpoonright \omega_{r+1}) < x_n < y_n < \omega_{r+1}$.

**Proof.** We can assume that $\omega_r$ and $\sup \text{dom}(U \upharpoonright \omega_{r+1})$ are constants of the structure $H$. Then use (1). $\square(2)$

Inside $K[U]$, let $M^* = J_r[F, U] \in K[U]$ be some $U$-mouse such that $\mathcal{P}(\kappa) \vDash \text{lp}(M^*)$ and $F$ is countably complete. We can assume that $M^*$ is a constant of $H$. By 2.6, $M := J_r[U \cup F, \emptyset]$ is a $\emptyset$-mouse inside $K[U]$, and by 2.7, $M$ is a $\emptyset$-mouse in the universe.

For $n < \omega$ set $X_n := H[\bigcup \{W_i | n \leq i < \omega\}]$. For $m < n < \omega$ let $X_m \neq X_n$ and $X_n \neq X_m$. Let $\pi_n: H_n \rightarrow X_n$, $H_n$ transitive. For $m \leq n < \omega$ let $\pi_{nm} := \pi_m^{-1} \circ \pi_n: H_m \rightarrow H_n$. For $n < \omega$ let $\kappa_n := \pi_n^{-1}(\kappa)$, $U_n := \pi_n^{-1}(U)$, $M_n := J_{\kappa_n}(U_n \cup F_n, \emptyset) := \pi_n^{-1}(M)$, $K_n := (K[U_n])^{\kappa_n}$.

(3) $\kappa_n \leq \kappa_m$, for $m \leq n < \omega$.

$M_n$ is a $\emptyset$-mouse (2.8). $\pi_{nm} | M_n: M_n \rightarrow M_m$ is an elementary embedding, and so by 2.12(iii):

(4) $M_n \vDash \Delta M_m$, for $m \leq n < \omega$.

Let us write $\leq^*$ for $\leq^* \emptyset$ and $\sim$ for $\sim \emptyset$. Since $\leq^*$ is a pre-wellordering (2.12(ii))
there exists some \( n_0 < \omega \) so that:

(5) \( M_n \sim M_{n_0} \) and \( \kappa_n = \kappa_{n_0} \) for all \( n \geq n_0 \).

Fix \( n \geq n_0 \) so that \( \omega_2 \leq \omega_1 \leq x_n < y_n < \omega_{r+1} \). Let \( \tilde{z}_i = \pi_n^{-1}(x_i) \), \( \tilde{y}_i = \pi_n^{-1}(y_i) \), \( \tilde{W}_i = \{ \tilde{x}_i, \tilde{y}_i \} \), for \( n \leq i < \omega \). Let \( \tilde{W}_i = \pi_n^{-1}(\omega_i) \), for \( i < \omega \).

Now set \( \tilde{X} = H_n[\tilde{x}_1 \cup \cdots \cup \{ \tilde{W}_i \mid n + 1 \leq i < \omega \}] \). Let \( \tilde{X} = \tilde{X}, \tilde{H} = \tilde{X} \), \( \tilde{H} \) transitive.

Let \( \tilde{H} : = \pi^{-1}(\kappa_n) \), \( U : = \pi^{-1}(U_n) \), \( \tilde{K} : = (K[\tilde{U}])^\tilde{H} \), and \( \tilde{M} : = J_{\tilde{X}}[\tilde{U} \cup \tilde{F}, \emptyset] : = \pi^{-1}(M_n) \). \( \tilde{M} \) is a \( \emptyset \)-mouse (2.8). By (1),

(6) \( \tilde{x}_n \in \tilde{X} \) and \( [\tilde{x}_n, \omega_{r+1}] \cap \tilde{X} = \emptyset \).

Hence:

(7) \( \tilde{x}_n \) is the critical point of \( \tilde{X} \) and \( \tilde{X}(\tilde{x}_n) = \tilde{\omega}_{r+1} \).

There is a unique elementary map \( \tilde{\sigma} : H_{n+1} \to \tilde{H} \) such that \( \tilde{X} \circ \tilde{\sigma} = \pi_{n+1, n} : \)

\[
\begin{array}{ccc}
H_{n+1} & \xrightarrow{\pi_{n+1, n}} & H_n \\
\sigma & \downarrow & \tilde{X} \\
\tilde{H} & \xleftarrow{\pi_{n+1, n}} & H_n \\
\end{array}
\]

\( \tilde{\sigma} \) is determined by: \( \pi_{n+1, n}(x_i) \to \pi^{-1}(x_i) \) and \( \pi_{n+1, n}(y_i) \to \pi^{-1}(y_i) \), \( n + 1 \leq i < \omega \).

By 2.12(iii) and (5),

(8) \( \tilde{M} \to M_n \).

Since \( \tilde{X} \upharpoonright \tilde{x}_n = \text{id} \),

(9) \( \tilde{U} \cup \tilde{F} \upharpoonright \tilde{x}_n \cap \tilde{M} \cap M_n = (U_n \cup F_n) \upharpoonright \tilde{x}_n \cap \tilde{M} \cap M_n \).

By 2.12(iv),

(10) \( \mathcal{P}(\tilde{x}_n) \cap \tilde{K} = \mathcal{P}(\tilde{x}_n) \cap \tilde{M} = \mathcal{P}(\tilde{x}_n) \cap M_n = \mathcal{P}(\tilde{x}_n) \cap K_n \).

By (10), and because \( \tilde{x}_n \) is the critical point of \( \tilde{X} \upharpoonright \tilde{K} : \tilde{K} \to K_n \):

(11) \( \tilde{x}_n \) is weakly inaccessible in \( \tilde{K} \).

(12) \( \tilde{x}_n \) is weakly inaccessible in \( K_n \), and so \( x_n \) and \( y_n \) are weakly inaccessible in \( K[U] \).

(13) \( \text{cof}(x_n) = \text{cof}(y_n) = \omega_r \).

Proof. Assume \( \text{cof}(x_n) < \omega_r \), \( \omega_2 \leq x_n \) and \( \sup \text{dom}(U \upharpoonright \omega_{r+1}) < x_n \) (2). By the covering property 3.20(i), \( x_n \) is singular in \( K[U] \), contradicting (12). \( \square \)(13)

(14) \( x_n \notin H[\omega_{r-1} \cup \{ y_n, x_{n+1}, y_{n+1}, \ldots \}] \).

Proof. Assume \( x_n = t(\tilde{y}, y_n, x_{n+1}, y_{n+1}, \ldots, y_l) \) for some \( H \)-term \( t \), \( \tilde{y} < \omega_{r-1} \), and \( l > n + 1 \). Since \( \text{cof}(x_n) = \omega_r \),

\[
x_n := \sup(x_n \cap \{ t(\tilde{v}, x_n, x_{n+1}, y_{n+1}, \ldots, x_l, y_l) \mid \tilde{v} < \omega_{r-1} \}) < x_n.
\]
Set

\[ y'_n := \sup (y_n \cap \{ t(\tilde{v}, y_n, x_{n+1}, y_{n+1}, \ldots, x_l, y_l) \mid \tilde{v} < \omega_{r-1} \}) \]
and

\[ z'_n := \sup (z_n \cap \{ t(\tilde{v}, z_n, x_{n+1}, y_{n+1}, \ldots, x_l, y_l) \mid \tilde{v} < \omega_{r-1} \}) \]

Since \( x_n = t(\tilde{v}, y_n, x_{n+1}, y_{n+1}, \ldots, x_l, y_l) \), we obtain \( x'_n < x_n < y'_n \). By simple indiscernibility arguments, \( x'_n, y'_n, z'_n \) are pairwise distinct and \( \langle W_0, \ldots, W_{n-1}, x'_n, y'_n, z'_n, W_{n+1}, W_{n+2}, \ldots \rangle \) is a homogeneous sequence for \( H \). This contradicts the minimal choice of \( x_n \). (14)

Set \( \check{X} := H_n[\omega_{r-1} \cup \{ \tilde{y}_n, \tilde{x}_{n+1}, \tilde{y}_{n+1}, \ldots \}] \). We shall carry out an analysis of \( \check{X} \) similar to the preceding one of \( \check{X} \). Let \( \check{X} : H = \hat{K} \), \( \hat{H} \) transitive. Let \( \check{k} := \check{x}^{-1}(\kappa_n) \),

\[ \check{U} := \check{A}^{-1}(U_n) \]

\[ \check{K} := (K[\check{U}])^{\check{U}} \]

and \( \check{M} := \check{J} \check{U} \cup \check{F} \cup \emptyset := \check{A}^{-1}(M_n) \). \( \check{M} \) is a \( \emptyset \)-mouse (2.8). By (14),

(15) \( \tilde{x}_n \in \tilde{X} \).

(16) \( \tilde{\omega}_r \not\in \check{X} \).

Proof. Assume \( \tilde{\omega}_r \in \tilde{X} \). Take \( f \in \hat{X} \), \( f : \tilde{\omega}_r \leftrightarrow \tilde{y}_n \). Then \( \tilde{x}_n \in \tilde{y}_n = f^* \tilde{\omega}_r \in \hat{X} \), contradicting (15). \( \Box \) (16)

Let \( \alpha \) be the critical point of \( \hat{k} \).

(17) \( \omega_{r-1} < \alpha < \tilde{\omega}_r \). \( [\alpha, \tilde{\omega}_r) \cap \hat{X} = \emptyset \). \( \hat{k}(\alpha) = \tilde{\omega}_r \).

Proof. Assume \( \beta \in \hat{X} \), \( \alpha \leq \beta < \tilde{\omega}_r \). Take \( f \in \hat{X} \), \( f : \tilde{\omega}_r \leftrightarrow \beta + 1 \). Then \( \alpha \in \beta + 1 = f^* \tilde{\omega}_r \in \hat{X} \). Contradiction. \( \Box \) (17)

In particular,

(18) \( \sup \text{dom}(U_n \upharpoonright \tilde{\omega}_r) < \alpha \).

As in the discussion of \( \hat{k} : \hat{H} \rightarrow H_n \) there is a unique elementary map \( \hat{\sigma} : H_{n+1} \rightarrow \hat{H} \) such that \( \hat{k} \circ \hat{\sigma} = \pi_{n+1}^{-1} \):

\[
\begin{array}{ccc}
H_{n+1} & \xrightarrow{\pi_{n+1}^{-1}} & H_n \\
\downarrow{\hat{\sigma}} & & \downarrow{\hat{k}} \\
\hat{H} & \xrightarrow{\hat{k}} & \hat{H}
\end{array}
\]

And as before we obtain:

(19) \( \hat{M} \sim M_n \).

(20) \( (\hat{U} \cup \hat{F}) \upharpoonright \alpha \cap \hat{M} \cap M_n = (U_n \cap F_n) \upharpoonright \alpha \cap M \cap M_n \).

(21) \( \mathcal{P}(\alpha) \cap \hat{K} = \mathcal{P}(\alpha) \cap K_n \).

(22) \( \alpha \) is weakly inaccessible in \( \hat{K} \).

(23) \( \tilde{\omega}_r \) is weakly inaccessible in \( K_n \), and \( \omega_r \) is weakly inaccessible in \( K[U] \).

By (12), \( \tilde{\omega}_r^{K_n} < \tilde{x}_n < \tilde{\omega}_{r+1}^{K_n} = \tilde{\omega}_r^{H_n} \), and since \( \hat{k}(\alpha) = \tilde{\omega}_r \):

(24) \( \alpha^+ \hat{\kappa} < \alpha^+ \hat{\eta} \).

(25) \( \mathcal{P}(\hat{\omega}) \cap K_n \subset \hat{\kappa} \).

**Proof.** Inside \( \hat{\kappa} \) form an iterate \( M^+ \) of \( \hat{\mathcal{M}} \) at points \( \geq \alpha \) so that \( \min(\text{meas}(M^+) \setminus \alpha) > \hat{\kappa} = K_n \supset \hat{\omega} \). Since \( \alpha \) is the critical point of \( \hat{\kappa} \) the ultrafilters of \( M^+ \) and \( M_n \) agree for all \( \nu < \hat{\omega} \). \( M^+ \sim \hat{\mathcal{M}} \sim M_n \), and by 2.12(iv),

\[ \mathcal{P}(\hat{\omega}) \cap K_n = \mathcal{P}(\hat{\omega}) \cap M_n = \mathcal{P}(\hat{\omega}) \cap M^+ \subset \hat{\kappa}. \quad \square(25) \]

Set \( D := \{ u \in \alpha | u \in \hat{\kappa} \wedge \alpha \in \hat{\pi}(u) \} \). \( D \) is a normal ultrafilter on \( \mathcal{P}(\alpha) \cap \hat{\kappa} \).

(26) \( \text{Ult}(\hat{\kappa}, D) \) is well-founded.

**Proof.** \( [f]_D \mapsto \hat{\pi}(f)(\alpha) \) defines a \( \Sigma_0 \)-preserving embedding of \( \text{Ult}(\hat{\kappa}, D) \) into \( K_n \). \( \square(26) \)

(27) \( D \in H_n \).

**Proof.** By (24), \( \alpha^+ \hat{\kappa} < \alpha^+ \hat{\eta} \). Since \( \hat{\kappa} \) satisfies the GCH (3.7), there is \( g \in \hat{\mathcal{M}}, g : \alpha \rightarrow \mathcal{P}(\alpha) \cap \hat{\kappa} \) onto. Then

\[ D := \{ (g(\nu)) | \nu < \alpha \wedge \alpha \in \hat{\pi}(g(\nu)) \} \]

\[ = \{ (\hat{\pi}(g)(\nu)) | \nu < \alpha \wedge \alpha \in \hat{\pi}(g)(\nu) \} \subset H_n. \quad \square(27) \]

(28) Set \( \bar{K} := (K_\omega[U_n])^{H_n} \). Then \( \text{Ult}(\bar{\kappa}, D) \) is well-founded.

**Proof.** By (25), \( \bar{\kappa} \subset \hat{\kappa} \), and \( \text{Ult}(\bar{\kappa}, D) \) is well-founded by (26). \( \square(28) \)

This fact holds as well inside \( H_n \). Let \( \alpha^* := \pi_n(\alpha), D^* := \pi_n(D) \). Since \( \pi_n : H_n \rightarrow H \) is elementary, \( \text{Ult}(K_\omega[U], D^*) \) is well-founded. By 3.25, there is an elementary embedding \( j : K[U \uparrow \omega,] \rightarrow K[U \uparrow \omega,] \) with critical point \( \alpha^* \). \( \alpha^* > \sup \text{dom}(U \uparrow \omega,) \) (by (18)), \( \omega, \) is regular and weakly inaccessible in \( K[U \uparrow \omega,] \) (23). By the embedding Theorem 3.18 there is a strong \( U^* >_\omega U \uparrow \omega, \) with \( \min \text{dom}(U^* \setminus \alpha^*) < \omega, \). But this is a contradiction to \( U \) being the canonical sequence. \( \square \)

8. Non-closure of the image model

8.1. **Theorem.** Assume there is an elementary embedding \( \pi : V \rightarrow M, M \) transitive with critical point \( \kappa \) such that \( ^\omega M \subset M \) and \( ^\omega M \neq M \). Then \( 0^{\text{long}} \) exists.

This strengthens a theorem of Sureson’s [20] who from the same assumption could show the existence of an inner model with \( \omega_1 \) measurable cardinals.

**Proof.** Assume \( \neg 0^{\text{long}} \) and work for a contradiction. Let \( U_0 := U_{\text{can}}, U' := \pi(U_{\text{can}}), \) and \( \bar{\pi} := \pi \upharpoonright K[U_0]. \bar{\pi} : K[U_0] \rightarrow K[U'] \) is elementary, and by 3.17, \( \bar{\pi} \) is the iteration map of a normal iterated ultrapower of \( K[U_0] \). This means that there
is a strictly increasing index \( \langle \kappa_i | i < \delta \rangle \) such that if \( \langle \langle K[U]_i | i < \delta \rangle, \langle \pi_{ij} | i < j < \delta \rangle \rangle \) is the corresponding iteration of \( K[U_0] \) by \( \langle \kappa_i | i < \delta \rangle \), then \( K[U'] = K[U_\delta] \) and \( \pi = \pi_{0\delta} \). We may of course assume that for every \( i < \delta \), \( \kappa_i \) is a measurable cardinal in \( K[U_i] \), \( \kappa_i \in \text{dom}(U_i) \). Then \( \kappa_0 = \kappa \), and since \( \text{not} \)\( \kappa_0 \),

(1) \( \text{otp dom}(U_0) < \kappa \).

The following claim is taken over from Sureson [20] with a slightly different presentation of the proof:

(2) \( \delta \geq \omega \).

**Proof.** Assume \( \delta < \omega \). Let \( X = \{ \pi(f)(\kappa_0, \ldots, \kappa_{\delta-1}) | f : \kappa_0 \times \cdots \times \kappa_\delta \rightarrow V \} \).

(2') \( X = M \).

**Proof.** For every \( \gamma \in \text{On}, \gamma = \pi_{0\delta}(f)(\kappa_0, \ldots, \kappa_{\delta-1}) = \pi(f)(\kappa_0, \ldots, \kappa_{\delta-1}) \) for some \( f \in K[U_0], f : \kappa_0 \times \cdots \times \kappa_\delta \rightarrow \text{On} \) by the 'representation properties' of the iterated ultrafilter \( K[U_\delta] \) (compare Definition 2.2(iii)). Hence \( \text{On} \subseteq X \).

Let \( z \in M \). Let \( z \in V_\alpha \) and take \( g : \beta \rightarrow V_\alpha \) onto. Then \( \pi(g) : \pi(\beta) \rightarrow V_{\pi(\alpha)} \) onto; let \( z = \pi(g)(\gamma) \) for some \( \gamma \in \text{On} \). \( \gamma = \pi(f)(\kappa_0, \ldots, \kappa_{\delta-1}) \) for some \( f : \kappa_0 \times \cdots \times \kappa_{\delta-1} \rightarrow V \). Define \( h : \kappa_0 \times \cdots \times \kappa_{\delta-1} \rightarrow V \) by: \( h(x_0, \ldots, x_{\delta-1}) = g(f(x_0, \ldots, x_{\delta-1})) \) if this is defined, and \( h(x_0, \ldots, x_{\delta-1}) = 0 \) else. Then

\[ z = \pi(g)(\pi(f)(\kappa_0, \ldots, \kappa_{\delta-1})) = \pi(h)(\kappa_0, \ldots, \kappa_{\delta-1}) \in X. \quad \Box(2') \]

(2') \( \text{M} \subseteq M \).

**Proof.** Let \( \{ x_\alpha | \alpha < \kappa \} \subseteq M \). Choose a sequence \( \{ f_\alpha | \alpha < \kappa \}, f_\alpha : \kappa_0 \times \cdots \times \kappa_{\delta-1} \rightarrow V \) such that \( x_\alpha = \pi(f_\alpha)(\kappa_0, \ldots, \kappa_{\delta-1}) \) for \( \alpha < \kappa \). Define \( F : \kappa_0 \times \cdots \times \kappa_{\delta-1} \rightarrow V \) by \( F(x_0, \ldots, x_{\delta-1}) = \langle f_\alpha(x_0, \ldots, x_{\delta-1}) | \alpha < \kappa \rangle \). For \( \alpha < \kappa \), \( \langle \pi(F)(\kappa_0, \ldots, \kappa_{\delta-1}) (\alpha) = \pi(f_\alpha)(\kappa_0, \ldots, \kappa_{\delta-1}) = x_\alpha \). So

\[ \langle x_\alpha | \alpha < \kappa \rangle = \langle \pi(F)(\kappa_0, \ldots, \kappa_{\delta-1}) \rangle | \kappa \in M. \quad \Box(2') \]

But (2') contradicts the non-closure property of \( M \). \( \Box(2') \)

Let \( C \) be a Prikry system for \( K[U_{\text{can}}] \) which satisfies the covering Theorem 3.23. Let \( C_0 := \bigcup \{ C(\gamma) | \gamma \in \text{dom}(U_{\text{can}}) \} \). Set \( C' := \pi(C), C'_0 := \pi(C_0) \). By (1), the ordertype of \( C'_0 \) is \( < \kappa \), and we get

(3) \( C'_0 = \pi''C_0 \subseteq \text{range}(\pi) \).

Set \( \tau := \sup \{ \kappa_i | i < \omega \}, \ X := \{ \kappa_i | i < \omega \}, \text{ and } D := C'_0 \cap \tau \). \( X \subseteq M \) since \( \text{not} \)\( \kappa \subseteq M \). In \( M \), apply the covering Theorem 3.23 to \( X \): There is \( f : (\tau)^{< \omega} \rightarrow \tau, f \in K[U'] \) and a \( \gamma < \tau \) such that

(4) \( X \subseteq \{ f(\bar{v}, \bar{u}) | \bar{v} < \gamma, \bar{u} \in D \} \).

Since the iteration by \( \langle \kappa_i | i < \delta \rangle \) is normal, \( \mathcal{P}(\tau) \cap K[U_\omega] = \mathcal{P}(\tau) \cap K[U_\delta] \) (compare 2.4(iii)). So \( f \in K[U_\omega] \). Choose \( i < \omega \) such that \( \kappa_i > \gamma \) and \( f \in
range(\(\pi_{\text{lm}}\)); let \(f = \pi_{\text{lm}}(\tilde{f})\). By (4), \(\kappa_i = f(\tilde{v}, \tilde{\mu})\) for some \(\tilde{v} < \kappa_i, \tilde{\mu} \in D\). By (3), \(D \subset \text{range}(\pi_{\text{lm}})\); let \(\tilde{\mu} = \pi_{\text{lm}}(\tilde{\mu})\). Then

\[
\kappa_i = \pi_{\text{lm}}(\tilde{f})(\tilde{v}, \pi_{\text{lm}}(\tilde{\mu})) = \pi_{\text{lm}}(\tilde{f}(\tilde{v}, \tilde{\mu})) \in \text{range}(\pi_{\text{lm}}).
\]

But since \(\kappa_i\) is the critical point of \(\pi_{\text{lm}}\) this is impossible. \(\Box\)

References