Global Square and Mutual Stationarity at the $\aleph_n$

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Abstract

We give a proof of a theorem of Jensen and Zeman on the existence of Global $\square$ in the Core Model below a measurable cardinal κ of Mitchell order ("$o_M(\kappa)$") equal to κ^{++}, and use it to prove the following theorem on mutual stationarity at the $\aleph_n$.

Let ω_1 denote the first uncountable cardinal of V and set $\mathrm{Cof}(\omega_1)$ to be the class of ordinals of cofinality ω_1.

Theorem: *If every sequence $(S_n)_{n<\omega}$ of stationary sets $S_n \subseteq \mathrm{Cof}(\omega_1) \cap \aleph_{n+2}$, is mutually stationary, then there is an inner model with infinitely many inaccessibles $(\kappa_n)_{n<\omega}$ so that for every m the class of measurables λ with $o_M(\lambda) \geq \kappa_m$ is stationary in κ_n for all $n > m$. In particular, there is such a model in which for all sufficiently large $m < \omega$ the class of measurables λ with $o_M(\lambda) \geq \omega_m$ is, in V, stationary below $\aleph_{m+2}$.*

1 Introduction

This paper extends previous investigations into the nature of *mutual stationarity*, a concept introduced by M. Foreman and M. Magidor [6] in order to transfer some combinatorial aspects of stationary subsets of regular cardinals to singular cardinals. They made particular use of this in investigating the non-saturation of the non-stationary ideals of the form $\mathcal{P}_\kappa(\lambda)$.

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Our purpose here is to establish that the mutual stationarity property at $\aleph_\omega$ (or more precisely at the sequence of the first ω-many uncountable cardinals, $\langle \aleph_n \mid 0 < n < \omega \rangle$), is a *large cardinal* property, that is, it entails the consistency of *strong axioms of infinity* which concern measurable cardinals. The definition of mutual stationarity is more general than this however:

Definition 1.1 *Let $(\kappa_n)_{n<\omega}$ be a strictly increasing sequence of regular cardinals $\geqslant \aleph_2$ with $\kappa_\omega = sup_{n<\omega}\kappa_n$. A sequence $(S_n)_{n<\omega}$ is called* mutually stationary in *$(\kappa_n)_{n<\omega}$ if every first-order structure $\mathfrak{A}$ of countable type with $\kappa_\omega \subseteq \mathfrak{A}$ has an elementary substructure $\mathfrak{B} \prec \mathfrak{A}$ such that*

$$\forall n < \omega \sup |\mathfrak{B}| \cap \kappa_n \in S_n.$$

M. Foreman and M. Magidor, together with J. Cummings further investigated the status of such sequences in [2]. Note that if $(S_n)_{n<\omega}$ is mutually stationary in $(\kappa_n)_{n<\omega}$ then each $S_n \cap \kappa_n$ is stationary in κ_n. In the following we shall denote the class $\{\xi \in Ord | cf(\xi) = \lambda\}$ by Cof_λ.

Definition 1.2 *Let $(\kappa_n)_{n<\omega}$ be a strictly increasing sequence of regular cardinals and $\lambda < \kappa_0$, λ regular. The* mutual stationarity property *$MS((\kappa_n)_{n<\omega}, \lambda)$ is the statement: if $(S_n)_{n<\omega}$ is a sequence of stationary sets $S_n \subseteq Cof_\lambda \cap \kappa_n$, then $(S_n)_{n<\omega}$ is mutually stationary in $(\kappa_n)_{n<\omega}$.*

M. Foreman and M. Magidor [6] proved the following two theorems:

Theorem. *For $(\kappa_n)_{n<\omega}$ be any strictly increasing sequence of uncountable regular cardinals:*
(i) *$MS((\kappa_n)_{n<\omega}, \omega)$ holds.*
(ii) *$MS((\kappa_n)_{n<\omega}, \omega_1)$ implies $V \neq L$.*

This did not yet say that MS was a large cardinal property. That it was is the left to right direction of the following equivalence, proven in [12]:

Theorem 1.3 *The theories $ZFC + \exists (\kappa_n)_{n<\omega} MS((\kappa_n)_{n<\omega}, \omega_1)$ and $ZFC + \exists \kappa (\kappa$ measurable) are equiconsistent.*

The implication from right to left was first proven by Cummings, Foreman, and Magidor [3] *via* Prikry forcing. They proved more than this: they showed that a tail of the Prikry generic sequence satisfies $MS((\kappa_n)_{n<\omega}, \lambda)$ for any $\lambda < \kappa_0$ (or indeed the mutual stationarity of *any* sequence of stationary sets $S_n \subseteq \kappa_n$ irrespective of the cofinalities of the ordinals in the S_n). This is essentially obtained by utilising the fact that a tail of the Prikry generic sequence remains *coherently Ramsey* in the generic extension. The forward direction was proven in [12] using the core model K of A. J. Dodd and R. B. Jensen (see [5]). The deduction of the existence of $0^\sharp$ from $MS((\kappa_n)_{n<\omega}, \omega_1)$ was done in detail, and the extension to proving the existence of the inner model with a measurable was sketched, using the hyperfine structure of S.Friedman and the first author ([7]). The proof involved the global square principle $\Box$ in L and techniques from the Jensen Covering theorem for L (see [4]). The purpose of this paper is to give a full account of the interaction of the proof of global $\Box$ with the MS property, (insofar as we are able) thus filling in the details of the above argument, but significantly strengthening the result to obtain models with many measures of high Mitchell order, in the case $(\kappa_n)_{n<\omega}$ consists of consecutive sequences of cardinals mentioned in the abstract:

Theorem 1.4 *If $MS((\aleph_n)_{0<n<\omega}, \omega_1)$ holds then there is an inner model, K, and there is $2 < k < \omega$ so that for any n with $k < n < \omega$ each $\aleph_n$ is a Mahlo limit (in V) of ordinals κ which are, in K, measurable of Mitchell order $o_M(\kappa) = \omega_{n-2}$. In fact, for such $\aleph_n$ the ordinals $\alpha \in \mathrm{Cof}(\omega_{n-2})$ which are singular in K are, in V, non-stationary below $\aleph_n$.*

One might wonder whether increasing the cofinality of the independently chosen stationary sets might yield increased Mitchell order. Well, perhaps, but seemingly not by our methods. The following is a corollary to the proof of the above theorem.

Corollary 1.5 *Let m be fixed, $1 \leq m < \omega$. Then if $MS((\aleph_{n+m})_{0<n<\omega}, \omega_m)$ holds, exactly the same conclusion as that of Theorem 1.4 may be drawn.*

The methods here seem just short of allowing us to conclude that there is an inner model with a measurable κ with Mitchell order of κequal to κ : (“$o_M(\kappa) = \kappa$”).

It is important in the above statement that we use all the alephs below $\aleph_\omega$ (from some point on) since the first author has shown that omitting a cardinal above each one for which we wish to consider arbitrary stationary sets, has a much weaker consistency strength, (see [11]).

Theorem 1.6 *The theories $ZFC + MS((\aleph_{2n+1})_{n<\omega}, \omega_1)$ and $ZFC + \exists\kappa(\kappa$ a measurable cardinal) are equiconsistent.*

The model K in Theorem 1.4 can be taken to be the core model built using measures (partial or full) only on its constructing extender sequence.

We shall need the following formulation of the Weak Covering Lemma due to W.MITCHELL (*cf.* [13])

Theorem 1.7 (Weak Covering Lemma) *Assume there is no inner model with a measurable cardinal κ with $o_M(\kappa) = \kappa^{++}$. Let α be regular in K with $\omega_1 \leq \gamma = cf(\alpha) < card(\alpha)$. Then in K we have $o_M(\alpha) \geq \gamma$.*

We shall assume a development of the fine structure of such a core model K, as can be found in M. ZEMAN [17]. K is thus a model of the form $L[E]$ with E a sequence of partial or full extenders in the manner of Zeman's book. However no such extender requires any generator beyond that of its critical point. We shall need to consider the proof of the existence of global square $\Box$ in such a model. This is known to hold, *cf* [10]. The fine structural notation we shall adopt is that of the book (which is also that of the paper cited). The indexing of extenders will be the Friedman-Jensen indexing whereby an extender is placed on the E sequence of a hierarchy at precisely the successor cardinal of the image of the critical point by that extender. Again this is following [10].

Jensen and Zeman's method of proof for global $\Box$ is to define a "smooth category" of structures and maps from which it is known that a global $\Box$ sequence can be derived. This latter *derivation* is purely combinatorial and so requires no inspection of the fine structure of the original model. The burden of their proof is the *construction* of the smooth category itself. However that construction does not yield an explicit computation for the order types of the various C_ν sequences. (It is the latter derivation that does that). For our proof we need to have a construction of global $\Box$ where we can see (i) what those order types will be and how they are arrived at; and (ii) that order types for certain C_ν-like sequences will (on a tail) not be prolonged by iterations of the mouse from which they are defined. We give a proof of Global $\Box$ *ab initio* directly without going through the smooth category. This is done in Section 3. In section 2 we give some fine structural lemmas that form the hard work of Jensen and Zeman's account in [10] which establish the right forms of parameter preservation and appropriate condensation lemmata. We merely quote these as Condensation Lemmas (I) and (II). However in order to prove that the order types of C_ν sequences are not prolonged by iterations of the structure over which they are defined we need to prove the preservation of the d-parameters of [10]. This is at Lemma 2.8. The analysis of the Condensation Lemmata apart, we try to keep the rest of the proof as self-contained as possible. The proofs of Lemmas 3.9 and 3.11 in particular repeat the proofs of [10] 4.3 and 4.5. These are key lemmata on the relationships between singularising structures and the maps between them, and are, in the Σ^* terminology, the successors to [1] Lemmas 6.15 and 6.18. From Definition

3.17 onwards this is an account very much following that of [1] (and which will be in the forthcoming [15]), but modestly dressed in the appropriate J_s mouse notation. In Section 4 we see how to use features of this proof to get the main Theorem 1.4; the reader who is completely familiar with the $\Box$ proof and wants to discover the ideas in the application to mutual stationarity may wish to go straight there.

2 Fine structural prerequisites

For an acceptable J-structure M we assume familiarity with the notions of the uniformly defined Σ_1-Skolem function for M, h_M ,and of the class of parameter sequences Γ^M, and the parameter sets $P^n_M, P_M, P^*_M, R^n_M, R_M$, and R^*_M. We shall write ρ_M as usual for the Σ_1-projectum of M. Similarly we shall write for the *n+1'st projectum* $\rho^{n+1}_M =_{df} \min\{\rho_{M^{n,p}} \mid p \in \Gamma^n_M\}$. We may assume that parameters are finite sets of ordinals. This applies as well to the *n'th-standard parameter* and the *standard parameter* denoted here p^n_M, p_M respectively for a structure M as above. We wellorder $[On]^{<\omega}$ by $u <^* v \leftrightarrow \max(u\Delta v) \in v$. For $X \subseteq Ord$ a set, we write $ot(X)$ for its order type, and by X^* we mean the set of limit points of X. Our discussion of fine structure is entirely in the language of $\Sigma^{(n)}_k$ relations due to Jensen (for which see [17] or [15]). Boldface relations such as $\mathbf{\Sigma}^{(n)}_1(M)$ denote those definable using parameters (in this case from M.)

Definition 2.1 ($\Sigma^{(n)}_1$**-Skolem Functions)** *Let M be an acceptable J-structure, and let $p \in \Gamma^n_M$.*

(i) $h^{n,p}_M = h_{M^{n,p}}$*;*

(ii) $\tilde{h}^n_M(w^n, x^0) = g_0(g_1 \cdots g_{n-1}((w^n)_0, \langle (w^n)_1, x^0(n-1)\rangle) \cdots x^0(0)\rangle)$ *where, for $i \leq n$*

$$g_i(\langle j, y^{i+1}\rangle, p) = h_{M^{i,p\restriction i}}(j, \langle y^{i+1}, p(i)\rangle).$$

Then g_i is uniformly lightface $\Sigma^{(i)}_1(M)$ in the variables shown. Thus $\tilde{h}^n_M$ is $\Sigma^{(n-1)}_1$ uniformly over all M. The Σ_1 hull of a set $X \subseteq M^{n,p}$ we shall denote by $h^{n,p}_M(X)$ (and is thus the set $\{h^{n,p}_M(i,x)) \mid i \in \omega, x \in X\}$). Note that $\tilde{h}^1_M(\langle j, y^0\rangle, p(0)) = g_0(j, \langle y, p(0)\rangle) = h_M(j, \langle y, p(0)\rangle)$. If $p \in R^n_M$ then every $x \in M$ is of the form $\tilde{h}^n_M(z,p)$ for some $z \in H^n_M$. We may similarly form hulls using $\tilde{h}^n_M$: again if $X \subseteq M^{n,p}$ say, and $q \in M$ then the $\Sigma^{(n-1)}_1$ hull of $X \cup \{q\}$ is the set $\{\tilde{h}^n_M(x,q)) \mid x \in X\}$ (we again may write $\tilde{h}^n_M(X \cup \{q\})$ for this hull here). The following states some of these facts and are easy to establish (see [17] p.29):

Lemma 2.2 *Let M be acceptable, and $p \in R^n_M$;*
(i) If $\omega\rho^n_M \in M$ and $p \in R^n_M$ then $\tilde{h}^n_M$ is a good, uniformly defined, $\Sigma_1^{(n-1)}(M)$ function mapping $\omega\rho^n_M$ onto M.
(ii) (a) every $A \subseteq H^n_M$ which is $\mathbf{\Sigma}_1^{(n)}(M)$ is $\mathbf{\Sigma_1}(M^{n,p})$;
(b) $\rho^{n+1}_M = \rho_{M^{n,p}}$

Lemma 2.3 *Let M be an acceptable J-structure. Then (i) $\mathbf{\Sigma^*}(M) \subseteq \mathbf{\Sigma}_\omega(M)$. (ii) Let $p \in R^*_M$. Then $\mathbf{\Sigma^*}(M) = \mathbf{\Sigma}_\omega(M)$.*

Lemma 2.4 *Let $\overline{M}, M$ be acceptable structures, and suppose $\pi : \overline{M} \longrightarrow M$ is $\Sigma_1^{(n)}$-preserving, and is such that $\pi \restriction \omega\rho^{n+1}_M = id$ and $ran(\pi) \cap P^*_M \neq \varnothing$. Then π is Σ^*-preserving.*

Proof: This is [17] 1.11.2. Q.E.D.

Recall that a premouse M is *sound above* ν if $\omega\rho^{n+1}_M \leq \nu$ means that $\tilde{h}^{n+1}_M(\nu \cup \{p_M\}) = |M|$. We also say that it is *k-sound* if it is sound above $\omega\rho^k_M$.

In the next lemma there are various concepts that we shall quickly gloss: $o^N(\kappa)$ is the extender order of κ in the hierarchies under consideration (and roughly corresponds to Mitchell order of measures); the hat over a premouse, as in $\hat{N}$, indicates the *expansion* of the premouse structure N, to which extenders are usually applied (as, for example, when coiterations of premice are formed). The premice then act as *bookkeeping* premice for the indices that are being used, whilst the actual extenders are applied to these hatted expansions. We simply follow the conventions of [10] and we ignore the differences between these structures. The reader worried about these details may consult [10] Sect. 2 or Ch. 8 of [17].

Theorem 2.5 (Condensation Lemma I) *(*cf [10] 2.1*). Suppose there is no inner model for $o_M(\kappa) = \kappa^{++}$. Let N be a premouse, M a mouse and $\sigma : \hat{N} \longrightarrow_{\Sigma_0^{(n)}} \hat{M}$ with $\sigma \restriction \omega\rho^{n+1}_N = id$;* *Then N is a mouse; moreover if N is sound above $\nu = crit(\sigma)$ then one of the following holds:*

(i) N is the core of M above ν and σ is the iteration map, which is the corresponding core map;

(ii) N is a proper initial segment of M;

(iii) For some $\kappa < \nu$ $\beta =_{df} o^N(\kappa) \geq \nu$, and if $\zeta < \kappa^{+M}$ is maximal so that E^M_β measures all subsets of $\kappa = crit(E^M_\beta)$ which lie in $M\|\zeta$, then N is an initial segment of M^+ where $\pi : \widehat{M\|\zeta} \longrightarrow^*_{E^M_\beta} M^+$.

In order to have sufficient further condensation Jensen and Zeman require certain parameters associated with canonical witness structures to be in the range of their maps. We only remind the reader of this definition here, and refer to the paper for a full discussion of their significance.

Definition 2.6 *Suppose $\gamma \in p^n_M$ and let σ^M_γ be the canonical witness map corresponding to W^γ_M; if $\sup(ran(\sigma^M_\gamma)) \cap \omega\rho^n_M < \omega\rho^n_M$ we set $\delta(\gamma) = \sup(ran(\sigma^M_\gamma) \cap \omega\rho^n_M)$.*

We set $\widetilde{p}^n_M =_{df} \{\gamma \in p^n_M \mid \delta(\gamma)$ is defined$\}$, and appropriately $\widetilde{p}_M =_{df} \bigcup_n \tilde{p}^n_M$. Further set $d^n_M =_{df} \{\delta(\gamma) \mid \gamma \in \tilde{p}_M, k \leq n\}$ etc.

This finite (possibly empty) set d^n_M then collects together all those sups of those canonical witness maps σ_γ just for those γ for which the map is non-cofinal at the k'th levels for $k \leq n$. This allows for an appropriate form of the Condensation Lemma for hierarchies below mice M with any κ with $(o_M(\kappa) = \kappa^{++})^M$. The following is again taken from [10].

Theorem 2.7 (Condensation Lemma II) (*cf* [10] 3.1) *Suppose there is no inner model for $o_M(\kappa) = \kappa^{++}$. Let N, M be mice and $\sigma : \hat{N} \longrightarrow_{\Sigma^{(n)}_1} \hat{M}$. Suppose further that $\sigma(\bar{\alpha}) = \alpha, \sigma(\bar{p}) = p_M \backslash \alpha$, and*

(i) $\omega\rho^{n+1}_M \leq \alpha < \omega\rho^n_M$ and M is sound above α;

(ii) $d^n_M \subseteq ran(\sigma)$.

Then $\bar{p} = p_N \backslash \bar{\alpha}$; N is sound above $\bar{\alpha}$, $\sigma(\tilde{p}_N \backslash \bar{\alpha}) = p_M \backslash \alpha$ and $\sigma(\delta^N(\gamma)) = \delta^M(\sigma(\gamma))$ whenever $\gamma \in \tilde{p}_N \backslash \bar{\alpha}$.

We shall need a lemma on preservation of these d-parameters under normal iterations. We prove this here.

Lemma 2.8 *Suppose $\pi : M \longrightarrow N$ is a normal iteration of M. Then $\pi(d_M) = d_N$.*

Proof: This would be by induction on the length of the iteration, but we simply do a one step ultrapower by an extender E with critical point κ and the reader can form the general and direct limit argument herself. This does not follow quite immediately from Condensation Lemma II as the latter assumes d_N is in the range of the map. We know that $\pi(p_M) = p_N$. We may express

$$\overline{p}_M = \{\nu \in p_M \mid \text{If } \nu \in [\omega\rho^{k+1}_M, \omega\rho^k_M) \text{ then the canonical witness map is non-cofinal into } \omega\rho^k_M\}.$$

And: $d_M = \{\delta^M(\nu) | \nu \in \overline{p}_M\}$.

Then if $\delta(\nu) \in d_M$ with $\nu \in [\omega\rho^{k+1}_M, \omega\rho^k_M)$ we have as in [10]:

$$(*) \quad \forall\xi^k\forall\zeta^k(\xi^k < \nu \wedge \zeta^k = \widetilde{h}^{k+1}_M(\xi^k, p_M\backslash(\nu+1)) \longrightarrow \zeta^k \leq \delta(\nu)).$$

This is $\Pi^{(k)}_1$ in $\nu, \delta(\nu)$, and p_M. If $crit(E) = \kappa \in [\omega\rho^{n+1}_M, \omega\rho^n_M)$ then π is $\Sigma^{(n)}_0$ preserving and cofinal into $\omega\rho^n_M$, hence $\Sigma^{(n)}_1$-preserving. If $k < n$ then it is

$\Sigma_2^{(k)}$preserving. Consequently wherever ν lies we have from these preservation properties:

(1) $\nu \in \bar{p}_M \longrightarrow \pi(\nu) \in \bar{p}_N \wedge \pi(\delta^M(\nu)) \geq \delta^N(\pi(\nu))$.

We want equality here. For $k < n$ $\Sigma_2^{(k)}$preservation suffices to guarantee this: if

$$\exists \delta^k < \pi(\delta^M(\nu))[\forall \xi^k \forall \zeta^k(\xi^k < \pi(\nu) \wedge \zeta^k = \widetilde{h}_N^{k+1}(\xi^k, \pi(p_M) \backslash (\pi(\nu)+1)) \longrightarrow \zeta^k \leq \delta^k]$$

then this would go down to M and give a contradiction. For $k = n$ we can reason as follows. Suppose $\bar{\delta} = \pi(f)(\kappa) = \delta^N(\pi(\nu)) < \pi(\delta^M(\nu))$. As at $(*)$:

$$\forall \xi^n \forall \zeta^n(\xi^n < \pi(\nu) \wedge \zeta^n = \widetilde{h}_N^{n+1}(\xi^n, p_N \backslash (\pi(\nu)+1)) \longrightarrow \zeta^n \leq \bar{\delta}).$$

By using a Łoš Lemma we should have that:

$$\{\alpha < \kappa | \forall \xi^n \forall \zeta^n(\xi^n < \nu \wedge \zeta^n = \widetilde{h}_M^{n+1}(\xi^n, p_M \backslash (\nu+1)) \longrightarrow \zeta^n \leq f(\alpha))\}$$

was of E-measure 1. But on a set of measure 1 $f(\alpha) < c_{\delta^M(\nu)}(\alpha) = \delta^M(\nu)$ so this contradicts the definition of $\delta^M(\nu)$. Hence

(3) $\nu \in \bar{p}_M \longrightarrow \pi(\delta^M(\nu)) = \delta^N(\pi(\nu))$.

Now note:

(4) $\pi(\nu) \in \bar{p}_N \longrightarrow \nu \in \overline{p}_M$ and hence again $\pi(\delta^M(\nu)) = \delta^N(\pi(\nu))$.

For $k < n$ this follows from $\Sigma_2^{(k)}$ preservation. For $k = n$ this follows from the cofinality of π into $\omega\rho_N^n$: if $\delta^N(\pi(\nu))$ is defined, then it is less than some $\pi(\delta)$ and the formula $(*)$ written out for N and $\pi(\delta)$ then goes down to M, so this suffices.

Q.E.D.

We shall also be assuming familiarity with the construction of fine-structural pseudo-ultrapowers, for which see [17] or [15]. We shall be using various "lift-up" lemmas. These are in the following form.

Definition 2.9 *Let M be an acceptable J-structure, and $\nu \in M$ a regular cardinal of M. Then $k(M,\nu)$ is defined to be the least k (if it exists) so that there is a good $\Sigma_1^{(k)}$-definable function whose domain is a bounded subset of ν and whose range is unbounded in ν. (Such a function is said to* singularize *ν and we say that ν is* $\mathbf{\Sigma_1^{(k)}}(M)$-singularized over *$M$.)*

Definition 2.10 *Let $\bar{M}, \bar{\nu}, k = k(\bar{M}, \bar{\nu})$ be as above with $\bar{\nu}$ regular in $\bar{M}$. Let $\overline{Q} =_{df} J_{\bar{\nu}}^{\bar{M}}$. Define $\Gamma^k_{\bar{M},\bar{\nu}} =_{df}$*

$$\{f \mid dom(f) \in \overline{Q} \wedge ran(f) \subseteq \overline{M}) \wedge (n < k \wedge f \in \mathbf{\Sigma}_1^{(n)}(\overline{M}) \wedge \omega\rho_{\overline{M}}^{n+1} \geq \overline{\nu})\}.$$

Theorem 2.11 (Pseudo-Ultrapower Theorem) *Let $\bar{M}$ be an acceptable J-structure, $\bar{\nu}$ a regular cardinal of $\bar{M}$ but with $k = k(\bar{M}, \bar{\nu})$ defined. Let $\overline{Q} =_{df} J_{\bar{\nu}}^{\bar{M}}$. Then there is a map $\tilde{\sigma} : \bar{M} \longrightarrow_{\Sigma_0} M$ (the "canonical k-extension" of $\sigma : \overline{Q} \longrightarrow_{\Sigma_0} Q$) satisfying:*

(i) $\tilde{\sigma}$ *is Q-preserving,* M *is an acceptable end extension of* Q*, and* $M = \{\tilde{\sigma}(f)(u) \mid u \in \sigma(dom(f)), f \in \Gamma\}$.
(ii)a) $\tilde{\sigma}$ *is* $\Sigma_2^{(n)}$ *preserving for* $n < k$;
b) $\rho_k = \rho_M^k$*, and* $\tilde{\sigma}$ *is* $\Sigma_0^{(k)}$ *preserving and cofinal (thus* $\Sigma_1^{(k)}$*-preserving);*
(iii) $\tilde{\sigma}(\bar{\nu}) = \nu$ *and the latter is regular in* M *;*
(iiv) $k = k(M, \nu)$*:* k *is least so that there is a* $\boldsymbol{\Sigma_1^{(k)}}(M)$ *map cofinalising* ν*.*

Lemma 2.12 (Interpolation Lemma) *Suppose* $\overline{M} = \langle J_{\overline{\beta}}^{\overline{A}}, \overline{B}\rangle$ *is a structure such that* $\overline{\nu}$ *is regular in* $\overline{M}$*, but with* $k = k(\bar{M}, \bar{\nu})$ *defined. Suppose further that* $f : \overline{M} \longrightarrow_{\Sigma_1^{(k)}} M = \langle J_\beta^A, B\rangle$*. Let* $\widetilde{\nu} = \sup f``\overline{\nu}$*. Then there is a structure* $\widetilde{M} = \langle J_{\widetilde{\beta}}^{\widetilde{A}}, \widetilde{B}\rangle$*, a map* $\widetilde{f} : \overline{M} \longrightarrow \widetilde{M}$ *with* $\widetilde{f} \supseteq f \restriction J_{\overline{\nu}}^{\overline{A}}$ *and* $\widetilde{f}, \Sigma_0^{(k)}$*-cofinal (and hence* $\Sigma_1^{(k)}$*-preserving), and a unique* $f' : \widetilde{M} \longrightarrow_{\Sigma_0^{(k)}} M$*, with* $f = f' \circ \widetilde{f}$ *and* $f' \restriction \tilde{\nu} = id \restriction \tilde{\nu}$*.*

3 Global $\Box$ in K.

Definition 3.1 *Let Sing* $= \{\beta \in \mathrm{Ord} \mid \lim(\beta) \wedge \mathrm{cf}(\beta) < \beta\}$ *be the class of singular limit ordinals.* Global $\Box$ *is the assertion: there is a system* $(C_\beta)_{\beta \in \mathrm{Sing}}$ *satisfying:*
(a) C_β *is a closed cofinal subset of* β*;*
(b) $\mathrm{ot}(C_\beta) < \beta$;
(c) if $\overline{\beta}$ *is a limit point of* C_β *then* $\overline{\beta} \in \mathrm{Sing}$ *and* $C_{\overline{\beta}} = C_\beta \cap \overline{\beta}$*.*

Jensen [8] introduced the principle and proved it held in L. The format of the proof we shall follow will be that of [1], which was a proof in the setting of generalised $L[A]$ hierarchies suitable for use Jensen's Coding Theorem. The second author [14] proved in the Dodd-Jensen core model K. The first proof of $\Box$ which used the Baldwin-Mitchell arrangement of the $L[E]$ hierarchy, was for Jensen's model for K with measures of order zero, and was by Wylie [16]. From the order types of the square sequences C_ξ we shall define stationary sets S_n to which we shall apply the MS-principle.

We consider how a global $\Box$ sequence can be derived in K. For clarity we shall assume there is no inner model with a measure of Mitchell order $o_M(\kappa) = \kappa^{++}$ (see [10]) and that K is built under this assumption. We assume for the rest of this section $V = K$. Jensen and Zeman prove (more than) the following.

Theorem 3.2 *Let* S *be the class of all singular limit ordinals that are limits of admissibles. There is a uniformly definable class* $\langle C_\nu | \nu \in S\rangle$ *so that:*

(i) C_ν is a set of ordinals closed below ν and, if $cf(\nu) > \omega$, then it is also unbounded;

(ii) $ot(C_\nu) < \nu$;

(iii) $\overline{\nu} \in C_\nu \longrightarrow \overline{\nu} \in S \wedge C_{\bar{\nu}} = \overline{\nu} \cap C_\nu$;

It is well known that once one has a global sequence defined on the singular ordinals of some cub class that contains all singular cardinals and is cub beneath each successor cardinal, then this can be filled out to a global sequence on *all* singular ordinals to satisfy Definition 3.1. Hence proving the above theorem suffices. As $V = K = L[E]$ for $E = E^K$ a fixed sequence of extenders, if ν is a singular ordinal, then there will be a least level $J^E_{\beta(\nu)}$ of the J^E-hierarchy over which ν is definably singularised, *i.e.* there will be a partial $\boldsymbol{\Sigma_\omega}(J^E_{\beta(\nu)})$ definable good function mapping a subset of some γ cofinally into ν. This level of the hierarchy $J^E_{\beta(\nu)}$ will also be our main singularising structure M_ν. Note that by Lemma 2.3 and the soundness of the K hierarchy, any such function is also $\Sigma_1^{(n)}(J^E_{\beta(\nu)})$ for some n. That is, $k(\nu, J^E_{\beta(\nu)})$ in the sense of Definition 2.9 is defined.

However there will be many other mice over which ordinals are singularized and we must consider these in addition.

Definition 3.3 *S^+ is the class of $s = \langle \nu_s, M_s \rangle$ where*

(a) $\nu_s \in Sing$;

(b) M_s is a mouse satisfying the following:

(i) ν_s is regular in M_s and $J_s =_{df} J^{E^{M_s}}_{\nu_s}$ is a union of admissible sets $J^{E^{M_s}}_\tau$;

(ii) for some m, ν_s is $\boldsymbol{\Sigma_1^{(m)}}(M_s)$ singularised, that is $k(\nu_s, M_s)$ is defined;

(iii) $M_s$is sound above ν_s, and if $\nu_s = \kappa^{+M_s}$ where $\kappa \in Card^{M_s}$, then M_s is sound above κ.

Recall that if $M = \langle J^E_\alpha, \in \rangle$ and $\nu \leq \alpha$ then $M||\nu =_{df} \langle J^E_\nu, \in E_\nu \rangle$. We then note the following facts:

Lemma 3.4 *(i) If $\langle \nu, M \rangle, \langle \nu, N \rangle$ satisfy (b)(i),(ii) above but are both sound above ν, with $M||\nu = N||\nu$, then $M = N$.*

(ii) If $\langle \nu, M \rangle, \langle \nu, N \rangle \in S^+$and $J^{E^M}_\nu = J^{E^N}_\nu$ then $M = N$.

Proof: Straightforward iteration and comparison. Q.E.D.

The following definition encapsulates the essential concepts associated with singularising structures.

Definition 3.5 *Let $s \in S^+$. Then we associate the following to ν_s:*

a) $n_s =_{df} k(\nu_s, M_s)$, *the least* $n \in \omega$ *so that* ν_s *is* $\boldsymbol{\Sigma}_1^{(n)}(M_s)$ *singularised over* M_s.

b) $M_s^l =_{df} M_s^{l, p_{M_s} \restriction l}$ *for* $l \leq k_s, n_s$.

c) $h_s^l =_{df} h_{M_s}^{l, p_{M_s} \restriction l}$; $\quad h_s =_{df} h_s^{n_s}$; $\quad \widetilde{h}_s =_{df} \widetilde{h}_{M_s}^{n_s+1}$.

d) $\kappa_s \simeq$ *the largest cardinal of* J_s, *if such exists;* $\omega\rho_s =_{df} On \cap M_s^{n_s}$;
$\beta(s) =_{df} On \cap M_s$.

e) $p_s =_{df} p_{M_s} \backslash \nu_s$ *if* ν_s *is a limit cardinal of* J_s; $p_s =_{df} p_{M_s} \backslash \kappa_s$ *otherwise;*
$q_s =_{df} p_s \cap \omega\rho_{M_s}^{n_s}$;
$d_s =_{df} d_{M_s}$

f) $\alpha_s =_{df} \max\{\alpha < \nu | \nu \cap \tilde{h}_s(\alpha \cup \{p_s\}) = \alpha\}$, *setting* $\max \emptyset = 0$.

g) $\gamma_s \simeq \min\{\gamma < \nu | \exists f (f$ *a good* $\Sigma_1^{(n_s) M_s}(\{p_s\})$ *function singularising* ν *with domain* $\subseteq \gamma)\}$.

Thus if $\nu_s = \kappa_s^+$, we may have κ_s in p_s. Note that the closure of the set in *f)* ensures that α_s is always defined; note also that α_s must be strictly less than the first ordinal γ_s partially mapped by $\widetilde{h}_s$ (with parameter p_s) cofinally into ν_s. Note also that if we set $\gamma' = \max\{\gamma_s, (p_{M_s} \cap \nu_s)+1\}$ ($\max\{\gamma_s, (p_{M_s} \cap \kappa_s)+1\}$ if κ_s is defined), and then $\widetilde{h}_s(\gamma' \cup p_s)$ must be cofinal in ν_s since we shall have enough parameters in the domain of this hull to define our cofinalising map).

Lemma 3.6 $\quad \omega\rho_{M_s}^{n_s} \geq \nu \geq \omega\rho_{M_s}^{n_s+1}$

Proof Let $n = n_s$, $\nu = \nu_s$. Suppose the first inequality failed. Then $n > 0$, and we have some parameter q with a $\Sigma_1^{(n-1)}(M_s)(\{q\})$ partial map f of some $\gamma < \nu$ cofinal into ν.

Pick such a $\gamma > \omega\rho_{M_s}^n$. Let $\pi : \widetilde{M} \longrightarrow M_s$ have range $\widetilde{h}_s^n(\gamma \cup \{q, p_s\})$, with $\widetilde{M}$ transitive. By the leastness of n, $ran(\pi)$ cannot be unbounded in ν. By Lemma 2.4, since $\pi \restriction \omega\rho_{M_s}^n = id$ and $p_{M_s} \in ran(\pi)$, π is Σ^* elementary. However then $ran(f) \subseteq ran(\pi)$, with the former unbounded in ν. A contradiction! If the second inequality failed, then the partial function $\boldsymbol{\Sigma}_1^{(n)}(M_s)$ singularising ν would be a subset of ν and thus a bounded subset of $\omega\rho_{M_s}^{n+1}$ belonging to M_s. Q.E.D.

Definition 3.7 *For $s, \bar{s} \in S^+$:(i) We set $f : \bar{s} \Longrightarrow s$ if there is $|f|$ with $|f| : J_{\bar{s}} \longrightarrow_{\Sigma_1} J_s$, and $|f|$ is the restriction of some $f^* : M_{\bar{s}} \longrightarrow_{\Sigma_1^{(n)}} M_s$ where $n =$*

$n_s, \nu_s = f^*(\nu_{\bar{s}})$(if $\nu_s \in M_s$); $\kappa_s \in ran(|f|)$ (if κ_s is defined); α_s, p_s, d_s are all in $ran(f^*)$.

(ii) $\mathbb{F} = \{\langle \bar{s}, |f|, s\rangle | f : \bar{s} \Longrightarrow s\}$; we write here $\bar{s} = d(f), s = r(f)$;

(iii) If $\nu_s \in M_s$, we set:

$p(s) =_{df} p_s \cup \{d_s, \alpha_s, \nu_s, \kappa_s\}$ (if κ_s is defined); otherwise

$p(s) =_{df} p_s \cup \{d_s, \alpha_s, \kappa_s\}$ (again including κ_s only if it is defined).

(iv) $f_{(\delta,q,s)}$ is the inverse of the transitive collapse of the hull $\widetilde{h}_s(\delta, \{p(\nu)\})$ in M_s.

(Lemma 3.9 will justify in the final clause *(iv)* that there is some $\bar{s}$ so that $\langle \bar{s}, |f_{(\delta,q,s)}|, s\rangle \in \mathbb{F}$).

Lemma 3.8 *If $\exists \bar{s}(f : \bar{s} \Longrightarrow s)$ then $|f|$ and f^* are uniquely determined by $ran(|f|) \cap \nu_s$.*

Proof: As M_s is sound above ν_s, we have by our definitions, that $\widetilde{h}_s(\omega\nu_s \cup \{p_s\}) = M_s$. We have a $\Delta_1(J_s)$ onto map $g : \omega\nu_s \twoheadrightarrow J_s$. Thus, if $Y = \widetilde{h}_s(\omega\nu_s \cap ran(|f|) \cup \{p_s\})$, then $Y = \widetilde{h}_s(ran(|f|) \cup \{p_s\})$=$ran(f^*)$. Q.E.D.

Lemma 3.8 justifies us in calling f^* the *canonical extension* of f, (or rather $|f|$) and sometimes we abuse notation and write $f^* : J_{\bar{s}} \longrightarrow_{\Sigma_1} J_s$ where more correctly we should write $f^* \restriction J_{\bar{s}} : J_{\bar{s}} \longrightarrow_{\Sigma_1} J_s$. By virtue of the last lemma, this does not cause any ambiguity.

The next two lemmata are fundamental and concern relationships between singularising structures, and associated maps between them.

Lemma 3.9 *Let $f : \overline{M} \longrightarrow_{\Sigma_1^{(n)}} M_s$; suppose $f(\bar{d}, \bar{\alpha}, \bar{p}) = d_s, \alpha_s, p_s$, and (where appropriate) $f(\bar{\kappa}, \bar{\nu}) = \kappa_s, \nu_s$. (The latter if $\nu_s \in M_s$; if $\nu_s = On \cap M_s$ then we take $\bar{s} = On \cap M_{\bar{s}}$.) Then $\bar{s} = (\bar{\nu}, \overline{M}) \in S^+$ and thus $f : \bar{s} \Longrightarrow s$; moreover $n, \bar{d}, \bar{\alpha}, \bar{p}, \bar{\kappa}$ (the latter defined if κ_s is) are $n_{\bar{s}}, d_{\bar{s}}, \alpha_{\bar{s}}, p_{\bar{s}}, \kappa_{\bar{s}}$.*

Proof We shall show that $\overline{M}$ is a singularising structure for $\bar{\nu} =_{df} \nu_{\bar{s}}$ and the other mentioned parameters have the requisite properties to satisfy the relevant definitions, and are moved correctly by f. We set $\nu = \nu_s$.

(1) *$\bar{p} = p_{\bar{M}} \backslash \overline{\nu}$; $\bar{d} = d^n_{\bar{M}} \backslash \bar{\nu}$ and $\overline{M}$ is sound above $\bar{\nu}$.*

Proof: Directly by the Condensation Lemma II Theorem 2.7 Q.E.D. (1)

Let $\overline{h}$ have the same functionally absolute definition over $\overline{M}$ as $\widetilde{h}_s$ does over M_s. $\overline{h}$ is thus $\mathbf{\Sigma_1^{(n)}}(\overline{M})$.

(2) *$\overline{\alpha}$ is defined from $\overline{M}$ as α was defined from M_s.*

Proof: Set $H(\xi^n,\zeta^n) \longleftrightarrow \widetilde{h}_s(\omega\xi^n \cup \{p_s\}) \cap \nu \subseteq \zeta^n$
$\overline{H}(\xi^n,\zeta^n) \longleftrightarrow \bar{h}(\omega\xi^n \cup \{\bar{p}\}) \cap \bar{\nu} \subseteq \zeta^n$.

Then H is $\Pi_1^{(n)M_s}(\{p_s\})$, and $\overline{H}$ $\Pi_1^{(n)\bar{M}}(\{\bar{p}\})$, by the same definition. As $M_s \models H(\alpha,\alpha)$ it follows that $\overline{M} \models \overline{H}(\bar{\alpha},\bar{\alpha})$. However for any ξ with $\bar{\alpha} < \xi^n < \bar{\nu}$ we must have $\overline{M} \models \neg\overline{H}(\xi^n,\xi^n)$, because $M_s \models \neg H(f(\xi^n), f(\xi^n))$ since $\alpha < f(\xi^n) < \nu$. Hence $\bar{\alpha}$ is defined in the requisite way. Q.E.D. (2)

(3) $\exists\bar{\xi}^n < \bar{\nu}(\tilde{h}_s(f(\bar{\xi}^n) \cup \{p_s\})$ *is unbounded in* $\nu)$.

If (3) were to hold for some $\bar{\xi}$ then $\bar{h}(\xi \cup \{\bar{p}\})$ would be cofinal in $\bar{\nu}$, since the following is a $\Pi_2^{(n)}$ expression which thus would go down to $\overline{M}$. It would then be a statement about the parameters $\bar{p}, \bar{h}$, $\bar{\xi}$ and $\bar{\nu}$ (the latter if $\nu = f(\bar{\nu}) < \omega\rho_s$):
$M_s \models (\forall\zeta^n < \nu)(\exists\delta^n < f(\bar{\xi}^n))(\exists i < \omega)(\zeta^n < \tilde{h}_s(i, \langle\delta^n, p_s\rangle) < \nu)$.
This would show that $\bar{h}$ is a singularising function for $\bar{\nu}$ over the structure $\overline{M}$ and that $n \geq n_{\bar{s}}$. We need to show that (3) holds. Suppose not. This has the consequence that $\tau =_{df} \sup f``\bar{\nu} \leq \gamma_s < \nu$. As $\alpha_s \in ran(f \restriction \bar{\nu})$ we have that $\alpha_s < \tau$. So by definition of α_s itself:

(4) $\tau \neq \nu \cap \tilde{h}_s(\tau \cup \{p_s\})$.

Hence the following is true in M_s:

$$\exists i \in \omega\exists\xi^n < \tau(\nu > \tilde{h}_s(i, \langle\xi^n, p_s\rangle) \geq \tau).$$

Let i, ξ^n witness this, and pick $\bar{\delta} < \bar{\nu}$ so that $f(\bar{\delta}) > \xi^n$. Then for any $\bar{\mu} < \bar{\nu}$, as $f(\bar{\mu}) < \tau$:
$M_s \models (\exists\zeta^n < f(\bar{\delta}))(\nu > \tilde{h}_s(i, \langle\zeta^n, p_s\rangle) \geq f(\bar{\mu}))$.
This is $\Sigma_1^{(n)}$ and hence, for all $\bar{\mu} < \bar{\nu}$, goes down to $\overline{M}$, yielding:
$\overline{M} \models \forall\bar{\mu} < \bar{\nu}\exists\zeta^n < \bar{\delta}(\bar{\nu} > \bar{h}(i, \langle\zeta^n, \bar{p}\rangle \geq \bar{\mu})$.
Hence $\bar{h}$ is a singularising function for $\bar{\nu}$. Thus whether (3) holds or not we have established the existence of suitable $\Sigma_1^{(n)}(\overline{M})$ singularising function.
(5) $n = n_{\bar{s}}$.
We are left with showing $n \leq n_{\bar{s}}$, as the above shows that $n \geq n_{\bar{s}}$. Suppose $m < n$ and that $\bar{g}$ is a $\Sigma_1^{(m)}(\overline{M})$ good function in the parameter $\bar{r}$. Let g be $\Sigma_1^{(m)}(M_s)$ using the same functionally absolute definition and the parameter $f(\bar{r})$. Suppose $\bar{\delta} < \bar{\nu}$. By the $\Sigma_1^{(n)}$-elementarity of f we have the following $\Sigma_1^{(n)}$ statement holds in M_s (as $\mathrm{ran}(g \restriction f(\bar{\delta}))$ is bounded in ν):

$$(\exists\xi^n < \nu)(\forall\zeta^m < f(\bar{\delta}))(\forall\eta^m < \nu)(g(\zeta^m) = \eta^m \longrightarrow \eta^m < \xi^n)$$

(assuming $\nu < \beta$; otherwise drop the bound ν.) As f is $\Sigma_1^{(n)}$-preserving, we have in $\overline{M}$:

$$(\exists \xi^n < \bar{\nu})(\forall \zeta^m < \bar{\delta})(\forall \eta^m < \bar{\nu})(\bar{g}(\zeta^m) = \eta^m \longrightarrow \eta^m < \xi^n)$$

As $\bar{\delta}$ was arbitrary, we conclude $ran(\bar{g} \restriction \xi)$ is bounded on any $\xi < \bar{\nu}$. Hence $n \leq n_{\bar{s}}$. Q.E.D.(5) and Lemma.

Definition 3.10 *Suppose* $f : \bar{s} \Longrightarrow s$. *Then let* $\lambda(f) =_{df} \sup f``\overline{\nu}$; $\rho(f) =_{df} \sup f``\rho_{\overline{\nu}}$.

Lemma 3.11 *Suppose* $f : \bar{s} \Longrightarrow s$, *and let* $\lambda = \lambda(f)$. *Then* $\lambda \in Sing$ *and there exists a unique* $f_0 : \bar{s} \Longrightarrow s' = s|\lambda$ *with* $f \restriction \overline{\nu} = f_0 \restriction \overline{\nu}$.

Proof: Let $n = n_s$. We apply directly the Interpolation Lemma with λ as $\widetilde{\nu}$, $M_{\bar{s}}$, M_s as $\overline{M}, M$ respectively, and using $f^* : M_{\overline{s}} \longrightarrow_{\Sigma_1^{(n)}} M_s$ (where f^* is the canonical extension of f) we have the structure $\widetilde{M} = M_{s'}$ and maps $\widetilde{f}, f'$ as specified.

(1) $s' = \langle \lambda, \widetilde{M} \rangle \in S^+, n = n_{s'}$.

By the comment above $\gamma_{\bar{s}}$ is defined and $n = n_{\bar{s}}$. As $\widetilde{h}_{\bar{s}}(\gamma_{\bar{s}} \cup \{p_{M_{\bar{s}}}, r\})$ is cofinal in $\overline{\nu}$ for some parameter r then $\lambda \cap \widetilde{h}^{n+1}_{\widetilde{M}}(\widetilde{f}(\gamma_{\bar{s}}) \cup \{p', \widetilde{f}(r)\})$ is cofinal in λ (setting $p' = \widetilde{f}(p_{\bar{s}}) = f'^{-1}(p_s)$). Thus λ is $\Sigma_1^{(n)}$-singularised over $\tilde{M}$. Hence $n \geq n_{s'}$. We need to show that λ is not $\Sigma_1^{(n-1)}$-singularised over $\tilde{M}$. Suppose this fails and thus that $\{\alpha | \sup(\lambda \cap \widetilde{h}^n_{\widetilde{M}}(\alpha \cup \{r\})) = \alpha\}$ is bounded in λ, by α' say, for some choice of a parameter $r \in \widetilde{M} = M_{s'}$. By the construction of the pseudo-ultrapower we may assume that r is of the form $\widetilde{f}(\overline{g}_0)(\eta)$ for some good $\mathbf{\Sigma}_1^{(n-1)}(M_{\overline{s}})$ function $\overline{g}_0$ and some $\eta < \lambda$. Define

$$\widetilde{H}(\xi^n, \zeta^n, d) \longleftrightarrow \widetilde{h}^n_{\tilde{M}}(\omega\xi^n \cup \{d\}) \cap \lambda \subseteq \zeta^n; \bar{H}(\xi^n, \zeta^n, d) \longleftrightarrow \widetilde{h}^n_{\bar{s}}(\omega\xi^n \cup \{d\}) \cap \bar{\nu} \subseteq \zeta^n.$$

These are (uniformly defined) $\Pi_1^{(n)}$ relations over their respective structures - in the parameters $\lambda, \bar{\nu}$. By the leastness in the definition of $n_{\bar{s}}$ we have that there are arbitrarily large $\bar{\tau}^n < \bar{\nu}$ with $\widetilde{h}^n_{\bar{s}}(\omega\bar{\tau}^n \cup \{p_{\bar{s}}\}) \cap \bar{\nu} \subsetneq \bar{\tau}^n$; using the soundness of $M_{\bar{s}}$ above $\bar{\nu}$, this implies that for arbitrary $\zeta^n < \bar{\tau}$: $\widetilde{h}^n_{\bar{s}}(i, \overline{\xi}^n, \bar{g}_0(\zeta^n)) \cap \bar{\nu} \subseteq \bar{\tau}^n$. In other words:

$\forall \zeta^n < \bar{\tau}^n \overline{H}(\bar{\tau}^n, \bar{\tau}^n, \bar{g}_0(\zeta^n))$.

As the substituted $\bar{g}_0$ is good $\Sigma_1^{(n-1)}$ we have that this is a $\Pi_1^{(n)}$ statement, and so is preserved upwards to $M_{s'}$:

$\forall \zeta^n < \widetilde{f}(\bar{\tau}^n) \tilde{H}(\widetilde{f}(\bar{\tau}^n), \widetilde{f}(\bar{\tau}^n), \widetilde{f}(\bar{g}_0)(\zeta^n))$.

However as $\widetilde{f} \upharpoonright \bar{\nu}$ is cofinal into λ, we may choose $\bar{\tau}^n$ so that $\widetilde{f}(\bar{\tau}^n) > \max\{\alpha', \eta\}$. This contradicts our definition of α'. Q.E.D.(1)

(2) $p' = p_{s'}$.

By the pseudo-ultrapower construction, we have $\widetilde{M} = \widetilde{h}^{n+1}_{\widetilde{M}}(\lambda \cup p') = \widetilde{h}^{n+1}_{\widetilde{M}}(\tilde{\kappa} \cup p')$ (where $\tilde{\kappa} = \widetilde{f}(\kappa_{\overline{s}})$ if $\kappa_{\overline{s}}$ is defined) and is sound above λ (or $\tilde{\kappa}$). The solidity of $p_{\overline{s}}$ above $\bar{\nu}$ transfers via the $\Sigma_1^{(n)}$-preserving map f' to show that p' is solid above λ (see [17] 3.6.8). Then the minimality of the standard parameter and the definition of $p_{s'}$ shows that $p_{s'} \leq^* p'$. However if $p_{s'} <^* p'$ held, we should have for some $i \in \omega, \vec{\xi}$ that $p' = \widetilde{h}^{n+1}_{\widetilde{M}}(i, \langle \vec{\xi}, p_{s'} \rangle)$, and thus $p_s = \widetilde{h}^{n+1}_{s}(i, \langle f'(\vec{\xi}), f'(p_{s'}) \rangle)$ whence $M_s = \widetilde{h}^{n+1}_{s}(\nu \cup f'(p_{s'}))$. This is a contradiction as $f'(p_{s'}) <^* p_s$. Q.E.D.(2)

(3) If $\widetilde{d} =_{df} \widetilde{f}(d_{\overline{s}})$ then $\widetilde{d} = d_{s'}$.

Proof: This is very similar to Lemma 2.8, using the $\Sigma_1^{(n)}$-preservation properties of $\widetilde{f}$, and is left to the reader. Q.E.D.(3)

(4) If $\widetilde{\alpha} =_{df} \widetilde{f}(\alpha_{\overline{s}})$ then $\widetilde{\alpha} = \alpha_{s'}$.

That $\widetilde{\alpha}$ is sufficiently closed, and hence $\widetilde{\alpha} \leq \alpha_{s'}$, is proven as in (2) of Lemma 3.9 using: $\widetilde{H}(\xi^n, \zeta^n) \longleftrightarrow h_{s'}(\omega\xi^n \cup \{p_\lambda\}) \cap \lambda \subseteq \zeta^n$; $\bar{H}(\xi^n, \zeta^n) \longleftrightarrow h_{\bar{s}}(\omega\xi^n \cup \{p_{\bar{s}}\}) \cap \bar{\nu} \subseteq \zeta^n$. For $\widetilde{\alpha} < \eta^n < \lambda$ we set $\bar{\eta} = f^{-1}``\eta^n$. Then we have $\neg\bar{H}(\bar{\eta}, \bar{\eta})$ (as $\bar{\eta} > \alpha_{\bar{s}}$). Hence for some $i \in \omega$, some $\bar{\xi} < \bar{\eta}$ we have $\eta \leq h_{\bar{s}}(i, \langle \bar{\xi}, p_{\bar{s}} \rangle) < \bar{\nu}$. As $f(\bar{\eta}) \geq \eta^n$ and as $\widetilde{f}$ is $\Sigma_0^{(n)}$-preserving we have $\eta^n \leq h_{s'}(i, \langle \widetilde{f}(\bar{\xi}), p_{s'} \rangle) < \lambda$. Q.E.D.(4)

We have shown enough now to set that $f_0^* = \widetilde{f}$. Q.E.D.(Lemma)

Lemma 3.12 *Suppose $f : \bar{s} \Longrightarrow s$ and $k_s = n_s$. Then $\lambda(f) < \nu_s \longleftrightarrow \rho(f) < \rho_s$.*

Proof: ($\rightarrow$) Suppose $\rho(f) = \rho_s$. Let $\lambda = \lambda(f)$. Then, in the notation of the previous Lemma the map f' is not only $\Sigma_0^{(n)}$ but is cofinal at the n'th level, and thus $\Sigma_1^{(n)}$-preserving. We also have that $f'(\langle \lambda, p_{s'} \rangle) = \langle \nu, p_s \rangle$. This implies that $\nu \cap f'``h_{s'}(\lambda \cup p_{s'}) \subseteq \nu \cap h_s(\lambda \cup p_s) = \lambda$. Were $\lambda < \nu$ this would contradict the fact that $\lambda > \alpha_s$ as the latter is by supposition, in $ran(f)$.

($\leftarrow$) Suppose $\lambda =_{df} \lambda(f) = \nu$. Again in the same notation, suppose $\rho' =_{df} \rho(f) < \rho_s$, It is then easy to see that a good $\Sigma_1^{(n)}$ function, $\overline{F}$ say, singularizing ν definable in some parameter $\overline{q}$ is taken by the $\Sigma_0^{(n)}$-preserving f^* to a good $\Sigma_1^{(n)}(M_s)$ function F in $q = f(\overline{q})$ singularizing λ, with all the parameters of the form x^n needed to define the values $F(\xi)$ in $ran(f^*)$. However if $\rho' =< \rho_s$ we should have that $F \in M_s$. However $\lambda = \nu$! Contradiction! Q.E.D.

The construction of the C_s-sequences attached to $s = (\nu_s, M_s)$ will follow in essence the construction in [15]. The main point is that we can give an estimate to the length of the C_s sequence.

We may state immediately what the C_s-sequences for $s = (\nu_s, M_s) \in S^+$ will be:

Definition 3.13 *Let $s \in S^+$; $C_s^+ =_{df} \{\lambda(f) \mid s\}; C_s =_{df} C_s^+ \backslash \{\nu_s\}$.*

Definition 3.14 *Let $f : \bar{s} \Longrightarrow s$. Then $\beta(f) =_{df} \max\{\beta \leq \alpha_{\nu_s} \mid f \upharpoonright \beta = id \upharpoonright \beta\}$.*

By elementary closure considerations show that $\beta(f)$ is defined, and that $\beta(f) = \alpha_{\nu_s}$ iff $f = id_{\nu_s}$ iff $f(\beta) \not> \beta$. if $\beta(f)$ were singular in $M_{\bar{\nu}}$ using some cofinal function $g : \beta' \longrightarrow \beta$ with $\beta' < \beta$, we should have that then $\beta(f) > \sup(ran(g)) = \beta$. Hence $M_{\bar{\nu}} \models$"$\beta(f)$ is a regular cardinal".

The next lemma lists some properties of $f_{(\gamma,q,s)}$ which were defined at 3.7. Firstly a *minimality property* of $f_{(\gamma,q,s)}$.

Lemma 3.15 *(i) If $\gamma \leq \nu_s$ then $f_{(\gamma,q,s)}$ is the least f such that $f \upharpoonright \gamma = id \upharpoonright \gamma$ with $q, p(s) \in ran(f^*)$, in that if g is any other such with these two properties, (meaning that $g \Longrightarrow s$ with extension g^* so that $\gamma \cup \{q, p(s)\} \subseteq ran(g^*)$) then $g^{-1} f_{(\gamma,q,\nu_s)} \in \mathbb{F}$.*

(ii) $f_{(\gamma,q,s)} = f_{(\beta,q,s)}$ where $\beta = \beta(f_{(\gamma,q,s)})$.

(iii) $f_{(\nu,0,s)} = id_s$;

(iv) Let $f : \bar{s} \Longrightarrow s$ with $\bar{\gamma} \leq \nu_{\bar{s}}, f$ "$\bar{\gamma} \subseteq \gamma \leq \alpha_\nu, \bar{q} \in J_{\bar{s}}, f^*(\bar{q}) = q$, then $ran(f^* f^*_{(\bar{\gamma},\bar{q},\bar{s})}) \subseteq ran(f^*_{(\gamma,q,s)})$.

With (i) this implies: if $\beta(f) \geq \gamma$ then $f f_{(\bar{\gamma},\bar{q},\bar{s})} = f_{(\gamma,q,s)}$.

(v) Set $g = f_{(\gamma,q,s)}$; $\lambda = \lambda(g)$ and $g_0 = red(g)$. Then $q \in J_{s|\lambda}$ and $g_0 = f_{(\gamma,q,s|\lambda)}$.

Proof: *(i) -(iv)* are easy consequences of the definitions. (For *(i)* note this makes sense since we have specified in effect that $ran(g^*) \supseteq ran(f_{(\gamma,q,s)})$.) We establish *(v)*. We know that $g_0 \Longrightarrow s|\lambda$. Set $g_0' = f_{(\gamma,q,s|\lambda)}$ and we shall argue that $g_0 = g_0'$. Let $k = g_0^{-1} g_0'$. The argument of Lemma 3.11 shows that $d(g_0) = d(g)$; as $g_0 \upharpoonright \gamma = id \upharpoonright \gamma$, and $q \in ran(g_0)$ by *(i)* the minimality of $g_0' \Longrightarrow s|\lambda$ implies we have such a k defined. Thus $k \in \mathbb{F}$. But $k \Longrightarrow d(g_0)$ so we conclude, as $d(g_0) = d(g)$, that $gk \in \mathbb{F}$. But $ran((gk)^*) \cap \lambda = ran(g^*) \cap \lambda$. So, using that $gk \upharpoonright \gamma = id \upharpoonright \gamma$, and $q, p(s) \in ran(gk)$, and then *(i)* again, we have $(gk)^{-1} g = k^{-1} \in \mathbb{F}$. Hence $k = id_{d(g_0')}$ and thus $g_0 = g_0'$. Q.E.D.

Our definitions are preserved through $\Longrightarrow$ when a map f is *cofinal*, meaning that $|f|$ is cofinal into $r(f)$:

Lemma 3.16 *Let* $f : \bar{s} \Longrightarrow s$ *with* $\lambda(f) = \nu$. *Set* $\bar{\nu} = \nu_{\bar{s}}$, $\nu = \nu_s$, *and let* $\bar{\gamma} < \bar{\nu}, \gamma = f(\bar{\gamma}), \bar{q} \in J_{\bar{s}}, f(\bar{q}) = q$. *Set*
$\bar{g} = f_{(\bar{\gamma},\bar{q},\bar{s})}$; $g = f_{(\gamma,q,s)}$. *Then*
(i) $\lambda(\bar{g}) < \bar{\nu} \longleftrightarrow \lambda(g) < \nu$;
(ii) If $\lambda(\bar{g}) < \bar{\nu}$ *then* $f(\lambda(\bar{g})) = \lambda(g)$ *and* $f(\beta(\bar{g})) = \beta(g)$.

Proof: Assume $\lambda(\bar{g}) < \bar{\nu}$. Set $\overline{h} = \tilde{h}_{\bar{s}}$, $\lambda' = f(\lambda(\bar{g}))$. The following is $\Pi_1^{(n)M_{\bar{s}}}(\{\lambda(\bar{g}), \bar{\gamma}, p(\bar{s})\})$:
$\forall x^n \forall \xi^n < \bar{\gamma} \forall i < \omega(x^n = \overline{h}(i, \langle \xi^n, \overline{q}, p(\bar{s})\rangle) \wedge x^n < \bar{\nu} \longrightarrow x^n < \lambda(\bar{g}))$; if $\bar{\nu} = On \cap M_{\bar{s}}$ then we drop the conjunct $x^n < \bar{\nu}$. Then
$\forall x^n \forall \xi^n < \gamma \forall i < \omega(x^n = \tilde{h}_s(i, \langle \xi^n, q, p(s)\rangle) \wedge x^n < \nu \longrightarrow x^n < \lambda')$
as f is $\Pi_1^{(n)}$-preserving. Hence $\lambda' \geq \lambda(g)$.

Claim 1: $\lambda' \leq \lambda(g)$.

As $\lambda(\bar{g}) < \bar{\nu}$ we have $\omega\rho(\bar{g}) < \omega\rho_{\bar{s}}$ by Lemma 3.12. Hence if we set $A = A^{n,p_{\bar{s}}\restriction n}$, and $\bar{N} = \langle J^A_{\rho(\bar{g})}, A \cap J_{\rho(\bar{g})}\rangle$ we have that $\overline{N} \in M_{\bar{s}}$ and is an amenable structure, with $\lambda(\bar{g}) = \sup(\bar{\nu} \cap h_{\overline{N}}(\bar{\gamma} \cup \{\bar{q}, p(\bar{s}) \cap \omega\rho_{\bar{s}}\})$.
Applying f^*, and with $N = f(\overline{N})$, we have $\lambda' = \sup(\nu \cap h_N(\gamma \cup \{q, p(s) \cap \omega\rho_\nu\})$.

For amenable structures (such as N) we have a uniform definition of the canonical $\Sigma_1(N)$ Skolem function h_N. From $\langle N, A_N\rangle \subseteq \langle M^n_s, A^n_s\rangle$, we have that $h_N \subseteq h_s$, and thus

$$\lambda' = \sup(\nu \cap h_s(\gamma \cup \{q, p(s) \cap \omega\rho_s\})) = \sup(\nu \cap \tilde{h}_s(\gamma \cup \{q, p(s)\})).$$

Thus $\lambda' \leq \lambda(g)$ and *Claim 1* is finished.

Claim 2 $f(\beta(\bar{g})) = \beta(g)$

Let $\beta = f(\beta(\bar{g}))$; as $\bar{g} = f_{(\beta(\bar{g}),\bar{q},\bar{s})}$ we have $\beta(\bar{g}) \notin ran(\overline{g})$. $\beta = f(\beta(\bar{g})) = f(\sup\{\bar{\delta} < \bar{\nu} \mid \bar{\delta} \subseteq ran(\overline{g})\}) = f(\sup\{\bar{\delta} < \bar{\nu} \mid \bar{\delta} \subseteq h_{\overline{N}}(\bar{\delta} \cup \{\bar{q}, p(\bar{s}) \cap \omega\rho\})\}) = \sup\{\delta < \nu \mid \delta \subseteq h_{\bar{N}}(\delta \cup \{q, p(s) \cap \omega\rho_s\})\}$. By the above $\beta \leq \sup\{\delta < \nu \mid \delta \subseteq h_\nu(\delta \cup \{q, p(s) \cap \omega\rho_s\}) = \beta(g)$. Suppose however $\beta < \beta(g)$. Then in M_s we have:
$\forall \beta^n \leq \beta \exists \xi^n < \gamma \exists i < \omega(\beta^n = \tilde{h}_s(i, \langle \xi, q, p(s)\rangle))$.
However f is $\Sigma_1^{(n)}$-preserving, so this goes down to $M_{\bar{s}}$ as:
$\forall \bar{\beta}^n \leq \beta(\overline{g}) \exists \bar{\xi}^n < \bar{\gamma} \exists i < \omega(\bar{\beta}^n = \tilde{h}_{\bar{s}}(i, \langle \bar{\xi}^n, \bar{q}, p(\bar{s})\rangle))$.
But this, with $\bar{\beta}^n \leq \beta(\bar{g})$ implies $\beta(\bar{g}) \in ran(\bar{g})$ which is a contradiction! This finishes *Claim 2* and *(ii)*. Finally, just note for $(\leftarrow)$ of *(i)* as $\rho(f) = \rho_s$, if $\lambda(g) < \nu$ then by Lemma 3.12 there is $\eta = f(\bar{\eta}) < \rho(f)$ with $\tilde{h}_s(\gamma \cup \{q, p(s)\}) \cap \omega\rho_s \subseteq \eta$. This $\Pi_1^{(n)}$ statement goes down to $M_{\bar{s}}$ as $\tilde{h}_{\bar{s}}(\bar{\gamma} \cup \{\bar{q}, p(\bar{s})\}) \cap \omega\rho_{\bar{s}} \subseteq \bar{\eta}$. Hence $\lambda(\bar{g}) < \lambda$. Q.E.D.

From this point onwards in the proof we are very much following, almost verbatim, the development of [1]: the fine structural arguments specific to our level of mice have all been dealt with, and the rest is very much combinatorial reasoning that is common to whatever model we are trying to define a $\Box$ sequence for.

Definition 3.17 *Let* $s = \langle \nu_s, M_s \rangle \in S^+, q \in J_{\nu_s}$. $B(q,s) =_{df} B^+(q,s) \backslash \{\nu_s\}$ *where*

$$B^+(q,s) =_{df} \{\beta(f_{(\gamma,q,s)}) \mid \gamma \leq \nu_s\}.$$

$B(q,s)$ is thus the set of those $\beta < \nu_s$ so that $\beta = \beta(f)$ where $f = f_{(\beta,q,s)}$.

Lemma 3.18 *Let f abbreviate $f_{(\gamma,q,s)}$. Assume $q \in J_s$. (i) Suppose $\gamma \in B(q,s)^*$. Then $ran(f) = \bigcup_{\beta \in B(q,s) \cap \gamma} ran(f_{(\beta,q,s)})$.*
(ii) Let $\gamma \leq \alpha_s$. Suppose $\bar{s}$ is such that $f : \bar{s} \Longrightarrow s$ with $f(\bar{q}) = q$. Then $\gamma \cap B(q,s) = B(\bar{q}, \bar{s})$.
(iii) Let $\lambda = \lambda(f)$; $f_0 = red(f)$. Then $\gamma \cap B(q, s|\lambda) = \gamma \cap B(q,s)$.

Proof: *(i)* is clear; *(ii)* follows from Lemma 3.15*(iv)*, and *(iii)* from *(ii)* and Lemma 3.15*(v)*. Q.E.D.

Definition 3.19 *Let $s \in S^+, q \in J_s$.*

$$\Lambda^+(q,s) =_{df} \{\lambda(f_{(\gamma,q,s)}) | \gamma \leq \nu_s\}; \Lambda(q,s) =_{df} \Lambda^+(q,s) \backslash \{\nu_s\}.$$

The sets $\Lambda(q,s) \subseteq C_s$ are first approximations to C_s if q is allowed to vary. We first analyse these sets.

Lemma 3.20 *Let $s \in S^+, q \in J_s$. (i) $\Lambda(q,s)$ is closed below ν_s; (ii) $ot(\Lambda(q,s)) \leq \nu_s$; (iii) if $\lambda \in \Lambda(q,s)$ then $q \in J_{s|\lambda}$ and $\Lambda(q, s|\lambda) = \lambda \cap \Lambda(q,s)$.*

Proof: Set $\Lambda = \Lambda(q,s)$. *(i):* Let $\eta \in \Lambda^*$. We claim that $\eta \in \Lambda^+(q,s)$. For each $\lambda \in \Lambda(q,s) \cap \eta$ pick $\beta_\lambda \in B(q,s)$ with $\lambda(f_{(\beta,q,s)}) = \lambda$. Clearly $\lambda \leq \lambda' \longrightarrow \beta_{\lambda'} \leq \beta_\lambda$. Let γ be the supremum of these β_λ. As $B(q,s)$ is closed (by *(i)* of Lemma 3.18), $\lambda(f_{(\gamma,q,s)}) = \sup_\lambda \lambda(f_{(\beta_\lambda,q,s)}) = \eta$.

(ii) is obvious; *(iii)*: Let $\lambda \in \Lambda$,and $g = \lambda(f_{(\gamma,q,s)})$, where we take $\beta = \beta(g)$. Suppose $g : \bar{s} \Longrightarrow s$. Let $g(\overline{q}) = q$ and set $g_0 = red(g)$. Then by Lemma 3.15*(v)* $g_0 = \lambda(f_{(\beta,q,s)}))$. If $\gamma \geq \beta$ then $\lambda = \lambda(f_{(\gamma,q,s|\lambda)}) \leq \lambda(f_{(\gamma,q,s)})$. If $\gamma \leq \beta$ then

$$|f_{(\gamma,q,s|\lambda)})| = |g_0||f_{(\gamma,\bar{q},\bar{s})}| = |g||f_{(\gamma,\bar{q},\bar{s})}| = |f_{(\gamma,q,s)}|$$

where the first equality is justified by Lemma 3.15*(v)*. Q.E.D.

Lemma 3.21 *If $f : \bar{s} \Longrightarrow s$, $\mu = \lambda(f), \overline{q} \in J_{\bar{s}}, f(\bar{q}) = q$, then:*

(i) $\Lambda(\overline{q}, \bar{s}) = \varnothing \longrightarrow \mu \cap \Lambda(q,s) = \varnothing$,

(ii) f "$\Lambda(\overline{q}, \bar{s}) \subseteq \Lambda(q, s|\mu)$,

(iii) If $\overline{\lambda} = \max \Lambda(\overline{q}, \bar{s})$ and $\lambda = f(\bar{\lambda})$ then $\lambda = \max(\mu \cap \Lambda(q,s))$.

Proof: *(i)* By its definition, if $\Lambda(\overline{q},\bar{s}) = \varnothing$ then $f_{(0,\overline{q},\bar{s})}$ is cofinal into $\bar{\nu}$. Hence $ran(ff_{(0,\overline{q},\bar{s})})$ is both cofinal in μ, and contained in $ran(f_{(0,q,s)})$ by Lemma 3.15*(iv)*, thus $\mu \cap \Lambda(q, \nu s) = \varnothing$. This finishes *(i)*. Note that By 3.20*(iii)* $\Lambda(q, s|\mu) = \mu \cap \Lambda(q,s)$. Let $f_0 = red(f)$.

(ii) Let $\overline{\lambda} = \lambda(f_{(\bar{\beta},\overline{q},\bar{s})}) \in \Lambda(\overline{q},\bar{s})$, and let $f(\overline{\beta},\overline{\lambda}) = \beta, \lambda = f_0(\overline{\beta},\overline{\lambda})$. Then $f_0(\lambda(f_{(\bar{\beta},\bar{q},\bar{s})})) = \lambda(f_{(\beta,q,s|\mu)}) \in \Lambda(q,s|\mu)$.

(iii) Let $\overline{\beta} = \sup\{\gamma|\lambda(f_{(\gamma,\bar{q},\bar{s})}) \leq \overline{\lambda}\}$. Then $\lambda(f_{(\bar{\beta},\bar{q},\bar{s})}) = \overline{\lambda}$, and by the assumed maximality of $\overline{\beta}$ we have $\lambda(f_{\bar{\beta}+1,}\bar{q},\bar{s})) = \bar{\nu}$. Set $\beta = f(\bar{\beta}) = f_0(\bar{\beta})$, then by (IV)(2), $\lambda = f_0(\bar{\lambda}) = \lambda(f_{\beta,q,s|\mu})$. However $\lambda(f_{\beta+1,q,s|\mu}) \geq \mu$, since, again by Lemma 3.15*(iv)*, $ran(f_0 f_{(\bar{\beta}+1,}, \bar{q},\bar{s})) \subseteq ran(f_{(\beta+1,q,s|\mu)})$. Thus $\lambda = \max(\Lambda(q,s|\mu)) = \max(\mu \cap \Lambda(q,s))$. Q.E.D.

The p.r. definitions of $\lambda(f)$, $B(q,s)$, $\Lambda(q,s)$, are uniform in the appropriate parameters. If $s = \langle \mu, M_\mu \rangle \in S^+$, then if we may define $F_s = \{f_{(\gamma,q,s|\nu)} | \nu \in S \cap \mu, q \in J_{s|\nu}, \gamma \leq \nu\}$, $E_s = \{\langle \nu, M_{s|\nu}, p(s|\nu), \tilde{h}_{s|\nu}\rangle | \nu \in S \cap \mu\}$, $G_s = \{\langle\langle s|\nu, q\rangle, \Lambda(q, s|\nu\rangle | q \in J_{s|\nu}, \nu \in S \cap \mu\}$. We then have:

Lemma 3.22 *(i) E_s, F_s, G_s are uniformly $\Delta_1(J_s)$ for $s \in S^+$;*
(ii) $\mu' < \mu \Longrightarrow E_{\mu'}, F_{\mu'}, G_{\mu'} \in J_s$.

Lemma 3.23 *Let $f : \bar{s} \Longrightarrow s$ with $\overline{q} \in J_{\bar{s}}, f(\overline{q}) = q$. Then*
(i) If f is cofinal then $|f| : \langle J_{\bar{s}}, \Lambda(\bar{q},\bar{s})\rangle \longrightarrow_{\Sigma_1} \langle J_s, \Lambda(q,s)\rangle$;
(ii) Otherwise: $|f| : \langle J_{\bar{s}}, \Lambda(\bar{q},\bar{s})\rangle \longrightarrow_{\Sigma_0} \langle J_s, \Lambda(q,s)\rangle$

Proof: *(i)* It suffices to show that $|f|(\Lambda(\bar{q},\bar{s}) \cap \overline{\tau}) = \Lambda(q,s) \cap f(\tau)$ for arbitrarily large $\overline{\tau} < \nu_{\bar{s}}$. However this follows from the last lemma and 3.21.

However, if $\bar{\lambda} \in \Lambda(\bar{q},\bar{s})$, then $\Lambda(\bar{q},\bar{s}) \cap \bar{\lambda} = \Lambda(\bar{q},\bar{s}|\bar{\lambda})$ by Lemma 3.20, and by the last lemma, if $f(\bar{\lambda}) = \lambda$, we have $f(\Lambda(\bar{q},\bar{s}|\bar{\lambda})) = \Lambda(q,s|\lambda) = \lambda \cap \Lambda(q,s)$ (with the latter equality by Lemma 3.20 again). If $\Lambda(\bar{q},\bar{s})$ is unbounded in $\nu_{\bar{s}}$, this suffices; if it is empty or bounded, then the Lemma 3.21 takes care of these cases.

For non-cofinal maps *(ii)* we still have, if $\lambda(f) = \mu$, that

$$|f_0| : \langle J_{\bar{s}}, \Lambda(\bar{q},\bar{s})\rangle \longrightarrow_{\Sigma_1} \langle J_{s|\mu}, \Lambda(q,s|\mu)\rangle$$

where $f_0 = red(f)$. But $\Lambda(q,s|\mu) = \mu \cap \Lambda(q,s)$, and $|f_0| = |f|$. Q.E.D.

The C_s sets may be decomposed into a finite sequence of sets of the form $\Lambda(l^i_s, s)$.

Definition 3.24 *Let* $s \in S^+, \eta \leq \nu_s$. $l^i_{\eta s} < \nu_s$ *is defined for* $i < m_{\eta s} \leq \omega$ *by induction on* i :

$$l^0_{\eta s} = 0; \qquad l^{i+1}_{\eta s} \simeq \max(\eta \cap \Lambda(l^i_{\eta s}, s)).$$

We also write l^i *for* $l^i_{\eta s}$ *if the context is clear; also we set* $l^i_s \simeq l^i_{\nu_s s}$; $m_s = m_{\nu_s s}$.

Some facts about this definition may be easily checked:

Fact • $l^i_{\eta s} \leq l^{i+1}_{\eta s}$ $(i < m_{\eta s})$ is monotone • $i > 0 \longrightarrow l^i_{\eta s} \in \eta \cap C_s$.

• Let $l^i_{\eta s}$ be defined, and suppose $l^i_{\eta s} < \mu \leq \eta$. Then $l^i_{\eta s} = l^i_{\mu s}$.

(The last here is by induction on i.)

Lemma 3.25 *Let* $f : \bar{s} \Longrightarrow s$. *(i) If* $\lambda = \lambda(f)$ *then* $l^i_{\lambda s} \simeq f(l^i_{\bar{s}})$;

(ii) let $\overline{\eta} < \nu_{\bar{s}}, f(\bar{\eta}) = \eta$; *then* $l^i_{\eta s} \simeq f(l^i_{\bar{\eta}\bar{s}})$.

Proof *(i)* By induction on i. If $i = 0$ this is trivial. Suppose $i = j + 1$. Then, as inductive hypothesis $l^j_{\lambda s} = f(l^j_{\bar{s}})$, and thus $|f| : \langle J_{\bar{s}}, \Lambda(l^j_{\bar{s}}, \bar{s})\rangle \longrightarrow_{\Sigma_1} \langle J_{s|\lambda}, \Lambda(l^j_{\lambda s}, s|\lambda)\rangle$, by the last lemma, as $|red(f)| = |f|$. However $\Lambda(l^j_{\lambda s}, s|\lambda) = \lambda \cap \Lambda(l^j_{\lambda s}, s)$, by 3.20. Hence: $f(l^i_{\bar{s}}) \simeq f(\max \Lambda(l^j_{\bar{s}}, \bar{s})) \simeq \max(\lambda \cap \Lambda(l^j_{\lambda s}, s)) \simeq l^i_{\lambda s}$ with the middle equality holding by Lemma 3.21(iii). *(ii)* is proved similarly. Q.E.D.

Corollary 3.26 *(i) Let* $f : \bar{s} \Longrightarrow s$ *cofinally. Then* $l^i_s \simeq f(l^i_{\bar{s}})$.

(ii) Let $\lambda \in C_s$. *Then* $l^i_{\lambda s} \simeq l^i_{s|\lambda}$.

Proof *(i)* is immediate. For *(ii)* choose $f : \bar{s} \Longrightarrow s$ with $\lambda = \lambda(f)$, and set $f_0 = red(f)$. Then $l^i_{\lambda s} \simeq f(l^i_{\bar{s}}) \simeq f_0(l^i_{\bar{s}}) \simeq l^i_{s|\lambda}$ with the last equality holding from *(i)*. Q.E.D.

Lemma 3.27 *Let* $\eta \leq \nu, \lambda = \min(C^+_s \backslash \eta)$. *Then* $l^i_s \simeq l^i_{\lambda s} \simeq l^i_{\eta s}$ *(for any* $i < \omega$ *for which either side is defined).*

Proof Induction on i, again $i = 0$ is trivial. Suppose $l^j_s = l^j_{\eta s} = l^j_{\lambda s}$ and $i = j + 1$. Set $l = l^j_{\eta s}$, then we have: $\Lambda(l, s) \cap \eta = \Lambda(l, s) \cap \lambda$, since $\Lambda(l, s) \subseteq C_s$ and $C_s \cap [\eta, \lambda) = \varnothing$. Suppose, without loss of generality that $l^i_{\eta s}$ is defined. Then $l^i_{\eta s} = \max(\eta \cap \Lambda(l, s)) = \max(\lambda \cap \Lambda(l, s)) = l^i_{\lambda s} = l^i_{s|\lambda}$. Q.E.D.

Lemma 3.28 *Let* $j \leq i < m_s$. *Set* $l = l^i_s$. *Then* $l^j_s \in ran(f_{0,l,s})$.

Proof Set $f = f_{(0,l,s)}$. Suppose $f : \bar{s} \Longrightarrow s$, and $\lambda = \lambda(f)$. Then $l^j_{\lambda s} \simeq f(l^j_{\bar{s}})$ by Lemma 3.25*(i)*. But l^j_s exists, and $l^j_s < \lambda \leq \nu_s$. Hence $l^j_s = l^j_{\lambda s} = f(l^j_{\bar{s}})$. Q.E.D.

Importantly the $\langle l^j_{\lambda s}\rangle$ sequences are finite.

Lemma 3.29 *Let $s \in S^+, \eta \leq \nu_s$. Then $m_{\eta s} < \omega$.*

Proof Suppose this fails. Then for some $\eta \leq \nu_s$ we have that $l^i_{\eta s}$ is defined for $i < \omega$. Let $\lambda = \min(C^+_s \backslash \eta)$. then $l^i_{\lambda s} = l^i_{\eta s}$ by Lemma 3.27. Choose $f : \bar{s} \Longrightarrow s$ with $\lambda = \lambda(f)$. Then $l^i_{\lambda s} = l^i_{s|\lambda} = f(l^i_{\bar{s}})$ for $i < \omega$ by Cor. 3.26*(ii)* & Lemma 3.25(i). Taking λ for ν_s, we assume, without loss of generality, that l^i_s is defined for $i < \omega$ for some $s \in S$. We obtain an infinite descending chain of ordinals by showing that as i increases, and with it l^i_s, the maximal β^i that must be contained in the range of any $f :\Longrightarrow s$ together with l^i_s in order for $ran(f)$ to be unbounded in s strictly *decreases.* This is absurd.

Set $l = l^i_s$. Define: $\beta^i = \beta^i_s =_{df} \max\{\beta | \lambda(f_{(\beta,l,s)}) < \nu_s\}$. By the definition of l^{i+1}_s we have that $\lambda(f_{(\beta,l,s)}) < \nu_s \longleftrightarrow \lambda(f_{(\beta,l,s)}) \leq l^{i+1}_s$. Furthermore, by the definition of β^i :

(1) $\lambda(f_{(\beta^i,l,s)}) \leq l^{i+1}_s$;

(2) $\lambda(f_{(\beta^i+1,l,s)}) = \nu_s$.

Claim $\beta^{i+1} < \beta^i$ for $i < \omega$.

Proof Set $f = f_{(\beta^{i+1},l^{i+1},s)}$. Then $\lambda(f) = l^{i+2}$, dropping the subscript ν. Let $f : \bar{s} \Longrightarrow s$. Then $l^j_{\bar{s}}$ exists and $f(l^j_{\bar{s}}) = l^j_{l^{i+1},s} = l^j_s$for $j \leq i+1$ since $l^j < l^{i+1} < \nu_s$ (with the first equality from Lemma 3.25*(i)* and (1), the second from Lemma 3.27).

(3) $\beta^i \geq \beta^{i+1}$.

Proof of (3): Suppose not, then $(\beta^i+1) \cup \{l^i\} \subseteq ran(f)$. Hence $ran(f_{(\beta^i+1,l^i,s)}) \subseteq ran(f)$. hence by (2), $\lambda(f) = \nu_s > l^{i+2}$. Contradiction!

(4) $\beta^i \neq \beta^{i+1}$.

Proof of (4): Suppose not. As β^{i+1} is the first ordinal moved by f we conclude that $f(\beta^i) > \beta^i$. Set $g = f_{(\beta^i,l,s)}, \bar{g} = f_{(\beta^i,\bar{l},\bar{s})}$ where $\bar{l} = l^i_{\bar{s}}$. Then $g = f\bar{g}$, since $f \restriction \beta^i = id, f(\bar{l}) = l (= l^i_{\bar{s}})$. Hence $l^{i+1} = \lambda(g) = \lambda(f\bar{g}) < l^{i+2} = \lambda(f)$. Hence $\lambda(\bar{g}) < \nu_{\bar{s}}$. Now we set: $g' = f_{(f(\beta^i),l,\nu)}$ and $g_0 = f_{(\beta^i,l,s|l^{i+2})}$. If further $f_0 = red(f)$, then we have also $g_0 = f_0\overline{g}$ by 3.15*(iv)*. As $l^{i+1} = \lambda(g) < l^{i+2}$, Lemma 3.16*(ii)* applies and:

$$f(\beta(\bar{g})) = f_0(\beta(\bar{g})) = \beta(g_0) = \beta(g) = \beta^i.$$

Hence $\beta^i \in ran(f)$ which is a contradiction. This proves the *Claim* and hence the Lemma. Q.E.D.

We now set $l_{\eta s} = l^{m-1}_{\eta s}$, where $m = m_{\eta s}$. Again we write l_s for $l_{\nu_s s}$. Notice that then $\Lambda(l_{\eta s}, \nu_s) \cap \eta$ is either unbounded in η or is empty. We first analyze the latter case.

Lemma 3.30 *Suppose $\Lambda(l_{\eta s}, s) \cap \eta = \varnothing$. Set $l = l_{\eta s}$. Then:*

(i) $l = 0 \longrightarrow C_s \cap \eta = \varnothing$,
(ii) $l > 0 \longrightarrow l = \max(C_s \cap \eta)$,
(iii) $\eta \in C_s^+ \longrightarrow \eta = \lambda(f_{(0,l,s)})$.

Proof Set $\rho = \min(C_s^+ \backslash (l+1)$.

(1) $l = l_{\rho s}$.

Proof: Set $n = m_{\eta s} - 1$. Then $l = l^n_{\eta s} < l + 1 < \eta$. Hence (by Fact after 3) $l = l^n_{l+1,s}$. But $\Lambda(l,s) \cap (l+1) = \varnothing$. Hence $l^{n+1}_{l+1,s}$ is undefined and $l = l_{l+1,s}$. Hence $l = l_{\rho,s}$ by Lemma 3.27. Q.E.D.(1)

(2) $\lambda(f_{(0,l,s)}) = \rho$.

Proof: Choose $f : \bar{s} \Longrightarrow s$, with $\lambda(f) = \rho$ witnessing that $\rho \in C_s$. Then, by Lemma 3.25(i), $f(l_{\bar{s}}) = l_{\rho s} = l$. Set $\bar{l} = l_{\bar{s}}$. Now note that we must have that $\lambda(f_{(0,\bar{l},\bar{s})}) = \bar{s}$. For, if this failed then $f(\lambda(f_{(0,\bar{l},\bar{s})})) = \lambda(f_{(0,l,s)}) < \rho$ by Lemma 3.16 and so the latter is in $C_s^+ \cap (l,\rho)$, which is absurd! Then $\lambda(f_{(0,l,s)}) = \lambda(f f_{(0,\bar{l},\bar{s})}) = \lambda(f) = \rho$. Q.E.D.(2)

From (2) and the definition of l as $l_{\eta s}$) it follows that $\rho \geq \eta$. There are thus three alternatives:

If $l = 0$ then *(i)* holds: $\rho = \min(C_s^+ \backslash 1) = \min(C_s^+) \geq \eta$. If $l > 0$ then $l = \max(C_s \cap \eta)$ since $(C_s \cap \eta)\backslash(l+1) \subseteq (C_s \cap \rho)\backslash(l+1) = \varnothing$ and thus we have *(ii)*; finally for *(iii)* if $\eta \in C_s^+ \longrightarrow \eta = \max(C_s^+ \backslash (l+1) = \rho = \lambda(f_{(0,l,s)})$. Q.E.D.

We now get a characterisation of the closed sets C_s^+.

Lemma 3.31 *Let λ be an element or a limit point of C_s^+. Let $l = l_{\lambda s}$. Then there is β such that $\lambda = \lambda(f_{(\beta,l,s)})$. Hence C_s is closed in ν_s, and $C_s^+ = \{\lambda(f_{(\beta,l,s)}) \mid \beta \leq \nu_s, l < \nu_s\}$.*

Proof *Case 1* $\lambda \cap \Lambda(l,s) = \varnothing$

Then $C_s \cap \lambda = \varnothing$ or $l = \max(C_s \cap \lambda)$ by the last lemma. Hence λ is not a limit point of C_s^+. Hence $\lambda \in C_s^+$, and thus $\lambda = \lambda(f_{(0,l,s)})$ by *(iii)* of that lemma.

Case 2 $\lambda \cap \Lambda(l,s)$ is unbounded in λ.

Given $\mu \in \Lambda(l,s) \cap \lambda$, let β_μ be such that $\lambda(f_{(\beta_\mu,l,s)}) = \mu$. Then $\lambda(f_{(\beta,l,s)}) = \lambda$ where $\beta = \sup_\mu \beta_\mu$.

The last sentence is immediate from the previous one. Q.E.D.

We remark that we have just shown that the first conjunct of (i) of Theorem 3.2 holds. We move towards proving the other clauses. The following is (iii).

Lemma 3.32 $\lambda \in C_s \longrightarrow \lambda \cap C_s = C_{s|\lambda}$.

Proof Assume inductively the result proven for all ν' with $\nu' < \nu_s$ and $s|\nu' \in S$, (that is, the lemma is proven with $s|\nu'$ replacing s) and we prove the lemma for ν_s by induction on λ. Let $l = l_{\lambda s}$. Hence by Cor.3.26 $l = l_{s|\lambda}$. By Lemma 3.31 $\lambda \in \Lambda(l, s)$. Set $\Lambda = \lambda \cap \Lambda(l, s)$. Then by Lemma 3.20*(ii)* $\Lambda = \Lambda(l, s|\lambda)$.

Case 1 $\Lambda = \varnothing$.

If $l = 0$, then $C_{s|\lambda} \subseteq \lambda \cap C_s = \varnothing$ (the latter by Lemma 3.30). If $l > 0$, then $l = l_{s|\lambda} = \max(C_{s|\lambda} \cap \lambda) = \max(C_{s|\lambda}) = l_{\lambda s} = \max(\lambda \cap C_s)$ by the same lemma. As $l < \lambda$, we use the inductive hypothesis on λ: $l \cap C_s = C_{s|l} = l \cap C_{s|\lambda}$ where the second equality is the inductive hypothesis taking $\lambda = \nu' < \nu_s$. Hence $C_{s|\lambda} = \lambda \cap C_s = C_{s|l} \cup \{l\}$.

Case 2 Λ is unbounded in λ.

Then $\mu \in \Lambda \longrightarrow \mu \in C_s \cap C_{s|\lambda}$. Hence by the overall inductive hypothesis $C_{s|\mu} = \mu \cap C_{s|\lambda}$ and (as $\mu < \lambda$) $C_{s|\mu} = \mu \cap C_s$. Hence $C_{s|\lambda} = \lambda \cap C_s = \bigcup_{\mu \in \Lambda} C_{s|\mu}$.

Q.E.D.

Now (i) of the Theorem follows easily:

Lemma 3.33 $\sup(C_s) < \nu_s \longrightarrow cf(\nu_s) = \omega$.

Proof Let $l = \sup(C_s) = l_s$. Then $ran(f_{(0,l,\nu_s)})$ is countable, and cofinal in ν_s.

Q.E.D.

Lemma 3.34 *Let* $f : \bar{s} \Longrightarrow s$. *Then* $|f| : \langle J_{\bar{s}}, C_{\bar{s}} \rangle \longrightarrow_{\Sigma_0} \langle J_s, C_s \rangle$.

Proof: It suffices to show that for arbitrarily large $\tau < \nu_{\bar{s}}$ that $|f|(C_{\bar{s}} \cap \tau) = C_s \cap |f|(\tau)$. As usual we continue to write "f" for "$|f|$". Set $l_{\bar{s}} = \bar{l}$.

Case 1 $\Lambda(\bar{l}, \nu_{\bar{s}})$ is unbounded in $C_{\bar{s}}$.

If $\bar{\lambda} \in C_{\bar{s}}$ and $\lambda = f(\bar{\lambda})$ then by 3.21 (and 3.20) $\lambda \in \Lambda(f(\bar{l}), s) \subseteq C_s$. By Lemma 3.22 we have $E_{\bar{s}|\bar{\lambda}} \in J_{\bar{s}}$ and $f(E_{\bar{s}|\overline{\lambda}}) = E_{s|\lambda}$. By Lemma 3.30 $C_{\bar{s}|\bar{\lambda}} = \{\lambda(f_{(0,l,\bar{s})}) < \bar{\lambda} | l < \bar{\lambda}\} \in J_{\bar{s}}$ and is uniformly Σ_0 from $E_{\bar{s}|\bar{\lambda}}$over $J_{\bar{s}}$. Consequently $|f|(C_{\bar{s}|\overline{\lambda}}) = C_{s|\lambda}$, by Σ_1-elementarity of $|f|$. But $C_{\bar{s}|\overline{\lambda}} = \overline{\lambda} \cap C_{\bar{s}|\bar{\nu}}$, $C_{s|\lambda} = \lambda \cap C_s$.

Case 2 $\Lambda(\bar{l}, \bar{\nu}) = \varnothing$.

Let $f(\bar{l}) = l$. Then $l = l_{\lambda\nu}$ where $\lambda = \lambda(f)$. However $\lambda(f_{0,\bar{l},\bar{s}}) = \nu_{\bar{s}}$ by our case hypothesis. Thus $\lambda(f_{(0,l,s)}) = \lambda(ff_{(0,\bar{l},\bar{s})}) = \lambda$. Hence $\Lambda(l, \nu) \cap \lambda = \varnothing$. By Lemma 3.30 we are reduced to the following two subcases:

Case 2.1 $\bar{l} = l = 0$. Then, $C_{\bar{s}} = C_s \cap \lambda = \varnothing$, and so the result is trivial.

Case 2.2 $\bar{l} = \max C_{\bar{s}}$. Then $l > 0$ and thus $l = \max(C_s \cap \lambda)$. Hence for sufficiently large $\bar{\tau} > \bar{l}$ $f(\bar{\tau} \cap C_{\bar{s}}) = f(C_{\bar{s}}) = f(C_{\bar{s}} \cap \bar{l} \cup \{\bar{l}\}) = (C_s \cap l) \cup \{l\} = C_s \cap \lambda = f(\bar{\tau}) \cap C_s$.

Q.E.D.

We now proceed towards calculating the order types of the C_s-sequences. This is done (in a somewhat speedy manner) in [1], but the following comes from [9]. We first generalise the definition of β^i.

Definition 3.35 *For* $\eta \leq \nu_s set : \beta^i_{\eta s} \simeq \max\{\beta | \lambda(f_{(\beta, l^i_{\eta s}, s)}) < \eta\}$.

In very close analogy to the $\beta^i = \beta^i_s$ we have parallel properties for the $\beta^i_{\eta s}$:

1. $\lambda(f_{(\beta, l^i_{\eta s}, s)}) < \nu_s \longleftrightarrow \lambda(f_{(\beta, l^i_{\eta s}, s)}) \leq l^{i+1}_{\eta s}$.

2. $\beta^i_{\eta s}$ is defined if and only if $l^{i+1}_{\eta s}$ is defined - *i.e.* $i + 1 < m_{\eta s}$.

3. $\beta^i_{\eta s} \simeq \beta^i_{\lambda s}$ if $\lambda = \min(C^+_s \backslash \eta)$. $\lambda(f_{(\beta, l^i_{\eta s}, s)}) < \eta \longleftrightarrow \lambda(f_{(\beta, l^i_{\eta s}, s)}) < \lambda$.

4. $\beta^{i+1}_{\eta s} < \beta^i_{\eta s}$ when defined. (By the same argument as for $\beta^{i+1} < \beta^i$.)

Now we set $b_\eta = b_{\eta s} =_{df} \{\beta^i_{\eta s} | i + 1 < m_{\eta s}\}$. For $\eta \in C_s$ we then set $d_\eta = d_{\eta s} =_{df} b_{\eta^+ s}$ where $\eta^+ = \min(C^+_s \backslash (\eta + 1))$. The subscript s on ordinals remains unaltered throughout the rest of the proof so we shall drop it. Then we have:

5. Let $\eta \in C_s$, with $l^i_{\eta^+} < \eta$. Then by induction on i: $l^i_{\eta^+} = l^i_\eta$.
6. Let $\eta \in C_s$, with $l^i_{\eta^+} < \eta$ then:

$$l^{i+1}_{\eta^+} = \eta \text{ if } \eta \in \Lambda(l^i_\eta, s), \text{ and } = l^{i+1}_s \text{ otherwise }.$$

Proof of 6: $l^i_{\eta^+} = l^i_\eta$ by 5. If $\eta \in \Lambda(l^i_\eta, s)$ then η is maximal in this set below η^+. So the first alternative holds. Note that $i \neq m_{\eta s} - 1$ (otherwise by Lemma 3.31 for some β, $\eta = \lambda(f_{(\beta, l^i_{\eta s}, s)}) \in \Lambda(l^i_\eta, s)$). Thus l^{i+1}_η is defined and $l^{i+1}_{\eta^+}$ must equal this.

Lemma 3.36 *Let* $\eta, \mu \in C_s$*, with* $\eta < \mu$*. Then* $d_\eta <^* d_\mu$.

Proof Let $\eta^+ = \min(C^+_s \backslash (\eta + 1))$, $\mu^+ = \min(C^+_s \backslash (\mu + 1))$. Let i be maximal so that $l^i_{\mu^+} = l^i_{\eta^+}$.Then $\beta^j_{\mu^+} = \beta^j_{\eta^+}$ for $j < i$. As $l^i_{\mu^+} \leq \eta < \mu$, we have by 6. above that $l^{i+1}_{\mu^+}$ is defined and $l^{i+1}_{\mu^+} = \mu$ or l^{i+1}_μ. Moreover then $\beta^i_{\mu^+}$ is defined, and by maximality of i, $l^{i+1}_{\eta^+} \neq l^{i+1}_{\mu^+}$.

Claim $l^{i+1}_{\eta^+} < l^{i+1}_{\mu^+}$.

That $l^{i+1}_{\mu^+} < \eta^+$ is ruled out: otherwise $l^{i+1}_{\eta^+} = l^{i+1}_{\mu^+}$ again). So $l^{i+1}_{\eta^+} < \eta^+ \leq l^{i+1}_{\mu^+}$.

Q.E.D. *Claim.*

As $\beta^i_{\mu^+}$ is defined, if $\beta^i_{\eta^+}$is undefined, then we'd be finished. Set $l = l^i_{\mu^+} = l^i_{\eta^+}$. Then $\lambda(f_{(\beta^i_{\eta^+}, l, s)}) = l^{i+1}_{\eta^+}$ and $\lambda(f_{(\beta^i_{\mu^+}, l, s)}) = l^{i+1}_{\mu^+}$. Hence $\beta^i_{\eta^+} < \beta^i_{\mu^+}$ and thus $d_\eta <^* d_\mu$ as required. Q.E.D.

Lemma 3.37 *Let α be p.r. closed so that for some $\alpha_0 < \alpha$ $\lambda(f_{(\alpha_0,0,s)}) = \nu_s$. Then $ot(C_s) < \alpha$.*

Proof: First note that $ot(\langle[\alpha]^{<\omega}, <^*\rangle) = \alpha$. Let $\alpha_0 < \alpha$ be such, with the property that $\lambda(f_{(\alpha_0,0,s)}) = \nu_s$. Then $\{\beta^i_{\eta s} \mid \eta \leq \nu_s, i+1 < m_{\eta s}\} \subseteq \alpha_0$. Thus $ot\langle\{d_\eta \mid \eta \in C_s\}, <^*\rangle \leq ot(\langle[\alpha]^{<\omega}, <^*\rangle) < \alpha$. Thus $ot(C_s) < \alpha$. Q.E.D.

To obtain the requisite $\langle C_\nu \mid \nu \in S\rangle$ for a Global sequence in K, we assign the appropriate level $K_{\beta(\nu)}$ as M_s over which ν is definably singularised. Then $s = \langle \nu, K_{\beta(\nu)}\rangle \in S^+$. Q.E.D.(Global $\Box$)

4 Obtaining Inner Models with measurable cardinals

We assume that we have a Global $\Box$ sequence $\langle C_\nu | \nu \in S\rangle$ in K constructed as in the last section. We have:

Theorem 4.1 *Assume $n > 3$ and $\{\alpha < \omega_n \mid \alpha \in \mathrm{Cof}(\omega_{n-2}) \cap K\text{-}Sing\}$ is, in V, stationary below ω_n. Then*

$$T_n =_{df} \{\beta \in \mathrm{Cof}(\omega_1) \cap \omega_n \mid \mathrm{ot}(C_\beta) \geq \omega_{n-3}\}$$

is stationary in ω_n.

Proof Let $C \subseteq \omega_n$ be an arbitrary closed and unbounded set in ω_n. Take $\gamma \in C^* \cap \mathrm{Cof}(\omega_{n-2})$ with γ a K-singular; in other words with C_γ defined. As $cf(\gamma) > \omega$, C_γ is cub in γ. Then $C \cap C_\gamma$ is closed unbounded in γ of ordertype $\geq \omega_{n-2}$. Take $\beta \in (C \cap C_\gamma)^*$ such that $cf(\beta) = \omega_1$ and $\mathrm{ot}(C \cap C_\gamma \cap \beta) \geq \omega_{n-3}$. By the coherency property 3.1(c), $C_\beta = C_\gamma \cap \beta$. Thus $\beta \in C \cap T_n \neq \emptyset$. $\Box$

Note that $(T_n)_{3<n<\omega}$ as above would be a sequence of sets to which we could apply the MS-principle, if we knew that they were (in V) stationary beneath the relevant $\aleph_n$. This is what the assumption in the above theorem achieves. The following is essentially our main Theorem 1.4.

Theorem 4.2 *If $MS((\aleph_n)_{1<n<\omega}, \omega_1)$ holds then there exists $k < \omega$ so that for all $n > k$, there is D_n, closed and unbounded in ω_n, so that*

$$D_n \cap \mathrm{Cof}(\omega_{n-2}) \subseteq \{\alpha < \omega_n \mid o^K(\alpha) \geq \omega_{n-2}\}.$$

Proof: We suppose not. Then for arbitrarily large $n < \omega$ $S^0_n =_{df} \{\alpha < \omega_n \mid \alpha \in \mathrm{Cof}(\omega_{n-2}) \wedge Sing^K(\alpha)\}$ is stationary in ω_n by appealing to Mitchell's Weak Covering Lemma for K, 1.7.

We shall define a sequence $(S_n)_{1<n<\omega}$ of stationary sets. By Theorem 4.1, for arbitrarily large $n < \omega$, T_n is stationary in ω_n; for such n (which we shall call *relevant*) let $S_n = T_n$; for all other $n > 1$ take $S_n = \mathrm{Cof}(\omega_1) \cap \omega_n$.

Define the first-order structure $\mathfrak{A} = (H_{\omega_{\omega+1}}, K_{\omega_{\omega+1}}, \in, \lhd, \langle f_n \rangle_{n<\omega}, \cdots)$ with a wellordering $\lhd$ of the domain of $\mathfrak{A}$, and the sequence of finitary functions f_n including a complete set of skolem functions for $\mathfrak{A}$. The mutual stationarity property yields some $X \prec H_{\omega_{\omega+1}}$ such that

$$\{\omega_n \mid n \leq \omega\} \subseteq X, \quad \forall n > 2\,(\sup X \cap \omega_n) \in S_n, \text{ and } \omega_2 \subseteq X.$$

(We may assume without loss of generality the latter clause, since a direct argument shows that all ordinals less than, say, ω_k may be added to the hull X without increasing the $\sup X \cap \omega_n$ for any $n > k$. (This goes as follows: let X_0 be a hull that satisfies the MS property and the first two requirements above: $\{\omega_n \mid n \leq \omega\} \subseteq X_0$, $\forall n > 2\,(\sup X_0 \cap \omega_n) \in S_n$. We now consider the enlarged hull of $X =_{df} X_0 \cup \omega_k$ in $\mathfrak{A}$. Let $n > k$. Consider for each m, and each $\vec{x} \in [X_0]^p$, $\sup\{f_m(\vec{\xi}, \vec{x}) \cap \omega_n \mid \vec{\xi} \in [\omega_k]^l\}$ where we have assumed that f_m is $l+p$-ary. But this is a supremum definable in X_0 from f_n, $\vec{x}$, ω_n, and ω_k. Hence it is less than $\sup(X_0 \cap \omega_n)$. By choice of $\langle f_n \rangle$, every $y \in X$ is of the form $f_m(\vec{\xi}, \vec{x})$ so this suffices.)

Let $\pi : (\bar{H}, \overline{K}, \in, \ldots) \cong (X, K \cap X, \in, \ldots)$, be the inverse of the transitive collapse, and $\beta_n =_{df} \pi^{-1}(\omega_n)$ for $n \leq \omega$. For each $2 < n < \omega : \beta_n > \aleph_2$ and $cof(\beta_n) = \omega_1$. Let $\beta_n^* =_{df} \sup(\pi``\beta_n)$. We now consider the coiteration of K with $\overline{K}$. Let $((M_i, \pi_{i,j}, \nu_i)_{i \leq j \leq \theta}\ ,\ (N_i, \sigma_{i,j}, \nu_i)_{i \leq j \leq \theta})$ be the resulting coiteration of $(K, \overline{K})$.

(1) *The first ultrapower on the K side is taken after a truncation. In fact $\pi_{0,1} : M_0^* \longrightarrow M_1$, where $\pi \neq id$ and M_0^* is a proper initial segment of K.*

Proof: Note that β_3 is a cardinal of $\overline{H}$, whilst $K_{\beta_3} = \bar{K}_{\beta_3}$ as $X \cap \omega_3$ is transitive. However $cf(\beta_3) = \omega_1$ and is thus not a true cardinal of K (by the Covering Lemma for K). Hence the first action of the comparison will be a truncation on the K side to a structure M_0^* in which β_3 is a cardinal., and thence the ultrapower map $\pi_{0,1}$ as stated. Q.E.D.(1)

(2) *On the $\overline{K}$ side of the coiteration all the maps $\sigma_{i,j}$ are the identity: $\forall i \leq \theta N_i = \overline{K}$.*

Proof: Suppose this is false for a contradiction and let ιbe the least index where an ultrapower of $N_\iota = \overline{K}$ is taken by some $E^N_{\nu_\iota}$ with critical point κ_ι. On the K side let ζ be least so that $\mathcal{P}(\kappa_\iota) \cap M_\iota \| \zeta = \mathcal{P}(\kappa_\iota) \cap N_\iota$. Let us set M^* to be this $M_\iota \| \zeta$. (Note that no truncation is taken in the comparison on the $\overline{K}$ side.). Note that since M_0^* was a truncate of K, we have that thereafter each M_i is sound above κ_i and that $\omega\rho_{M_i}^{n+1} \leq \kappa_i < \omega\rho_{M_i}^n$ for some $n = n(i)$.

As $E_{\nu_\iota}^N$ is a total measure on $N_\iota = \overline{K}$ we have that $\tilde{E} =_{df} E_{\pi(\nu_\iota)}^K = \pi(E_{\nu_\iota}^N)$ is a full measure in K with critical point $\tilde{\kappa} =_{df} \pi(\kappa_\iota)$.

We apply the measure $E_{\nu_\iota}^N$ to M^* itself and form the fine structural ultrapower $\widetilde{M} = Ult^*(M^*, E_{\nu_\iota}^N)$ with map $t : M^* \longrightarrow \widetilde{M}$. Note that by the weak amenability of $E_{\nu_\iota}^N$, $\widetilde{M} \cap \mathcal{P}(\kappa_\iota) = M^* \cap \mathcal{P}(\kappa_\iota)$, and that t is $\Sigma_0^{(n)}$ and cofinal.

We should like to compare M^* with $\widetilde{M}$ but for this we need the following Claim.

Claim 1 $\widetilde{M}$ is normally iterable above κ_ι.

Proof: First note:

(i) M^* and $\overline{K}$ agree up to ν_ι, hence if E_ι is the extender sequence on M_ι we have that $\pi \restriction J_{\nu_\iota}^{E_\iota} : J_{\nu_\iota}^{E_\iota} \longrightarrow J_{\widetilde{\nu}}^{E^K}$ cofinally for $\widetilde{\nu} =_{df} \sup \pi``\nu_\iota$.

(ii) $cf(\nu_\iota) > \omega$ and hence we have a canonical extension $\pi^* \supseteq \pi \restriction J_{\nu_\iota}^{E_\iota}$ with $\pi^* : M^* \longrightarrow M'$ with $\omega\rho_{M^*}^{n+1} \leq \kappa_\iota < \omega\rho_{M^*}^n$ implying that $\omega\rho_{M'}^{n+1} \leq \tilde{\kappa} < \omega\rho_{M'}^n$, M' sound above $\tilde{\kappa}$, and π^* $\Sigma_0^{(n)}$ preserving.

Proof: Note that $cf(\nu_\iota) = cf(\kappa_\iota^{+M_\iota}) > \omega$ since otherwise we have that $\kappa_\iota^{+M_\iota}$ is a $\overline{K}$ cardinal, which H will think, by Weak Covering, has uncountable cofinality equal to some β_i. As $cf(\beta_i) = \omega_1$ it would be a contradiction to have $cf(\nu_\iota) = \omega$. By the definition of ζ we have that $\omega\rho_{M^*}^{n+1} \leq \kappa_\iota < \omega\rho_{M^*}^n$ for some n and that M^* is sound above κ_ι. Consequently ν_ι is definably singularized over M^* and we have the right conditions to apply 2.11 with the other properties mentioned following from that. Q.E.D.(ii)

(iii) $\tilde{\kappa}$ a K-cardinal, $\omega\rho_{M'}^{n+1} \leq \tilde{\kappa}$, and M' sound above κ' imply that M' is an initial segment of K.

Applying the full measure $\widetilde{E}$ yields $\sigma : K \longrightarrow_{\tilde{E}} \widetilde{K}$. Let $\widetilde{M'} = \sigma(M')$, and this is also an initial segment of $\widetilde{K}$. As $\pi^* \supseteq \pi \restriction J_{\nu_\iota}^{E_\iota}$ we have:

(iv) $X \in E_{\nu_\iota}^N \longleftrightarrow \pi^*(X) = \pi(X) \in \widetilde{E}$.

Defining $\mathbb{D}(M^*, E_{\nu_\iota}^N)$ the term model for the ultrapower we have:

(v) (a) The map $d([f]) = \sigma \circ \pi^*(f)(\tilde{\kappa})$ is a structure preserving map $d : \mathbb{D}(M^*, E_{\nu_\iota}^N) \longrightarrow \widetilde{M'}$. (a) The map $k : \widetilde{M} \longrightarrow \widetilde{M'}$ is $\Sigma_0^{(n)}$-preserving with $k(\kappa_\iota) = \tilde{\kappa}$.

Proof: This is a standard computation for (a), and for (b) note that $\omega\rho_{\widetilde{M'}}^{n+1} \leq \sigma(\tilde{\kappa}) < \omega\rho_{\widetilde{M'}}^n$ by (ii) and the elementarity of σ. Q.E.D.(v)

By (v)(b) since $\widetilde{M'}$ is normally iterable above $\tilde{\kappa}$ $\widetilde{M}$ will be normally iterable above κ_ι, as required. Q.E.D. *Claim 1*.

Claim 2 $E^N_{\nu_\iota} = E^{M^*}_{\nu_\iota}$.

Proof: Since M^* and $\widetilde{M}$ agree up to ν_ι the coiteration of these two is above κ_ι. By *Claim 1* this coiteration is successful with iterations $i : \widetilde{M} \longrightarrow \widetilde{M}_\theta$ and $j : M^* \longrightarrow M^*_\theta$ say.

(vi) The iteration i of $\widetilde{M}$ is above $(\kappa^+_\iota)^{\widetilde{M}} = (\kappa^+_\iota)^{M^*}$.

Proof: $\overline{K}, M^*, \widetilde{M}$all agree up to ν_ι and forming $\widetilde{W} = Ult(J^{E^{M^*}}_{\nu_\iota}, E^N_{\nu_\iota})$ we see therefore that it is an initial segment of $\widetilde{M}$. From coherence of our extender sequences we know that

$$E^{\widetilde{M}} \restriction \nu_\iota = E^{\overline{K}} \restriction \nu_\iota = E^{M^*} \restriction \nu_\iota \text{ and } E^{\widetilde{M}}_{\nu_\iota} = \emptyset = E^{\widetilde{W}}_{\nu_\iota}.$$

By the initial segment property of extender sequences we have that there are no further extenders on the $E^{\widetilde{M}}$ sequence with critical point κ_ι. Hence all critical points used in forming the iteration map i are above $(\kappa^+_\iota)^{\widetilde{M}}$. Q.E.D. (vi)

The rest of the argument is fairly standard.

(vii) $\widetilde{M}_\theta = M^*_\theta$.

Proof: Let $A \in \Sigma^{(n)}_1(M^*)$ in p_{M^*} be such that $A \cap \kappa_\iota \notin M^*$, and then note that $A \cap \kappa_\iota \notin \tilde{M}$ as they agree about subsets of κ_ι. Hence if the iteration j is simple, then M^*_θ is not a proper initial segment of $\widetilde{M}_\theta$. But if j is non-simple then we reach the same conclusion as no proper initial segment of $\widetilde{M}_\theta$ can be unsound. Hence $\widetilde{M}_\theta$ is an initial segment of M^*_θ. But again we cannot have that it is a proper initial segment, since using the $\Sigma^{(n)}_0$ preservation property of t we'd have $A \cap \kappa_i$ in M^*_θ a contradiction as before. Q.E.D. (vii)

(viii) (i) $\omega\rho^{n+1}_{\widetilde{M}} = \omega\rho^{n+1}_{M^*} = \omega\rho^{n+1}_{M^*_\theta}$.

(ii) If $p = p_{M^*} \backslash \omega\rho^{n+1}_{M^*}$ then $i \circ t(p) = p_{M^*_\theta, n+1}$.

(iii) t is Σ^*-preserving.

Proof: These are standard arguments from the proof of solidity for mice - *cf.* [17] p153-4. In (ii) one first sees that $i \circ t(p) \in P^{n+1}_{M^*_\theta}$; a solidity argument on witnesses $W^{\alpha,p}_{M^*}$ shows that in fact $i \circ t(p) = p_{M^*_\theta, n+1}$.

(ix) $j \restriction \kappa = id = i \circ t \restriction \kappa$; however $crit(j) = \kappa_\iota$.

Proof: As the first clause is immediate, we argue that $j(\kappa_\iota) > \kappa_\iota$. As j is an iteration map $j(p) \in P^{n+1}_{M^*_\theta}$. By the Dodd-Jensen Lemma (*cf.* [17] Theorem 4.3.9) $j(p) \leq^* i \circ t(p)$, and hence by (8)(ii) we have $j(p) = i \circ t(p)$. By the soundness of above κ_ι we have that $\kappa = \widetilde{h}^{n+1}_{M^*}(i, \xi, p)$ for some $i < \omega$, some $\xi < \kappa_\iota$. Hence $j(\kappa_\iota) = \widetilde{h}^{n+1}_{M^*_\theta}(i, \xi, j(p))$. As $j(p) = i \circ t(p)$ we have

$j(\kappa_\iota) = i \circ t(\widetilde{h}^{n+1}_{M^*}(i, \xi, p)) = i \circ t(\kappa_\iota) > \kappa_\iota.$ Q.E.D. (ix)

Hence κ_ι is the first point moved by j and thus some measure $E^{M^*}_\gamma$ is applied as the first ultrapower on the M^* side of the coiteration with $crit(E^{M^*}_\gamma) = \kappa_\iota$ and γ least with $E^{M^*}_\gamma \neq E^{\widetilde{M}}_\gamma$. As $E^{M^*} \restriction \nu_\iota = E^{\widetilde{M}} \restriction \nu_\iota$ and (see the proof of (vi)) $E^{\widetilde{M}}_{\nu_\iota} = \emptyset$ we must have $\gamma = \nu_\iota$ here. But then

$$X \in E^{M^*}_{\nu_\iota} \longleftrightarrow \kappa_\iota \in j(X) \longleftrightarrow \kappa_\iota \in i \circ t(X) \longleftrightarrow \kappa_\iota \in t(X) \longleftrightarrow X \in E^N_{\nu_\iota}.$$

Hence $E^N_{\nu_\iota} = E^{M^*}_{\nu_\iota}$ which is our *Claim 2*. Q.E.D. (2)

At the θ'th stage therefore, M_θ is an end extension of $\overline{K}$. For $n < \omega$, let i_n be the least stage i where $\kappa_i \geq \beta_n$ if such an i exists, otherwise set $i_n = \theta$. Let $k_0 < \omega$ be the least k such that any truncations performed on the K iteration have been performed before stage i_k. We may also assume that from this point i_{k_0} on then, that the least $m > 0$ with $\omega\rho^m_{M_\iota} < \kappa_\iota$ is fixed for all $\iota \geq i_{k_0}$; for this m then, we set $\rho = \omega\rho^m_{M_\iota}$ for any $\iota \geq i_{k_0}$, and we shall have that any M_i is sound above κ_i for $\iota \geq i_{k_0}$, and thus that $M_\iota = \widetilde{h}^m_{M_\iota}(\kappa_\iota \cup \{p_{M_\iota}\})$. Further by choice of m note that for $n > k_0$, $\rho^{m-1}_{M_{i_n}} > \kappa_{i_n} \geq \beta_n$. As we have in the iteration that $\pi_{i,j}(\langle d_{M_i}, p_{M_i}\rangle) = \langle d_{M_j}, p_{M_j}\rangle$, and parameters are finite sequences, we may further assume that k_0 has also been chosen sufficiently large so that for any $n \geq k_0$: (i) $d_{M_{i_n}}, p_{M_{i_n}} \cap [\beta_{n-1}, \beta_n) = \emptyset$, (ii) k_0 is itself relevant.

(3) *Suppose $\langle \kappa_i | i < i_n\rangle$ is unbounded in β_n, where n is relevant. Then for no $i_0 < i_n$ do we have $\pi_{i_0,i}(\kappa_{i_0}) = \kappa_i$ for unboundedly many $\kappa_i < \kappa_{i_n}$.*

Proof: If the conclusion failed then we should have $\pi_{i,j}(\kappa_i) = \kappa_j$ for an ω_1-sub-sequence of the sequence of critical points $\langle \kappa_i | i < i_n\rangle$; let us choose such an ω_1-sub-sequence, and call the set of its elements $\overline{D}$ with the choice of $\overline{D}$ ensuring that $\overline{D}$ is closed below β_n. These are all inaccessible in $\overline{K}$. Applying π, if we set $D = \pi``\overline{D}$, then we have that D is a cub set of order type ω_1 below β^*_n of K-inaccessibles. Note that π is continuous on $\overline{D}$ since $\overline{H}$ is correct about whether any ordinal α has cofinality ω or not, since all the $\beta_n (n < \omega)$ have uncountable cofinality; hence, easily, if κ_λ is a limit point of $\overline{D}$, then it has cofinality ω in $\overline{H}$. If $f : \omega \longrightarrow \kappa_\lambda$ is the least function in $\overline{H}$ witnessing this, then $\pi(\kappa_\lambda) = \pi(\sup\{f``\omega\}) = \sup\{\pi(f(n)) | n \in \omega\}$. (We are using here that the MS property is formulated using *all* the $\aleph_n$'s and not just a subsequence.) But n is relevant so β^*_n is singular in K, but of uncountable cofinality. Thus the closed $C_{\beta^*_n}$ sequence of K of K-singular ordinals, has non-empty intersection with D, which is absurd.

(4) *If* $n \geq k_0$ *is relevant then* β_n *is* $\Sigma_1^{(m)}$ *singularised over* M_{i_n} *and the latter is sound above* β_n.

Proof: The last conjunct follows from the definition of $i_n : M_{i_n}$ is sound above $\widetilde{\kappa} =_{df} \sup\langle\kappa_i | i < i_n\rangle$ Divide into the two cases of $\tilde{\kappa} < \beta_n$ or $\tilde{\kappa} = \beta_n$. In the first case then $M_{i_n} = \widetilde{h}^m_{M_{i_n}}(\widetilde{\kappa} \cup \{p_{M_{i_n}}\})$ and hence β_n is so singularised over M_{i_n}; in the second case take $\delta < \beta_n$, $\delta \geq \omega\rho_{M_{i_n}}$. Take i minimal such that $\kappa_i \in [\delta, \beta_n)$. Then $M_i = \widetilde{h}^m_{M_i}(\delta \cup \{p_{M_i}\})$ and in particular $\kappa_i \in \widetilde{h}^m_{M_i}(\delta \cup \{p_{M_i}\})$. By (3) take $\gamma < \beta_n$ such that whenever $\kappa_j \in (\gamma, \beta_n)$ then $\kappa_j \neq \pi_{ij}(\kappa_i)$, and take some index j such that $\kappa_j \in (\gamma, \beta_n)$. By elementarity, $\pi_{ij}(\kappa_i) \in \widetilde{h}^m_{M_j}(\delta \cup \{p_{M_j}\})$. Since $\kappa_j > \pi_{ij}(\kappa_i)$, the point $\pi_{ij}(\kappa_i)$ is not moved in the further iteration past stage j, and so $\pi_{ij}(\kappa_i) \in \widetilde{h}^m_{M_{i_n}}(\delta \cup \{p_{M_{i_n}}\})$. We thus have that

$$(*) \quad \rho > \alpha_{\beta_n} =_{df} \max\{\alpha \mid \sup(\widetilde{h}^m_{M_{i_n}}(\alpha \cup \{p_{M_{i_n}}\}) \cap \beta_n) = \alpha\}.$$

But now there must be some $\gamma < \beta_n$ with $\sup(\widetilde{h}^m_{M_{i_n}}(\gamma \cup \{p_{M_{i_n}}\}) \cap \beta_n) = \beta_n$. Because if this failed we could choose a sequence

$$\gamma_0 = \rho, \gamma_{i+1} = \sup(\widetilde{h}^m_{M_{i_n}}(\gamma_i \cup \{p_{M_{i_n}}\}) \cap \beta_n) < \beta_n, \text{ and take } \gamma = \sup_i \gamma_i.$$

As $cf(\beta_n) > \omega$, $\gamma < \beta_n$. However we have then that

$$\gamma = sup(\widetilde{h}^m_{M_{i_n}}(\gamma \cup \{p_{M_{i_n}}\}) \cap \beta_n) < \beta_n$$

and simultaneously $\gamma > \alpha_{\beta_n}$. Contradiction! (4) is thus proven. Q.E.D.(4)

(5) *If* n *is relevant, then in the notation of* (4), *if* $m > 1$ *then for no smaller* $m' < m$ *is* β_n $\Sigma_1^{(m'-1)}$ *singularised over* M_{i_n}.

Proof: Just note that as $\rho^{m'-1}_{M_{i_n}} \geq \rho^{m-1}_{M_{i_n}} > \beta_n$, any purported $\Sigma_1^{(m'-1)}$-singularisation over M_{i_n} yields a cofinalising function *in* M_{i_n}. This is absurd as β_n is regular in M_{i_n}. Q.E.D.(5)

We thus have, by (4), that for relevant n, $s_n =_{df} \langle\beta_n, M_{i_n}\rangle \in S^+$. We therefore have C_{s_n} sequences associated to such s_n as in the Global $\Box$ proof of the previous section.

(6) *For relevant* $n \geq k_0$, *we have* $ot(C_{s_n}) \leq \widetilde{\beta}$ *where* $\widetilde{\beta}$ *is the least p.r.closed ordinal above* β_{k_0}.

Proof: Set $i = i_{k_0}$; $j = i_n$. Then by the usual property of ultrapowers $\pi_{i,j}$"$\omega\rho^{m-1}_{M_i}$ is cofinal in $\omega\rho^{m-1}_{M_j}$.

Set $s = s_{k_0}$ and let $\delta = \delta_{k_0}$ be least such that $\lambda(f_{(\delta,0,s)}) = \beta_{k_0}(= \nu_s)$ where $f_{(\delta,0,s)} \Longrightarrow s$. Then $\delta < \beta_{k_0}$. Let $Y =_{df} \pi_{i,j}$"$ran(f^*_{(\delta,0,s)})$. As $ran(f^*_{(\delta,0,s)})$ is a $\Sigma_1^{(m-1)}$ hull in $M_s(= M_i)$ we have that Y is a $\Sigma_1^{(m-1)}$ hull in $M_j(= M_{s_n})$. We note that α_s, α_{s_n} (in the sense of Definition 3.5 f)) are below ρ by (∗) of (4). Consequently if we define $\widetilde{Y} =_{df} ran(f^*_{(\beta_{k_0}+1,0,s_n)})$ then $\widetilde{Y}$ is a $\Sigma_1^{(m-1)}$ hull of M_j. However $\widetilde{Y} \supseteq Y$, as $\pi_{i,j}(p_s, d_s) = p_{s_n}, d_{s_n}$, $\pi_{i,j}$ is $\Sigma_1^{(m-1)}$-preserving, and $\pi_{i,j} \restriction \beta_{k_0} = id$. (We need Lemma 2.8 here on the preservation of the d_s parameters under iteration.)

By choice of δ and Lemma 3.12 $\rho(f_{(\delta,0,s)}) = \omega\rho_s$. Hence Y is cofinal in $\omega\rho_{s_n}$. However then $\widetilde{Y}$ is also so cofinal. That is $\rho(f_{(\beta_{k_0}+1,0,s_n)}) = \omega\rho_{s_n}$ which again by Lemma 3.12 implies $\lambda(f_{(\beta_{k_0}+1,0,s_n)}) = \nu_{s_n} = \beta_n$. By Lemma 3.37 this implies $ot(C_{s_n}) \leq \widetilde{\beta}$. Q.E.D.(6)

For relevant n we form the "lift-up" map $\pi^*_n : M_{i_n} \longrightarrow M^*_n$ which extends $\pi \restriction (\overline{K}|\beta^+_n)$ (where $\beta^+_n = (\beta^+_n)^{\overline{K}}$). We obtain the structure M^*_n and the map π^*_n as a pseudo-ultrapower.

(7)(a) *For relevant n, π^*_n is $\Sigma_1^{(m-1)}$-preserving, and β^*_n is $\Sigma_1^{(m-1)}$-singularised over M^*_n; further, if $m > 1$, then for no smaller $m' < m$, is β^*_n is $\Sigma_1^{(m'-1)}$-singularised over M^*_n.*

(b) *M^*_n is normally iterable above β^*_n.*

Proof : (a) The Pseudo-Ultrapower Theorem 2.11 (with $k = m-1$) shows the right degree of elementarity of π^*_n, *i.e.* that it is $\Sigma_0^{(m-1)}$ preserving. It further states that the map is cofinal and thus $\Sigma_1^{(m-1)}$-preserving, and that it yields that β^*_n is $\Sigma_1^{(m-1)}$-singularised over M^*_n, whilst β^*_n is $\Sigma_1^{(m'-1)}$-regular over M^*_n for any $m' < m$ (if $m > 1$). For (b) this is a standard argument about canonical extensions defined from pseudo-ultrapowers using the fact that $cf(\beta_n) = cf((\beta_n)^{+M_n}) > \omega$. (Note $cf(\beta^+_n)^{\overline{K}} = \omega_1$, either because $(\beta^+_n)^{\overline{K}} = (\beta^+_n)^{\overline{H}} = \beta_{n+1}$, or otherwise by applying the Weak Covering Lemma inside $\overline{H} : \overline{H} \models$"$cf(\beta^+_n)^{\overline{K}} = \beta_n$" , and β_n of course has cofinality ω_1.) See [17] Lemma 5.6.5. Q.E.D.(7)

(8) *M^*_n is an initial segment of K.*

Proof: Note that by construction $M^*_n \restriction \beta^*_n = K \restriction \beta^*_n$. By 7(i) $\rho^m_{M^*_n} \leq \beta^*_n$; again the pseudo-ultrapower construction shows M^*_n is sound above β^*_n and hence is coded by a $\Sigma_1^{(m-1)}(M^*_n)$ subset of β^*_n, A say. An elementary iteration and comparison argument shows that, when K is compared with M^*_n, to models N_η, M^*_η

then A is $\Sigma_1^{(m-1)}$ definable over N_η, and thus is in K itself. As M_n^* is a mouse in K, its soundness above β_n^* implies that after any supposedly necessary coiteration, we must have $N_\eta = M_\eta^*$ and hence $core(N_1) = core(N_\eta) = core(M_\eta^*) = M_n^*$. Hence M_n^* is an initial segment of K. Q.E.D.(8)

(9)(a) $s_n^* = \langle \beta_n^*, M_n^* \rangle \in S^+$;

(b) *M_n^* is the assigned K-singularising structure for β_n^*; hence in K, $C_{\beta_n^*}$ is defined over M_n^*, that is $C_{\beta_n^*} =_{df} C_{s_n^*}$.*

Proof: For (a), by (7)(a) M_n^* singularises appropriately, it is sound above β_n^*, and by (8) it is a mouse. For (b) we have shown that M_n^* is an initial segment of K, and thus conforms to the definition of the segment chosen to define the canonical C-sequence associated to β_n^* in K. Q.E.D.(9)

We thus conclude:

(10) *For relevant $n \geq k_0$ $ot(C_{\beta_n^*}) \leq \pi(\widetilde{\beta}) < \pi(\beta_{k_0+1}) = \omega_{k_0+1}$.*

Proof: By (6) $ot(C_{s_n}) \leq \widetilde{\beta}$ because $\tilde{h}_{s_n}(\beta_{k_0}+1, p(s_n))$ is cofinal in $\omega\rho_{s_n} = \omega\rho_{M_{in}}^{m-1}$. Set $\beta' = \pi_n^*(\beta_{k_0}+1)$.By the $\Sigma_1^{(m-1)}$-elementarity of π_n^* we shall have that $\pi_n^{*}``\tilde{h}_{s_n}(\beta_{k_0}+1, p(s_n)) \subset \tilde{h}_{s_n^*}(\beta', p(s_n^*))$. As $\pi_n^* \restriction \omega\rho_{s_n}$ is cofinal into $\omega\rho_{s_n^*}$, we deduce that $\rho(f_{(\beta',0,s_n^*)}) = \omega\rho_{s_n^*}$. By Lemma 3.12 this ensures that $\lambda(f_{(\beta',0,s_n^*)}) = \nu_{s_n^*} = \beta_n^*$. This in turn implies by Lemma 3.37, that $ot(C_{s_n^*})$ is less than the least p.r. closed ordinal greater than β'. However $\pi_n^* \restriction \beta_n^+$ extends $\pi \restriction \beta_n^+$, and thus this ordinal is $\pi(\widetilde{\beta})$. The final inequality is clear.

Now (10) yields the final contradiction, as for relevant n, S_n was chosen to consist of points β where $ot(C_\beta) \geq \omega_{n-3}$, whereas (10) establishes an ultimate bound on such order types of ω_{k_0+1}. Q.E.D.(Theorem 4.2)

We finally remark that the Corollary 1.5 is immediate: after shifting our attention to cardinals above $\aleph_k$ we still use the same hypothesis concerning sufficient singular ordinals in K in order to establish the stationarity of the T_n now contained in $\mathrm{Cof}(\omega_k)$. We take $\omega_k \subseteq X$ and now the analogues of the ordinals β_n have cofinality ω_k; H is correct about the cofinality of any ordinal whose V-cofinality is less than ω_k. The proof of (3) now shows that there is no closed ω_k subsequence of critical points κ_i unbounded in such a β_n, as the map π is now continuous at points of cofinality less than ω_k. Hence we can deduce (4) that the iterates are indeed singularizing structures for the β_n as required.

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