Abstract

We give a proof of a theorem of Jensen and Zeman on the existence of Global $\square$ in the Core Model below a measurable cardinal $\kappa$ of Mitchell order ($\omega_M(\kappa)$) equal to $\kappa^{++}$, and use it to prove the following theorem on mutual stationarity at the $\aleph_n$.

Let $\omega_1$ denote the first uncountable cardinal of $V$ and set $\text{Cof}(\omega_1)$ to be the class of ordinals of cofinality $\omega_1$.

**Theorem:** If every sequence $(S_n)_{n<\omega}$ of stationary sets $S_n \subseteq \text{Cof}(\omega_1) \cap \aleph_{n+2}$, is mutually stationary, then there is an inner model with infinitely many inaccessibles $(\kappa_n)_{n<\omega}$ so that for every $m$ the class of measurables $\lambda$ with $\omega_M(\lambda) \geq \kappa_m$ is stationary in $\kappa_n$ for all $n > m$. In particular, there is such a model in which for all sufficiently large $m < \omega$ the class of measurables $\lambda$ with $\omega_M(\lambda) \geq \omega_m$ is, in $V$, stationary below $\aleph_{m+2}$.

1 Introduction

This paper extends previous investigations into the nature of mutual stationarity, a concept introduced by M. Foreman and M. Magidor [6] in order to transfer some combinatorial aspects of stationary subsets of regular cardinals to singular cardinals. They made particular use of this in investigating the non-saturation of the non-stationary ideals of the form $\mathcal{P}_\kappa(\lambda)$.

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Our purpose here is to establish that the mutual stationarity property at $\aleph_\omega$ (or more precisely at the sequence of the first $\omega$-many uncountable cardinals, $\langle \aleph_n \mid 0 < n < \omega \rangle$), is a large cardinal property, that is, it entails the consistency of strong axioms of infinity which concern measurable cardinals. The definition of mutual stationarity is more general than this however:

**Definition 1.1** Let $(\kappa_n)_{n<\omega}$ be a strictly increasing sequence of regular cardinals $\geq \aleph_2$ with $\kappa_\omega = \sup_{n<\omega} \kappa_n$. A sequence $(S_n)_{n<\omega}$ is called mutually stationary in $(\kappa_n)_{n<\omega}$ if every first-order structure $A$ of countable type with $\kappa_\omega \subseteq A$ has an elementary substructure $B < A$ such that

$$\forall n < \omega \sup |B| \cap \kappa_n \in S_n.$$  

M. Foreman and M. Magidor, together with J. Cummings further investigated the status of such sequences in [2]. Note that if $(S_n)_{n<\omega}$ is mutually stationary in $(\kappa_n)_{n<\omega}$ then each $S_n \cap \kappa_n$ is stationary in $\kappa_n$. In the following we shall denote the class $\{\xi \in \text{Ord} \mid c\text{f}(\xi) = \lambda\}$ by $\text{Cof}_\lambda$.

**Definition 1.2** Let $(\kappa_n)_{n<\omega}$ be a strictly increasing sequence of regular cardinals and $\lambda < \kappa_0$, $\lambda$ regular. The mutual stationarity property $\text{MS}((\kappa_n)_{n<\omega}, \lambda)$ is the statement: if $(S_n)_{n<\omega}$ is a sequence of stationary sets $S_n \subseteq \text{Cof}_\lambda \cap \kappa_n$, then $(S_n)_{n<\omega}$ is mutually stationary in $(\kappa_n)_{n<\omega}$.

M. Foreman and M. Magidor [6] proved the following two theorems:

**Theorem.** For $(\kappa_n)_{n<\omega}$ be any strictly increasing sequence of uncountable regular cardinals:

(i) $\text{MS}((\kappa_n)_{n<\omega}, \omega)$ holds.

(ii) $\text{MS}((\kappa_n)_{n<\omega}, \omega_1)$ implies $V \neq L$.

This did not yet say that $\text{MS}$ was a large cardinal property. That it was is the left to right direction of the following equivalence, proven in [12]:

**Theorem 1.3** The theories $\text{ZFC} + \exists (\kappa_n)_{n<\omega} \text{MS}((\kappa_n)_{n<\omega}, \omega_1)$ and $\text{ZFC} + \exists \kappa (\kappa$ measurable) are equiconsistent.

The implication from right to left was first proven by Cummings, Foreman, and Magidor [3] via Prikry forcing. They proved more than this: they showed that a tail of the Prikry generic sequence satisfies $\text{MS}((\kappa_n)_{n<\omega}, \lambda)$ for any $\lambda < \kappa_0$ (or indeed the mutual stationarity of any sequence of stationary sets $S_n \subseteq \kappa_n$ irrespective of the cofinalities of the ordinals in the $S_n$). This is essentially
obtained by utilising the fact that a tail of the Prikry generic sequence remains coherently Ramsey in the generic extension. The forward direction was proven in [12] using the core model $K$ of A. J. Dodd and R. B. Jensen (see [5]). The deduction of the existence of $0^+$ from $MS(\langle \kappa_n \rangle_{n<\omega}, \omega_1)$ was done in detail, and the extension to proving the existence of the inner model with a measurable was sketched, using the hyperfine structure of S. Friedman and the first author ([7]).

The purpose of this paper is to give a full account of the interaction of the proof of global $\square$ with the MS property, (insofar as we are able) thus filling in the details of the above argument, but significantly strengthening the result to obtain models with many measures of high Mitchell order, in the case $\langle \kappa_n \rangle_{n<\omega}$ consists of consecutive sequences of cardinals mentioned in the abstract:

**Theorem 1.4** If $MS(\langle \aleph_n \rangle_{0<n<\omega}, \omega_1)$ holds then there is an inner model, $K$, and there is $2 < k < \omega$ so that for any $n$ with $k < n < \omega$ each $\aleph_n$ is a Mahlo limit (in $V$) of ordinals $\kappa$ which are, in $K$, measurable of Mitchell order $o_M(\kappa) = \omega_{n-2}$. In fact, for such $\aleph_n$ the ordinals $\alpha \in Col(\omega_{n-2})$ which are singular in $K$ are, in $V$, non-stationary below $\aleph_n$.

One might wonder whether increasing the cofinality of the independently chosen stationary sets might yield increased Mitchell order. Well, perhaps, but seemingly not by our methods. The following is a corollary to the proof of the above theorem.

**Corollary 1.5** Let $m$ be fixed, $1 \leq m < \omega$. Then if $MS(\langle \aleph_{n+m} \rangle_{0<n<\omega}, \omega_m)$ holds, exactly the same conclusion as that of Theorem 1.4 may be drawn.

The methods here seem just short of allowing us to conclude that there is an inner model with a measurable $\kappa$ with Mitchell order of $\kappa$ equal to $\kappa$ (“$o_M(\kappa) = \kappa$”).

It is important in the above statement that we use all the alephs below $\aleph_\omega$ (from some point on) since the first author has shown that omitting a cardinal above each one for which we wish to consider arbitrary stationary sets, has a much weaker consistency strength, (see [11]).

**Theorem 1.6** The theories $ZFC + MS(\langle \aleph_{2n+1} \rangle_{n<\omega}, \omega_1)$ and $ZFC + \exists \kappa (\kappa$ a measurable cardinal) are equiconsistent.

The model $K$ in Theorem 1.4 can be taken to be the core model built using measures (partial or full) only on its constructing extender sequence.
We shall need the following formulation of the Weak Covering Lemma due to W. Mitchell (cf. [13])

**Theorem 1.7 (Weak Covering Lemma)** Assume there is no inner model with a measurable cardinal \( \kappa \) with \( o_M(\kappa) = \kappa^{++} \). Let \( \alpha \) be regular in \( K \) with \( \omega_1 \leq \gamma = cf(\alpha) < card(\alpha) \). Then in \( K \) we have \( o_M(\alpha) \geq \gamma \).

We shall assume a development of the fine structure of such a core model \( K \), as can be found in M. Zeman [17]. \( K \) is thus a model of the form \( L[E] \) with \( E \) a sequence of partial or full extenders in the manner of Zeman’s book. However no such extender requires any generator beyond that of its critical point. We shall need to consider the proof of the existence of global square \( \square \) in such a model. This is known to hold, cf [10]. The fine structural notation we shall adopt is that of the book (which is also that of the paper cited). The indexing of extenders will be the Friedman-Jensen indexing whereby an extender is placed on the \( E \) sequence of a hierarchy at precisely the successor cardinal of the image of the critical point by that extender. Again this is following [10].

Jensen and Zeman’s method of proof for global \( \square \) is to define a “smooth category” of structures and maps from which it is known that a global \( \square \) sequence can be derived. This latter derivation is purely combinatorial and so requires no inspection of the fine structure of the original model. The burden of their proof is the construction of the smooth category itself. However that construction does not yield an explicit computation for the order types of the various \( C_\nu \) sequences. (It is the latter derivation that does that). For our proof we need to have a construction of global \( \square \) where we can see (i) what those order types will be and how they are arrived at; and (ii) that order types for certain \( C_\nu \)-like sequences will (on a tail) not be prolonged by iterations of the mouse from which they are defined. We give a proof of Global \( \square \) *ab initio* directly without going through the smooth category. This is done in Section 3. In section 2 we give some fine structural lemmas that form the hard work of Jensen and Zeman’s account in [10] which establish the right forms of parameter preservation and appropriate condensation lemmata. We merely quote these as Condensation Lemmas (I) and (II). However in order to prove that the order types of \( C_\nu \) sequences are not prolonged by iterations of the structure over which they are defined we need to prove the preservation of the \( d \)-parameters of [10]. This is at Lemma 2.8. The analysis of the Condensation Lemmata apart, we try to keep the rest of the proof as self-contained as possible. The proofs of Lemmas 3.9 and 3.11 in particular repeat the proofs of [10] 4.3 and 4.5. These are key lemmata on the relationships between singularising structures and the maps between them, and are, in the \( \Sigma^1 \) terminology, the successors to [1] Lemmas 6.15 and 6.18. From Definition
3.17 onwards this is an account very much following that of [1] (and which will
be in the forthcoming [15]), but modestly dressed in the appropriate $J_s$ mouse
notation. In Section 4 we see how to use features of this proof to get the main
Theorem 1.4: the reader who is completely familiar with the □ proof and wants
to discover the ideas in the application to mutual stationarity may wish to go
straight there.

2 Fine structural prerequisites

For an acceptable $J$-structure $M$ we assume familiarity with the notions of the
uniformly defined $\Sigma_1$-Skolem function for $M$, $h_M$, and of the class of parameter
sequences $\Gamma_M$, and the parameter sets $P^n_M$, $P^*_M$, $R^n_M$, $R^*_M$, and $R^*_M$. We shall
write $\rho_M$ as usual for the $\Sigma_1$-projectum of $M$. Similarly we shall write for the
$n+1$'st projectum $\rho^{n+1}_M = \min\{\rho_{M^p} \mid p \in \Gamma^0_M\}$. We may assume that parameters
are finite sets of ordinals. This applies as well to the $n$'th-standard parameter
and the standard parameter denoted here $p^n_M, p_M$ respectively for a structure $M$ as above. We wellorder $[On]^{<\omega}$ by $u <^* v \iff \max(u\Delta v) \in v$. For $X \subseteq \Ord$ a set, we write $ot(X)$ for its order type, and by $X^*$ we mean the set of limit points of $X$. Our discussion of fine structure is entirely in the language of $\Sigma_k^{(n)}$ relations
due to Jensen (for which see [17] or [15]). Boldface relations such as $\Sigma_1^{(n)}(M)$
denote those definable using parameters (in this case from $M$.)

**Definition 2.1 ($\Sigma_1^{(n)}$-Skolem Functions)** Let $M$ be an acceptable $J$-structure,
and let $p \in \Gamma^0_M$.
(i) $h^n_M = h_{M^p}$;
(ii) $\tilde{h}^n_M(w^n, x^0) = g_0(g_1 \cdot \cdot \cdot g_{n-1}((w^n)_0, ((w^n)_1, x^0(n-1)) \cdot \cdot \cdot x^0(0))$, where
for $i \leq n$

$$g_i((j, y^{i+1}), p) = h_{M^p, i}(j, (y^{i+1}, p(i))).$$

Then $g_i$ is uniformly lightface $\Sigma_1^{0_i}(M)$ in the variables shown. Thus $\tilde{h}^n_M$
is $\Sigma_1^{(n-1)}$ uniformly over all $M$. The $\Sigma_1$ hull of a set $X \subseteq M^{n-p}$ we shall denote
by $h^n_M(X)$ (and is thus the set $\{h^n_M(i, x) \mid i \in \omega, x \in X\}$). Note that
$\tilde{h}^n_M((j, y^0), p(0)) = g_0((j, (y, p(0))) = h_M(j, (y, p(0)))$. If $p \in R^n_M$ then every
$x \in M$ is of the form $h^n_M(z, p)$ for some $z \in H^n_M$. We may similarly form hulls
using $\tilde{h}^n_M$: again if $X \subseteq M^{n-p}$ say, and $q \in M$ then the $\Sigma_1^{(n-1)}$ hull of $X \cup \{q\}$ is
the set $\{h^n_M(x, q) \mid x \in X\}$ (we again may write $\tilde{h}^n_M(X \cup \{q\}$ for this hull here).
The following states some of these facts and are easy to establish (see [17] p.29):
Lemma 2.2 Let $M$ be acceptable, and $p \in R^n_M$:
(i) If $\omega p^n_M \in M$ and $p \in R^n_M$ then $\hat{\nu}^n_M$ is a good, uniformly defined, $\Sigma_1^{(n-1)}(M)$ function mapping $\omega p^n_M$ onto $M$.
(ii) (a) every $A \subseteq H^n_M$ which is $\Sigma_1^{(n)}(M)$ is $\Sigma_1(M^{n,p})$;
    (b) $\rho^{p+1}_M = \rho_{M^{n,p}}$

Lemma 2.3 Let $M$ be an acceptable $J$-structure. Then (i) $\Sigma^*(M) \subseteq \Sigma_\omega(M)$.
(ii) Let $p \in R^n_M$. Then $\Sigma^*(M) = \Sigma_\omega(M)$.

Lemma 2.4 Let $M, M'$ be acceptable structures, and suppose $\pi : M \rightarrow M'$ is $\Sigma_1^{(n)}$-preserving, and is such that $\pi \upharpoonright \omega p_M^{n+1} = id$ and $\text{ran}(\pi) \cap P_M^* \neq \emptyset$. Then $\pi$ is $\Sigma^*$-preserving.

Proof: This is [17] 1.11.2. Q.E.D.

Recall that a premouse $M$ is sound above $\nu$ if $\omega p^n_M \leq \nu$ means that $\hat{\nu}^n_M(\nu \cup \{p_M\}) = |M|$. We also say that it is $k$-sound if it is sound above $\omega p^k_M$.

In the next lemma there are various concepts that we shall quickly gloss: $o^N(\kappa)$ is the extender order of $\kappa$ in the hierarchies under consideration (and roughly corresponds to Mitchell order of measures); the hat over a premouse, as in $\hat{N}$, indicates the expansion of the premouse structure $N$, to which extenders are usually applied (as, for example, when coiterations of premice are formed). The premice then act as bookkeeping premice for the indices that are being used, whilst the actual extenders are applied to these hatted expansions. We simply follow the conventions of [10] and we ignore the differences between these structures. The reader worried about these details may consult [10] Sect. 2 or Ch. 8 of [17].

Theorem 2.5 (Condensation Lemma I) (cf [10] 2.1). Suppose there is no inner model for $o_M(\kappa) = \kappa^{++}$. Let $N$ be a premouse, $M$ a mouse and $\sigma : \hat{N} \rightarrow o^{(n)}_0$ $\hat{M}$ with $\sigma \upharpoonright \omega p^{n+1}_N = id$. Then $N$ is a mouse; moreover if $N$ is sound above $\nu = \text{crit}(\sigma)$ then one of the following holds:
(i) $N$ is the core of $M$ above $\nu$ and $\sigma$ is the iteration map, which is the corresponding core map;
(ii) $N$ is a proper initial segment of $M$;
(iii) For some $\kappa < \nu$ $\beta \equiv_M o^N(\kappa) \geq \nu$, and if $\zeta < \kappa^{+M}$ is maximal so that $E^M_\beta$ measures all subsets of $\kappa = \text{crit}(E^M_\beta)$ which lie in $M\upharpoonright\zeta$, then $N$ is an initial segment of $M^+$ where $\pi : \hat{M}\upharpoonright\zeta \rightarrow M^+$.

In order to have sufficient further condensation Jensen and Zeman require certain parameters associated with canonical witness structures to be in the range
of their maps. We only remind the reader of this definition here, and refer to the paper for a full discussion of their significance.

**Definition 2.6** Suppose $\gamma \in p^n_M$ and let $\sigma^M_\gamma$ be the canonical witness map corresponding to $W^M_\gamma$: if $\sup(\text{ran}(\sigma^M_\gamma)) \cap \omega p^n_M < \omega p^n_M$ we set $\delta(\gamma) = \sup(\text{ran}(\sigma^M_\gamma)) \cap \omega p^n_M$.

We set $\bar{p}^n_M = \{ \gamma \in p^n_M | \delta(\gamma) \text{ is defined} \}$, and appropriately $\bar{p}^n_M = \bigcup_n \bar{p}^n_M$.

Further set $d^n_M = \{ \delta(\gamma) | \gamma \in \bar{p}^n_M, k \leq n \}$ etc.

This finite (possibly empty) set $d^n_M$ then collects together all those sups of those canonical witness maps $\sigma_\gamma$ just for those $\gamma$ for which the map is non-cofinal at the $k$th levels for $k \leq n$. This allows for an appropriate form of the Condensation Lemma for hierarchies below mice $M$ with any $\kappa$ with $(o_M(\kappa) = \kappa^{++} )^M$. The following is again taken from [10].

**Theorem 2.7** (Condensation Lemma II) (cf [10] 3.1) Suppose there is no inner model for $o_M(\kappa) = \kappa^{++}$. Let $N, M$ be mice and $\sigma : N \rightarrow M$. Suppose further that $\sigma(\bar{\alpha}) = \alpha, \sigma(\bar{p}) = p_M \setminus \alpha$, and

$(i)$ $\omega p^n_M + 1 \leq \alpha < \omega p^n_M$ and $M$ is sound above $\alpha$;

$(ii)$ $d^n_M \subseteq \text{ran}(\sigma)$.

Then $\bar{p} = p_N \setminus \bar{\alpha} : N$ is sound above $\bar{\alpha}, \sigma(\bar{p}_N \setminus \bar{\alpha}) = p_M \setminus \alpha$ and $\sigma(\delta^N(\gamma)) = \delta^M(\sigma(\gamma))$ whenever $\gamma \in \bar{p}_N \setminus \bar{\alpha}$.

We shall need a lemma on preservation of these $d$-parameters under normal iterations. We prove this here.

**Lemma 2.8** Suppose $\pi : M \rightarrow N$ is a normal iteration of $M$. Then $\pi(d^n_M) = d^n_N$.

**Proof:** This would be by induction on the length of the iteration, but we simply do a one step ultrapower by an extender $E$ with critical point $\kappa$ and the reader can form the general and direct limit argument herself. This does not follow quite immediately from Condensation Lemma II as the latter assumes $d^n_N$ is in the range of the map. We know that $\pi(p_M) = p_N$. We may express $\bar{\nu} = \{ \nu \in p_M | \text{ If } \nu \in [\omega p^{k+1}_M, \omega p^k_M] \text{ then the canonical witness map is non-cofinal into } \omega p^k_M \}$.

And: $d^n_M = \{ \delta^M(\nu) | \nu \in \bar{\nu} \}$.

Then if $\delta(\nu) \in d^n_M$ with $\nu \in [\omega p^{k+1}_M, \omega p^k_M]$ we have as in [10]:

$(\ast) \ \forall \xi^k \exists \zeta^k (\xi^k < \nu \land \zeta^k = \bar{\nu}_{M}(\xi^k, p_M \setminus (\nu + 1)) \rightarrow \zeta^k \leq \delta(\nu))$.

This is $\Pi_1(k)$ in $\nu, \delta(\nu), \text{ and } p_M$. If $\text{crit}(E) = \kappa \in [\omega p^{n+1}_M, \omega p^n_M]$ then $\pi$ is $\Sigma^n_0$ preserving and cofinal into $\omega p^n_M$, hence $\Sigma^n_1$-preserving. If $k < n$ then it is
$\Sigma_2^{(k)}$ preserving. Consequently wherever $\nu$ lies we have from these preservation properties:

1. $\nu \in \bar{p}_M \longrightarrow \pi(\nu) \in \bar{p}_N \land \pi(\delta^M(\nu)) \geq \delta^N(\pi(\nu))$.

We want equality here. For $k < n \Sigma_2^{(k)}$ preservation suffices to guarantee this: if

$$\exists \delta^k \in \pi(\delta^M(\nu))[\forall \xi^k \forall \zeta^k (\xi^k < \pi(\nu) \land \zeta^k = \bar{h}_{N}^{-1}(\xi^k, \pi(p_M) \setminus (\pi(\nu) + 1)) \longrightarrow \zeta^k \leq \delta^k]$$

then this would go down to $M$ and give a contradiction. For $k = n$ we can reason as follows. Suppose $\delta = \pi(f)(\xi) = \delta^N(\pi(\nu)) < \pi(\delta^M(\nu))$. As at (\ast):

$$\forall \xi^n \forall \zeta^n (\xi^n < \pi(\nu) \land \zeta^n = \bar{h}_n^{-1}(\xi^n, p_N \setminus (\pi(\nu) + 1)) \longrightarrow \zeta^n \leq f(\alpha))$$

was of $E$-measure 1. But on a set of measure 1 $f(\alpha) < c_{\delta^M(\nu)}(\alpha) = \delta^M(\nu)$ so this contradicts the definition of $\delta^M(\nu)$. Hence

$$\forall \xi^n \forall \zeta^n (\xi^n < \pi(\nu) \land \zeta^n = \bar{h}_n^{-1}(\xi^n, p_N \setminus (\pi(\nu) + 1)) \longrightarrow \zeta^n \leq f(\alpha))$$

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$$\forall \xi^n \forall \zeta^n (\xi^n < \pi(\nu) \land \zeta^n = \bar{h}_n^{-1}(\xi^n, p_N \setminus (\pi(\nu) + 1)) \longrightarrow \zeta^n \leq f(\alpha))$$

was of $E$-measure 1. But on a set of measure 1 $f(\alpha) < c_{\delta^M(\nu)}(\alpha) = \delta^M(\nu)$ so this contradicts the definition of $\delta^M(\nu)$. Hence

$$\forall \xi^n \forall \zeta^n (\xi^n < \pi(\nu) \land \zeta^n = \bar{h}_n^{-1}(\xi^n, p_N \setminus (\pi(\nu) + 1)) \longrightarrow \zeta^n \leq f(\alpha))$$

was of $E$-measure 1. But on a set of measure 1 $f(\alpha) < c_{\delta^M(\nu)}(\alpha) = \delta^M(\nu)$ so this contradicts the definition of $\delta^M(\nu)$.

By using a Los Lemma we should have that:

$$\{ \alpha < \nu \forall \xi^n \forall \zeta^n (\xi^n < \pi(\nu) \land \zeta^n = \bar{h}_n^{-1}(\xi^n, p_N \setminus (\pi(\nu) + 1)) \longrightarrow \zeta^n \leq f(\alpha)) \}$$

was of $E$-measure 1. But on a set of measure 1 $f(\alpha) < c_{\delta^M(\nu)}(\alpha) = \delta^M(\nu)$ so this contradicts the definition of $\delta^M(\nu)$. Hence

$$\forall \xi^n \forall \zeta^n (\xi^n < \pi(\nu) \land \zeta^n = \bar{h}_n^{-1}(\xi^n, p_N \setminus (\pi(\nu) + 1)) \longrightarrow \zeta^n \leq f(\alpha))$$

was of $E$-measure 1. But on a set of measure 1 $f(\alpha) < c_{\delta^M(\nu)}(\alpha) = \delta^M(\nu)$ so this contradicts the definition of $\delta^M(\nu)$. Hence

$$\forall \xi^n \forall \zeta^n (\xi^n < \pi(\nu) \land \zeta^n = \bar{h}_n^{-1}(\xi^n, p_N \setminus (\pi(\nu) + 1)) \longrightarrow \zeta^n \leq f(\alpha))$$

was of $E$-measure 1. But on a set of measure 1 $f(\alpha) < c_{\delta^M(\nu)}(\alpha) = \delta^M(\nu)$ so this contradicts the definition of $\delta^M(\nu)$.

Now note:

$$\forall \xi^n \forall \zeta^n (\xi^n < \pi(\nu) \land \zeta^n = \bar{h}_n^{-1}(\xi^n, p_N \setminus (\pi(\nu) + 1)) \longrightarrow \zeta^n \leq f(\alpha))$$

was of $E$-measure 1. But on a set of measure 1 $f(\alpha) < c_{\delta^M(\nu)}(\alpha) = \delta^M(\nu)$ so this contradicts the definition of $\delta^M(\nu)$. Hence

$$\forall \xi^n \forall \zeta^n (\xi^n < \pi(\nu) \land \zeta^n = \bar{h}_n^{-1}(\xi^n, p_N \setminus (\pi(\nu) + 1)) \longrightarrow \zeta^n \leq f(\alpha))$$

was of $E$-measure 1. But on a set of measure 1 $f(\alpha) < c_{\delta^M(\nu)}(\alpha) = \delta^M(\nu)$ so this contradicts the definition of $\delta^M(\nu)$.

We shall also be assuming familiarity with the construction of fine-structural pseudo-ultrapowers, for which see [17] or [15]. We shall be using various “lift-up” lemmas. These are in the following form.

**Definition 2.9** Let $M$ be an acceptable $J$-structure, and $\nu \in M$ a regular cardinal of $M$. Then $k(M, \nu)$ is defined to be the least $k$ (if it exists) so that there is a good $\Sigma_2^{(k)}$-definable function whose domain is a bounded subset of $\nu$ and whose range is unbounded in $\nu$. (Such a function is said to singularize $\nu$ and we say that $\nu$ is $\Sigma_2^{(k)}(M)$-singularized over $M$.)

**Definition 2.10** Let $\bar{M}, \bar{\nu}, k = k(M, \bar{\nu})$ be as above with $\bar{\nu}$ regular in $\bar{M}$. Let $\bar{Q} = af J^M_{\bar{\nu}}$. Define $\Gamma^k_{\bar{M}, \bar{\nu}} = af$

$$\{ f \mid dom(f) \subseteq \bar{Q} \land ran(f) \subseteq M \} \land (n < k \land f \in \Sigma_1^{(n)}(M) \land \omega_{\bar{\nu}}^{n+1} \geq \bar{\nu}) \}.$$

**Theorem 2.11** (Pseudo-Ultrapower Theorem) Let $\bar{M}$ be an acceptable $J$-structure, $\bar{\nu}$ a regular cardinal of $\bar{M}$ but with $k = k(M, \bar{\nu})$ defined. Let $\bar{Q} = af J^M_{\bar{\nu}}$. Then there is a map $\bar{\sigma} : \bar{M} \longrightarrow \Sigma_0 M$ (the “canonical k-extension” of $\sigma : \bar{Q} \longrightarrow \Sigma_0 Q$) satisfying:

$$8$$
(i) $\bar{\sigma}$ is $Q$-preserving, $M$ is an acceptable end extension of $Q$, and $M = \{\bar{\sigma}(f)(u) \mid u \in \sigma(\text{dom}(f)), f \in \Gamma\}$.

(ii) $\bar{\sigma}$ is $\Sigma^0_2(n)$ preserving for $n < k$;

b) $\rho_k = \rho^k_{\bar{\sigma}}$, and $\bar{\sigma}$ is $\Sigma^0_k$ preserving and cofinal (thus $\Sigma^0_1$-preserving);

(iii) $\bar{\sigma}(\nu) = \nu$ and the latter is regular in $M$;

(iv) $k = k(M, \nu)$: $k$ is least so that there is a $\Sigma^0_k(M)$ map cofinalising $\nu$.

Lemma 2.12 (Interpolation Lemma) Suppose $M = (\mathcal{A}, B)$ is a structure such that $\bar{\sigma}$ is regular in $M$, but with $k = k(M, \nu)$ defined. Suppose further that $f: M \rightarrow \Sigma^0_1(M) = (\mathcal{A}, B)$. Let $\bar{\sigma} = \sup f \bar{\sigma}$. Then there is a structure $\tilde{M} = (\mathcal{A}, B)$, a map $\tilde{f}: M \rightarrow \tilde{M}$ with $\tilde{f} \supseteq f \restriction J^A_{\beta}$ and $\tilde{f}, \Sigma^0_0$-cofinal (and hence $\Sigma^0_1$-preserving), and a unique $f^\prime: \tilde{M} \rightarrow \Sigma^0_0(M)$, with $f = f^\prime \circ \tilde{f}$ and $f^\prime \restriction \nu = id \restriction \nu$.

3 Global $\square$ in $K$.

Definition 3.1 Let $\text{Sing} = \{\beta \in \text{Ord} \mid \lim(\beta) \wedge \text{cf}(\beta) < \beta\}$ be the class of singular limit ordinals. Global $\square$ is the assertion: there is a system $(C_\beta)_{\beta \in \text{Sing}}$ satisfying:

(a) $C_\beta$ is a closed cofinal subset of $\beta$;

(b) $\text{ot}(C_\beta) < \beta$;

(c) if $\beta$ is a limit point of $C_\beta$ then $\beta \in \text{Sing}$ and $C_\beta = C_\beta \cap \beta$.

Jensen [8] introduced the principle and proved it held in $L$. The format of the proof we shall follow will be that of [1], which was a proof in the setting of generalised $L[A]$ hierarchies suitable for use Jensen’s Coding Theorem. The second author [14] proved in the Dodd-Jensen core model $K$. The first proof of $\square$ which used the Baldwin-Mitchell arrangement of the $L[E]$ hierarchy, was for Jensen’s model for $K$ with measures of order zero, and was by Wylie [16]. From the order types of the square sequences $C_\xi$ we shall define stationary sets $S_\alpha$ to which we shall apply the $MS$-principle.

We consider how a global $\square$ sequence can be derived in $K$. For clarity we shall assume there is no inner model with a measure of Mitchell order $\sigma_M(\kappa) = \kappa^{++}$ (see [10]) and that $K$ is built under this assumption. We assume for the rest of this section $V = K$. Jensen and Zeman prove (more than) the following.

Theorem 3.2 Let $S$ be the class of all singular limit ordinals that are limits of admissibles. There is a uniformly definable class $(C_\nu | \nu \in S)$ so that:
(i) $C_\nu$ is a set of ordinals closed below $\nu$ and, if $\text{cf}(\nu) > \omega$, then it is also unbounded;

(ii) $\text{ot}(C_\nu) < \nu$;

(iii) $\mathfrak{p} \in C_\nu \longrightarrow \mathfrak{p} \in S \land C_\nu = \mathfrak{p} \cap C_\nu$.

It is well known that once one has a global sequence defined on the singular ordinals of some cub class that contains all singular cardinals and is cub beneath each successor cardinal, then this can be filled out to a global sequence on all singular ordinals to satisfy Definition 3.1. Hence proving the above theorem suffices. As $V = K = L[E]$ for $E = E^K$ a fixed sequence of extenders, if $\nu$ is a singular ordinal, then there will be a least level $J^E_\beta(\nu)$ of the $J^E$-hierarchy over which $\nu$ is definably singularised, i.e. there will be a partial $\Sigma_\omega(\beta(\nu))$ definable good function mapping a subset of some $\gamma$ cofinally into $\nu$. This level of the hierarchy $J^E_\beta(\nu)$ will also be our main singularising structure $M_\nu$. Note that by Lemma 2.3 and the soundness of the $K$ hierarchy, any such function is also $\Sigma_1^{(1)}(\beta(\nu))$ for some $n$. That is, $k(\nu, J^E_\beta(\nu))$ in the sense of Definition 2.9 is defined.

However there will be many other mice over which ordinals are singularized and we must consider these in addition.

**Definition 3.3** $S^+$ is the class of $s = \langle \nu_s, M_s \rangle$ where

(a) $\nu_s \in \text{Sing}$;

(b) $M_s$ is a mouse satisfying the following:

(i) $\nu_s$ is regular in $M_s$ and $J_s = \text{df} J^E_{\nu_s}$ is a union of admissible sets $J^E_{\nu_s}$;

(ii) for some $m$, $\nu_s$ is $\Sigma_1^{(m)}(M_s)$ singularised, that is $k(\nu_s, M_s)$ is defined;

(iii) $M_s$ is sound above $\nu_s$, and if $\nu_s = \kappa^{+M_s}$ where $\kappa \in \text{Card}^{M_s}$, then $M_s$ is sound above $\kappa$.

Recall that if $M = \langle J^E_\alpha, \in \rangle$ and $\nu \leq \alpha$ then $M||\nu = \text{df} \langle J^E_\nu, \in \rangle$ and $E_\nu$. We then note the following facts:

**Lemma 3.4**

(i) If $\langle \nu, M \rangle, \langle \nu, N \rangle$ satisfy (b), (i), (ii) above but are both sound above $\nu$, with $M||\nu = N||\nu$, then $M = N$.

(ii) If $\langle \nu, M \rangle, \langle \nu, N \rangle \in S^+$ and $J^E_\nu = J^E_\nu$ then $M = N$.

**Proof:** Straightforward iteration and comparison. Q.E.D.

The following definition encapsulates the essential concepts associated with singularising structures.
Definition 3.5 Let $s \in S^+$. Then we associate the following to $\nu_s$:

a) $n_s = df k(\nu_s, M_s)$, the least $n \in \omega$ so that $\nu_s$ is $\Sigma_1^{(n)}(M_s)$ singularised over $M_s$.

b) $M_s^l = df M_s^{1, pM_s \downarrow l}$ for $l \leq k_s, n_s$.

c) $h_s^l = df h_{M_s}^{1, pM_s \downarrow l}$; $h_s = df h_s^n$; $\tilde{h}_s = df \tilde{h}_s^{n+1}$.

d) $\kappa_s \simeq$ the largest cardinal of $J_s$, if such exists; $\omega p_s = df \text{On} \cap M_s^n$.

Let $\beta(s) = df \text{On} \cap M_s$.

e) $p_s = df p_{M_s} \setminus \nu_s$ if $\nu_s$ is a limit cardinal of $J_s$; $p_s = df p_{M_s} \setminus \kappa_s$ otherwise;

$\sigma_s = df p_s \cap \omega p_M^n$;

$\delta_s = df d_{M_s}$.

f) $\alpha_s = df \max \\{ \alpha < \nu \setminus \tilde{h}_s(\alpha \cup \{p_s\}) = \alpha \}$, setting $\max \emptyset = 0$.

g) $\gamma_s \simeq \min \{ \gamma < \nu | \exists f \text{ a good } \Sigma_1^{(n_s)}(M_s) \{ \{p_s\} \} \text{ function singularising } \nu \text{ with domain } \subseteq \gamma \}$.

Thus if $\nu_s = \kappa_s^+$, we may have $\kappa_s$ in $p_s$. Note that the closure of the set in $f$ ensures that $\alpha_s$ is always defined; note also that $\alpha_s$ must be strictly less than the first ordinal $\gamma_s$ partially mapped by $\tilde{h}_s$ (with parameter $p_s$) cofinally into $\nu_s$. Note also that if we set $\gamma' = \max \{ \gamma_s, (p_{M_s} \setminus \nu_s) + 1 \} \max \{ \gamma_s, (p_{M_s} \setminus \kappa_s) + 1 \}$ if $\kappa_s$ is defined, then $\tilde{h}_s(\gamma' \cup p_s)$ must be cofinal in $\nu_s$ since we shall have enough parameters in the domain of this hull to define our cofinalising map).

Lemma 3.6 \quad $\omega p_M^\nu_s \geq \nu \geq \omega p_{M_s}^{n+1}$

Proof Let $n = n_s$, $\nu = \nu_s$. Suppose the first inequality failed. Then $n > 0$, and we have some parameter $q$ with a $\Sigma_1^{(n-1)}(M_s)\{ \{ q \} \}$ partial map $f$ of some $\gamma < \nu$ cofinal into $\nu$.

Pick such a $\gamma > \omega p_{M_s}^n$. Let $\pi : \tilde{M} \rightarrow M_s$ have range $\tilde{h}_s^n(\gamma \cup \{q, p_s\})$, with $\tilde{M}$ transitive. By the leastness of $n$, $\text{ran}(\pi)$ cannot be unbounded in $\nu$. By Lemma 2.4, since $\pi \upharpoonright \omega p_{M_s}^n = \text{id}$ and $p_{M_s} \in \text{ran}(\pi)$, $\pi$ is $\Sigma^*$ elementary. However then $\text{ran}(f) \subseteq \text{ran}(\pi)$, with the former unbounded in $\nu$. A contradiction! If the second inequality failed, then the partial function $\Sigma_1^{(n)}(M_s)$ singularising $\nu$ would be a subset of $\nu$ and thus a bounded subset of $\omega p_{M_s}^{n+1}$ belonging to $M_s$. Q.E.D.

Definition 3.7 For $s, \bar{s} \in S^+: (i)$ We set $f : \bar{s} \implies s$ if there is $|f|$ with $|f| : J_{\bar{s}} \rightarrow_{\Sigma_1} J_s$, and $|f|$ is the restriction of some $f^* : M_{\bar{s}} \rightarrow_{\Sigma_1^{(n)}} M_s$ where $n =
\( n_s, \nu_s = f^*(\nu_s) (\text{if } \nu_s \in M_s) \); \( \kappa_s \in \text{ran}(|f|) \) (if \( \kappa_s \) is defined); \( \alpha_s, p_s, d_s \) are all in \( \text{ran}(f^*) \).

(ii) \( F = \{ (s, |f|, s) | f : s \rightarrow s \} \); we write here \( s = d(f), s = r(f) \);

(iii) If \( \nu_s \in M_s \), we set:
\[
p(s) = (d_p \cup \{ d_s, \alpha_s, \nu_s, \kappa_s \}) \quad (\text{if } \kappa_s \text{ is defined}); \quad \text{otherwise}
\]
\[
p(s) = (d_p \cup \{ d_s, \alpha_s, \kappa_s \}) \quad (\text{again including } \kappa_s \text{ only if it is defined}).
\]

(iv) \( f(\delta, s) \) is the inverse of the transitive collapse of the hull \( h_s(\delta, \{ p(\nu) \}) \) in \( M_s \).

(\text{Lemma 3.9 will justify in the final clause (iv) that there is some } s \text{ so that } \langle s, |f(\delta, s)|, s \rangle \in \mathbb{F} \).

**Lemma 3.8** If \( \exists s(f : s \rightarrow s) \) then \( |f| \) and \( f^* \) are uniquely determined by \( \text{ran}(|f|) \cap \nu_s \).

**Proof:** As \( M_s \) is sound above \( \nu_s \), we have by our definitions, that \( \bar{h}_s(\omega \nu_s \cup \{ p_s \}) = M_s \). We have a \( \Delta_1(J_s) \) onto map \( g : \omega \nu_s \rightarrow J_s \). Thus, if \( Y = \bar{h}_s(\omega \nu_s \cap \text{ran}(|f|) \cup \{ p_s \}) \), then \( Y = \bar{h}_s(\text{ran}(|f|) \cup \{ p_s \}) = \text{ran}(f^*) \). Q.E.D.

Lemma 3.8 justifies us in calling \( f^* \) the canonical extension of \( f \), (or rather \( |f| \)) and sometimes we abuse notation and write \( f^* : J_f \rightarrow \Sigma_1 J_s \) where more correctly we should write \( f^* \upharpoonright J_f : J_f \rightarrow \Sigma_1 J_s \). By virtue of the last lemma, this does not cause any ambiguity.

The next two lemmata are fundamental and concern relationships between singularising structures, and associated maps between them.

**Lemma 3.9** Let \( f : M \rightarrow \Sigma_1 M_s \); suppose \( f(d, \alpha, \bar{\nu}) = d_s, \alpha_s, p_s \), and (where appropriate) \( f(k, \bar{\nu}) = \kappa_s, \nu_s \). (The latter if \( \nu_s \in M_s \); if \( \nu_s = \Omega \cap M_s \) then we take \( s = \Omega \cap M_s \).) Then \( s = (\bar{\nu}, \tilde{M}) \in S^+ \) and thus \( f : s \rightarrow s \); moreover \( n, d, \alpha, \bar{\nu}, \kappa \) (the latter defined if \( \kappa_s \) is) are \( n_s, d_s, \alpha_s, p_s, \kappa_s \).

**Proof** We shall show that \( \tilde{M} \) is a singularising structure for \( \bar{\nu} = \bar{d}_t \nu_s \) and the other mentioned parameters have the requisite properties to satisfy the relevant definitions, and are moved correctly by \( f \). We set \( \nu = \nu_s \).

1. \( \bar{\nu} = p_M \setminus \varrho, \bar{d} = d_M \setminus \bar{\nu} \) and \( \tilde{M} \) is sound above \( \bar{\nu} \).

**Proof:** Directly by the Condensation Lemma II Theorem 2.7 Q.E.D. (1)

Let \( \tilde{h} \) have the same functionally absolute definition over \( \tilde{M} \) as \( \bar{h}_s \) does over \( M_s \). \( \tilde{h} \) is thus \( \Sigma_1^{(n)}(\tilde{M}) \).

2. \( \varrho \) is defined from \( \tilde{M} \) as \( \alpha \) was defined from \( M_s \).
Proof: Set $H(\xi^n, \zeta^n) \longmapsto \tilde{h}_s(\omega \xi^n \cup \{p_s\}) \cap \nu \subseteq \zeta^n$.

Then $H$ is $\Pi_1^{(n)M_s}(\{p_s\})$, and $\overline{H}(\xi^n, \zeta^n) \longmapsto \tilde{h}(\omega \xi^n \cup \{\bar{p}\}) \cap \bar{\nu} \subseteq \zeta^n$.

Hence the following is a consequence that $H(\alpha, \alpha)$ it follows that $M_s \models \overline{H}(\alpha, \alpha)$. However for any $\xi$ with $\alpha < \xi^n < \bar{\nu}$ we must have $M_s \models \overline{H}(\xi^n, \zeta^n)$, because $M_s \models \overline{H}(f(\xi^n), (f(\zeta^n))$ since $\alpha < f(\xi^n) < \nu$. Hence $\alpha$ is defined in the requisite way.

(3) $\exists \xi^n < \bar{\nu}(\tilde{h}_s(f(\xi^n) \cup \{p_s\})$ is unbounded in $\nu$).

If (3) were to hold for some $\xi$ then $\tilde{h}(\xi \cup \{\bar{p}\})$ would be cofinal in $\bar{\nu}$, since the following is a $\Pi_1^{(n)}$ expression which thus would go down to $\overline{M}$. It would then be a statement about the parameters $\bar{\nu}, \tilde{h}, \xi$ and $\bar{\nu}$ (the latter if $\nu = f(\bar{\nu}) < \omega p_s$):

$M_s \models (\forall \xi^n < \nu)(\exists \delta^n < f(\xi^n)(\exists i < \omega)(\xi^n < \tilde{h}_s(i, (\delta^n, p_s))) < \nu$).

This would show that $\tilde{h}$ is a singularising function for $\bar{\nu}$ over the structure $\overline{M}$ and that $n \geq n_s$. We need to show that (3) holds. Suppose not. This has the consequence that $\tau = g f \sup f^n \bar{\nu} \leq \gamma_s < \nu$. As $\alpha_s \in \operatorname{ran}(f \mid \bar{\nu})$ we have that $\alpha_s < \tau$. So by definition of $\alpha_s$ itself:

(4) $\tau \neq \nu \cap \tilde{h}_s(\tau \cup \{p_s\})$.

Hence the following is true in $M_s$:

$\exists i \in \omega \exists \xi^n < \tau(\nu > \tilde{h}_s(i, (\xi^n, p_s)) \geq \tau)$.

Let $i, \xi^n$ witness this, and pick $\delta < \bar{\nu}$ so that $f(\delta) > \xi^n$. Then for any $\bar{\nu} < \bar{\nu}$, as $f(\bar{\nu}) < \tau$:

$M_s \models (\exists \xi^n < f(\bar{\nu}))(\nu > \tilde{h}_s(i, (\xi^n, p_s)) \geq f(\bar{\nu}))$.

This is $\Sigma_1^{(n)}$ and hence, for all $\bar{\nu} < \bar{\nu}$, goes down to $\overline{M}$, yielding:

$\overline{M} \models \exists \bar{\nu} < \bar{\nu}(\exists \xi^n < \tau(\nu > \tilde{h}_s(i, (\xi^n, \bar{\nu})) \geq \bar{\nu})$.

Hence $\tilde{h}$ is a singularising function for $\bar{\nu}$. Thus whether (3) holds or not we have established the existence of suitable $\Sigma_1^{(n)}(\overline{M})$ singularising function.

(5) $n = n_s$.

We are left with showing $n \leq n_s$, as the above shows that $n \geq n_s$. Suppose $m < n$ and that $\tilde{g}$ is a $\Sigma_1^{(m)}(\overline{M})$ good function in the parameter $\bar{\nu}$. Let $g$ be $\Sigma_1^{(m)}(M_s)$ using the same functionally absolute definition and the parameter $f(\bar{\nu})$. Suppose $\delta < \bar{\nu}$. By the $\Sigma_1^{(m)}$-elementarity of $f$ we have the following $\Sigma_1^{(m)}(\overline{M})$ statement holds in $M_s$ (as $\operatorname{ran}(g \mid f(\delta))$ is bounded in $\nu$):

$\exists \xi^n < \nu)(\forall \xi^m < f(\delta))(\forall \eta^m < \nu)(g(\xi^m) = \eta^m \rightarrow \eta^m < \xi^n)$
Lemma 3.11

In other words:
exists a unique
and so is preserved upwards to
the canonical extension of

As \( \delta \) was arbitrary, we conclude \( \text{ran}(\bar{g} \upharpoonright \xi) \) is bounded on any \( \xi < \bar{\nu} \). Hence
\( n \leq n_s \).

Q.E.D. (5) and Lemma.

Definition 3.10

Suppose \( f : \bar{s} \Rightarrow s \). Then let \( \lambda(f) = \sup \text{ran} f \upharpoonright \mathfrak{P} \); \( \rho(f) = \sup f^{h_0} \).

Lemma 3.11

Suppose \( f : \bar{s} \Rightarrow s \), and let \( \lambda = \lambda(f) \). Then \( \lambda \in \text{Sing} \) and there exists a unique \( f_0 : \bar{s} \Rightarrow s \mid \lambda \) with \( f \upharpoonright \mathfrak{P} = f_0 \upharpoonright \mathfrak{P} \).

Proof: Let \( n = n_s \). We apply directly the Interpolation Lemma with \( \lambda \) as \( \bar{\nu}, M_s, M_s \) as \( \bar{M}, M \) respectively, and using \( f^* : M_{\bar{\nu}} \rightarrow \Sigma_1^{(n-1)} M_s \) (where \( f^* \) is the canonical extension of \( f \)) we have the structure \( \bar{M} = M_{s'} \) and maps \( \bar{f}, f' \) as specified.

1. \( s' = (\lambda, \bar{M}) \in S^+, n = n_{s'} \).

2. By the comment above \( \gamma_{\bar{s}} \) is defined and \( n = n_s \). As \( \bar{h}_{\bar{s}}(\gamma_{\bar{s}} \cup \{ p_{M_s}, r \}) \) is cofinal in \( \mathfrak{P} \) for some parameter \( r \) then \( \lambda \cap \bar{h}_{\bar{M}}^{s+1}(\bar{f}(\gamma_{\bar{s}}) \cup \{ p', f(r) \}) \) is cofinal in \( \lambda \) (setting \( p' = \bar{f}(p_s) = f^{-1}(p_s) \)). Thus \( \lambda \) is \( \Sigma_1^{(n-1)} \)-singularised over \( \bar{M} \). Hence \( n \geq n_{s'} \).

3. We need to show that \( \lambda \) is not \( \Sigma_1^{(n-1)} \)-singularised over \( M \). Suppose this fails and thus that \( \{ \alpha | \sup(\lambda \cap \bar{h}_{\bar{M}}^{s+1}(\alpha \cup \{ r \}) = \alpha \} \) is bounded in \( \lambda \), by \( \alpha' \) say, for some choice of a parameter \( r \) in \( \bar{M} = M_{s'} \). By the construction of the pseudo-ultrapower we may assume that \( r \) is of the form \( \bar{f}(\bar{g}_0)(\eta) \) for some good \( \Sigma_1^{(n-1)}(M_{\bar{\nu}}) \) function \( \bar{g}_0 \) and some \( \eta < \lambda \). Define

\[
\bar{H}(\xi^0, \zeta^n, d) \longleftrightarrow \bar{h}^n_s(\omega \xi^0 \cup \{ d \}) \cap \lambda \subseteq \zeta^n; \bar{H}(\xi^0, \zeta^n, d) \longleftrightarrow \bar{h}^n_s(\omega \xi^0 \cup \{ d \}) \cap \overline{\nu} \subseteq \zeta^n.
\]

These are (uniformly defined) \( \Pi_1^{(n)} \) relations over their respective structures - in the parameters \( \lambda, \bar{\nu} \). By the leastness in the definition of \( n_{\bar{s}} \) we have that there are arbitrarily large \( \bar{\tau}^n < \bar{\nu} \) with \( \bar{h}^n_s(\omega \bar{\tau}^n \cup \{ p_s \}) \cap \bar{\nu} \subseteq \overline{\bar{\tau}^n} \); using the soundness of \( M_\bar{s} \) above \( \bar{\nu} \), this implies that for arbitrary \( \zeta^n < \bar{\tau}^n \): \( \bar{h}^n_s(i, \bar{\xi}^n, \bar{g}_0(\bar{\xi}^n)) \cap \bar{\nu} \subseteq \bar{\tau}^n \).

In other words:

\[
\forall \zeta^n < \bar{\tau}^n \bar{H}(\bar{\tau}^n, \bar{\tau}^n, \bar{g}_0(\bar{\xi}^n)).
\]

As the substituted \( \bar{g}_0 \) is good \( \Sigma_1^{(n-1)} \) we have that this is a \( \Pi_1^{(n)} \) statement, and so is preserved upwards to \( M_{s'} \):

\[
\forall \zeta^n < \bar{f}(\bar{\tau}^n) \bar{H}(f(\bar{\tau}^n), \bar{f}(\bar{\tau}^n), \bar{f}(\bar{g}_0)(\bar{\xi}^n)).
\]
However as $\tilde{f} \upharpoonright \tilde{\nu}$ is cofinal into $\lambda$, we may choose $\tilde{\tau}^n$ so that $\tilde{f}(\tilde{\tau}^n) > \max\{\alpha', \eta\}$. This contradicts our definition of $\alpha'$. \hspace{1cm} \text{Q.E.D.}(1)

(2) $p' = p_{s'}$

By the pseudo-ultrapower construction, we have $\tilde{M} = \tilde{h}_{M_1}^{m+1} (\lambda \uplus p') = \tilde{h}_{M_1}^{m+1} (\tilde{\kappa} \cup p')$ (where $\tilde{\kappa} = \tilde{f}(\kappa_\tau)$ if $\kappa_\tau$ is defined) and is sound above $\lambda$ (or $\tilde{\kappa}$). The solidity of $\eta$ above $\tilde{\nu}$ transfers via the $\Sigma_1(n)$-preserving map $f'$ to show that $p'$ is solid above $\lambda$ (see [17] 3.6.8). Then the minimality of the standard parameter and the definition of $p_{s'}$ shows that $p_{s'} \leq^n p'$. However if $p_{s'} \prec^n p'$ held, we should have for some $i \in \omega$, $\tilde{\xi}$ that $p' = \tilde{h}_{M_1}^{m+1} (i, (\tilde{\xi}, p_{s'}))$, and thus $p_s = \tilde{h}_{M_1}^{m+1} (i, (\tilde{f}(\tilde{\xi}), f'(p_{s'})))$ whence $M_s = \tilde{h}_{M_1}^{m+1} (\nu \cup f'(p_{s'}))$. This is a contradiction as $f'(p_{s'}) \prec p_s$. \hspace{1cm} \text{Q.E.D.}(2)

(3) If $\tilde{d} = \eta f(d_{\tau})$ then $\tilde{d} = d_{s'}$

Proof: This is very similar to Lemma 2.8, using the $\Sigma_1(n)$-preservation properties of $f'$, and is left to the reader. \hspace{1cm} \text{Q.E.D.}(3)

(4) If $\tilde{\alpha} = \alpha f(\alpha)$ then $\tilde{\alpha} = \alpha_{s'}$

That $\tilde{\alpha}$ is sufficiently closed, and hence $\tilde{\alpha} \leq \alpha_{s'}$, is proven as in (2) of Lemma 3.9 using: $H(\xi^n, \zeta^n) \hookrightarrow h_s(\omega \xi^n \cup \{p_s\}) \cap \lambda \subseteq \zeta^n$; $H(\xi^n, \zeta^n) \hookrightarrow h_s(\omega \xi^n \cup \{p_s\}) \cap \tilde{\nu} \subseteq \zeta^n$. For $\tilde{\alpha} < \eta^n < \lambda$ we set $\tilde{\eta} = f^{-1} \eta^n$. Then we have $\tilde{\eta} = \lambda \neq \tilde{\eta} = f^{-1} \eta^n$. Then we have $H(\tilde{\eta}, \tilde{\eta})$ (as $\tilde{\eta} > \alpha_{s'}$). Hence for some $i \in \omega$, some $\tilde{\xi} < \tilde{\eta}$ we have $\eta \leq h_{s'}(i, (\tilde{\xi}, p_{s'})) < \tilde{\nu}$. As $f(\tilde{\eta}) \geq \eta^n$ and as $f$ is $\Sigma_2(n)$-preserving we have $\eta^n \leq h_s(i, (f(\tilde{\xi}), p_{s'})) < \lambda$. \hspace{1cm} \text{Q.E.D.}(4)

We have shown enough now to set that $f_{s'}^* = \tilde{f}$. \hspace{1cm} \text{Q.E.D.}(Lemma)

Lemma 3.12 Suppose $f : s \rightarrow s$ and $k_s = n_s$. Then $\lambda(f) < n_s \hookrightarrow \rho(f) < \rho_s$.

Proof: ($\rightarrow$) Suppose $\rho(f) = p_s$. Let $\lambda = \lambda(f)$. Then, in the notation of the previous Lemma the map $f'$ is not only $\Sigma_0(n)$ but is cofinal at the $n$th level, and thus $\Sigma_1(n)$-preserving. We also have that $f'(\lambda, p_{s'}) = (\nu, p_s)$. This implies that $\nu \uplus f'^* h_{s'}(\lambda \uplus p_{s'}) \subseteq \nu \cup h_{s'}(\lambda \uplus p_s) = \lambda$. Were $\lambda < \nu$ this would contradict the fact that $\lambda > \alpha_s$ as the latter is by supposition, in $\text{ran}(f)$.

($\leftarrow$) Suppose $\lambda = df \lambda = \nu$. Again in the same notation, suppose $\rho' = df \rho(f) < \rho_s$. It is then easy to see that a good $\Sigma_1(n)$ function, $\overline{F}$ say, singularizing $\nu$ definable in some parameter $\overline{\eta}$ is taken by the $\Sigma_0(n)$-preserving $f^*$ to a good $\Sigma_1(n)(M_s)$ function $F$ in $q = f(\overline{\eta})$ singularizing $\lambda$, with all the parameters of the form $x^n$ needed to define the values $F(\xi)$ in $\text{ran}(f^*)$. However if $\rho' < \rho_s$ we should have that $F \in M_s$. However $\lambda = \nu$. Contradiction! \hspace{1cm} \text{Q.E.D.}
The construction of the $C_s$-sequences attached to $s = (\nu_s, M_s)$ will follow in essence the construction in [15]. The main point is that we can give an estimate to the length of the $C_s$ sequence.

We may state immediately what the $C_s$-sequences for $s = (\nu_s, M_s) \in S^+$ will be:

**Definition 3.13** Let $s \in S^+; C_s^+ = \{ \lambda(f) \mid s \}; C_s = \{ g \mid \lambda(g) = \lambda(f) \} \setminus \{ \nu_s \}$.

**Definition 3.14** Let $f : s \implies s$. Then $\beta(f) = \max \{ \beta \mid f \setminus \beta = \id \setminus \beta \}$.

By elementary closure considerations show that $\beta(f)$ is defined, and that $\beta(f) = \alpha_{\nu_s}$ iff $f = \id_{\nu_s}$ iff $f(\beta) \not= \beta$. If $\beta(f)$ were singular in $M_\beta$ using some cofinal function $g : \beta' \rightarrow \beta$ with $\beta' < \beta$, we should have that then $\beta(f) > \sup(\ran(g)) = \beta$. Hence $M_\beta \models \text{"} \beta \text{ is a regular cardinal"}$. The next lemma lists some properties of $f(\gamma, q, s)$ which were defined at 3.7. Firstly a minimality property of $f(\gamma, q, s)$.

**Lemma 3.15** (i) If $\gamma \leq \nu_s$ then $f(\gamma, q, s)$ is the least $f$ such that $f \mid \gamma = \id \mid \gamma$ with $q, p(s) \in \ran(f^*)$, in that if $g$ is any other such with these two properties, (meaning that $g \implies s$ with extension $g^*$ so that $\gamma \cup \{ q, p(s) \} \subseteq \ran(g^*)$) then $g^{-1}f(\gamma, q, \nu_s) \in \mathbb{F}$.

(ii) $f(\gamma, q, s) = f(\beta, q, s)$ where $\beta = \beta(f(\gamma, q, s))$.

(iii) $f(\nu_s, q, s) = \id_{s}^q$.

(iv) Let $f : s \implies s$ with $\gamma \leq \nu_s, f^s \subseteq \gamma \leq \alpha_{\nu_s}, q \in J_\gamma, f^*_\gamma(q) = q$, then $\ran(f^*_\gamma) \subseteq \ran(f^*_s)$.

With (i) this implies: if $\beta(f) \geq \gamma$ then $f f(\gamma, q, s) = f(\gamma, q, s)$.

(v) Set $g = f(\gamma, q, s)$; $\lambda = \lambda(g)$ and $g_0 = \red(g)$. Then $q \in J_{\delta} \setminus \lambda$ and $g_0 = f(\gamma, q, s) \setminus \lambda$.

**Proof:** (i)-(iv) are easy consequences of the definitions. (For (i) note this makes sense since we have specified in effect that $\ran(g^*) \supseteq \ran(f(\gamma, q, s))$.) We establish (v). We know that $g_0 \implies s \setminus \lambda$. Set $g_0' = f(\gamma, q, s) \setminus \lambda$ and we shall argue that $g_0 = g_0'$. Let $k = g_0^{-1}g_0'$. The argument of Lemma 3.11 shows that $d(g_0) = d(g)$; as $g_0 \setminus \gamma = \id \setminus \gamma$, and $q \in \ran(g_0)$ by (i) the minimality of $g_0' \implies s \setminus \lambda$ implies we have such a $k$ defined. Thus $k \in \mathbb{F}$. But $k \implies d(g_0)$ so we conclude, as $d(g_0) = d(g)$, that $gk \in \mathbb{F}$. But $\ran((gk)^*) \cap \lambda = \ran(g^*) \cap \lambda$. So, using that $gk \setminus \gamma = \id \setminus \gamma$, and $q, p(s) \in \ran(gk)$, and then (i) again, we have $(gk)^{-1}g = k^{-1} \in \mathbb{F}$. Hence $k = id_{d(g_0')}$ and thus $g_0 = g_0'$.

Q.E.D.

Our definitions are preserved through $\implies$ when a map $f$ is cofinal, meaning that $|f|$ is cofinal into $r(f)$:
Lemma 3.16 Let $f : \tilde{s} \rightarrow s$ with $\lambda(f) = \nu$. Set $\hat{\nu} = \nu_{\tilde{s}}$, $\nu = \nu_{s}$, and let $\gamma < \hat{\nu}$, $\gamma = f(\gamma), \tilde{\eta} \in J_s, f(\tilde{\eta}) = \eta$. Set

$g = f(\gamma, \tilde{\eta}); g = f(\gamma, \tilde{\eta})$. Then

(i) $\lambda(g) = \nu \rightarrow \lambda(g) < \nu$;

(ii) If $\lambda(\hat{g}) < \hat{\nu}$ then $f(\lambda(\hat{g})) = \lambda(g)$ and $f(\beta(\hat{g})) = \beta(g)$.

Proof: Assume $\lambda(\hat{g}) < \hat{\nu}$. Set $\tilde{h}_s = \tilde{h}_s$, $\lambda' = f(\lambda(\hat{g}))$. The following is $\Pi^{n}_1(M_{\tilde{s}})(\{\lambda(\hat{g}), \gamma, p(\tilde{s})\})$:

$\forall x^n \forall \xi^n < \gamma \forall i < \omega(x^n = \tilde{h}_s(i, \xi^n, \tilde{q}, \tilde{p}))) \wedge x^n < \tilde{\nu} \rightarrow x^n < \lambda(\tilde{g})$; if $\tilde{\nu} = \text{On} \cap M_{\tilde{s}}$ then we drop the conjunct $x^n < \tilde{\nu}$. Then

$\forall x^n \forall \xi^n < \gamma \forall i < \omega(x^n = \tilde{h}_s(i, \xi^n, q, p(s))) \wedge x^n < \nu \rightarrow x^n < \lambda'(i)$

as $f$ is $\Pi^{n}_1$-preserving. Hence $\lambda' \geq \lambda(g)$.

Claim 1: $\lambda' \leq \lambda(g)$.

As $\lambda(\hat{g}) < \hat{\nu}$ we have $\omega(\hat{g}) < \omega(\tilde{s})$ by Lemma 3.12. Hence if we set $A = A^n, p_s^n$, and $N = (\lambda^A_p, \lambda \cap J_p(\tilde{g}))$ we have that $\lambda(\hat{g}) = \sup(\tilde{\nu} \cap \tilde{h}_N(\gamma \gamma, \tilde{q}, \tilde{p})))$. Applying $f$, and with $N = f(N)$, we have $\lambda' = \sup(\nu \cap h_N(\gamma \gamma, q, p(s)))$.

For amenable structures (such as $N$) we have a uniform definition of the canonical $\Sigma_1(N)$ Skolem function $h_N$. From $(N, A_N) \subseteq \langle M^n_s, A^n_s \rangle$, we have that $h_N \subseteq h_s$, and thus

$\lambda' = \sup(\nu \cap h_s(\gamma \gamma, q, p(s))) = \sup(\nu \cap h_s(\gamma \gamma, q, p(s))))$.

Thus $\lambda' \leq \lambda(g)$ and Claim 1 is finished.

Claim 2 $f(\beta(\hat{g})) = \beta(g)$

Let $\beta = f(\beta(\hat{g}))$; as $\tilde{g} = f(\beta(\hat{g}, \tilde{g}, \tilde{s})$ we have $\beta(\tilde{g}) \notin \text{ran}(\tilde{g})$. $\beta = f(\beta(\hat{g})) = f(\sup(\delta < \nu \mid \delta \subseteq \lambda(g))) = f(\sup(\delta < \nu \mid \delta \subseteq \lambda(\hat{g}))) = \sup(\delta < \nu \mid \delta \subseteq \lambda(\hat{g})))$. The above $\beta \leq \sup(\delta < \nu \mid \delta \subseteq h_N(\delta \subseteq \tilde{q}, \tilde{p}(s) \cap \rho(\tilde{g})))$. Hence $\lambda < \beta$. Then in $M_N$ we have:

$\forall \beta^n \leq \beta \exists \xi^n < \gamma \forall i < \omega(\beta^n = \tilde{h}_s(i, \xi^n, q, p(s))))$.

However $f$ is $\Sigma_1(n)$-preserving, so this goes down to $M_{\tilde{s}}$ as:

$\forall \beta^n \leq \beta(\hat{g}) \exists \xi^n < \gamma \forall i < \omega(\beta^n = \tilde{h}_s(i, \xi^n, q, p(s))))$.

But this, with $\beta^n \leq \beta(\hat{g})$ implies $\beta(\hat{g}) \in \text{ran}(\tilde{g})$ which is a contradiction! This finishes Claim 2 and (ii). Finally, just note for ($\rightarrow$ of (i)) as $\rho(f) = p_s$, if $\lambda(g) < \nu$ then by Lemma 3.12 there is $\eta = f(\hat{g}) < \rho(f)$ with $\tilde{h}_s(\gamma \gamma, q, p(s))) \cap \rho(\tilde{g}) \subseteq \eta$. This $\Pi^{n}_1$ statement goes down to $M_{\tilde{s}}$ as $\tilde{h}_s(\gamma \gamma, q, p(s))) \cap \rho(\tilde{g}) \subseteq \hat{\eta}$. Hence $\lambda(\hat{g}) < \lambda$.

From this point onwards in the proof we are very much following, almost verbatim, the development of [1]: the fine structural arguments specific to our level of mice have all been dealt with, and the rest is very much combinatorial.
reasoning that is common to whatever model we are trying to define a □ sequence for.

**Definition 3.17** Let \( s = (\nu_s, M_s) \in S^+, q \in J_{\nu_s} \). Then \( B(q, s) = \{ f \in B^+(q, s) \mid \nu_s \} \), where

\[
B^+(q, s) = \{ \beta(f_{(\gamma, q, s)}) \mid \gamma \leq \nu_s \}.
\]

\( B(q, s) \) is thus the set of those \( \beta < \nu_s \) so that \( \beta = \beta(f) \) where \( f = f_{(\beta, q, s)} \).

**Lemma 3.18** Let \( f \) abbreviate \( f_{(\gamma, q, s)} \). Assume \( q \in J_s \). (i) Suppose \( \gamma \in B(q, s)^{\ast} \). Then \( \text{ran}(f) = \bigcup_{\beta \in B(q, s) \cap \gamma} \text{ran}(f_{(\beta, q, s)}) \).

(ii) Let \( \gamma \leq \alpha_s \). Suppose \( \bar{s} \) is such that \( f : \bar{s} \Rightarrow s \) with \( f(\bar{q}) = q \). Then \( \gamma \cap B(q, s) = B(\bar{q}, \bar{s}) \).

(iii) Let \( \lambda = \lambda(f) ; f_0 = \text{red}(f) \). Then \( \gamma \cap B(q, s|\lambda) = \gamma \cap B(q, s) \).

**Proof:** (i) is clear; (ii) follows from Lemma 3.15(iv), and (iii) from (ii) and Lemma 3.15(v). Q.E.D.

**Definition 3.19** Let \( s \in S^+, q \in J_s \).

\[
\Lambda^+(q, s) = \{ \lambda(f_{(\gamma, q, s)}) \mid \gamma \leq \nu_s \}; \Lambda(q, s) = \{ \lambda(f_{(\gamma, q, s)}) \mid \gamma \leq \nu_s \}.
\]

The sets \( \Lambda(q, s) \subseteq C_s \) are first approximations to \( C_s \) if \( q \) is allowed to vary. We first analyse these sets.

**Lemma 3.20** Let \( s \in S^+, q \in J_s \). (i) \( \Lambda(q, s) \) is closed below \( \nu_s \). (ii) \( \text{ot}(\Lambda(q, s)) \leq \nu_s \); (iii) if \( \lambda \in \Lambda(q, s) \) then \( q \in J_s|\lambda \) and \( \Lambda(q, s|\lambda) = \lambda \cap \Lambda(q, s) \).

**Proof:** Set \( \Lambda = \Lambda(q, s) \). (i) Let \( \eta \in \Lambda^{\ast} \). We claim that \( \eta \in \Lambda^+(q, s) \). For each \( \lambda \in \Lambda(q, s) \) pick \( \beta \in B(q, s) \) with \( \lambda(f_{(\beta, q, s)}) = \lambda \). Clearly \( \lambda \leq \chi \rightarrow \beta \chi \leq \beta \lambda \). Let \( \gamma \) be the supremum of these \( \beta \lambda \). As \( B(q, s) \) is closed (by (i) of Lemma 3.18), \( \lambda(f_{(\gamma, q, s)}) = \sup \lambda \lambda(f_{(\beta \lambda, q, s)}) = \eta \).

(ii) is obvious; (iii) Let \( \lambda \in \Lambda \) and \( g = \lambda(f_{(\gamma, q, s)}) \), where we take \( \beta = \beta(g) \). Suppose \( g : \bar{s} \Rightarrow s \). Let \( g(\bar{q}) = q \) and set \( g_0 = \text{red}(g) \). Then by Lemma 3.15(v) \( g_0 = \lambda(f_{(\beta, q, s)}) \). If \( \gamma \geq \beta \) then \( \lambda = \lambda(f_{(\gamma, q, s)}) \). \( \lambda = \lambda(f_{(\gamma, q, s)}) \).

If \( \gamma \leq \beta \) then \( |f_{(\gamma, q, s)})| = |g_0|f_{(\gamma, q, s)}| = |g||f_{(\gamma, q, s)}| = |f_{(\gamma, q, s)}| \) where the first equality is justified by Lemma 3.15(v). Q.E.D.

**Lemma 3.21** If \( f : \bar{s} \Rightarrow s, \mu = \lambda(f), \bar{q} \in J_s, f(\bar{q}) = q \), then:

(i) \( \Lambda(\bar{q}, \bar{s}) = \emptyset \rightarrow \mu \cap \Lambda(q, s) = \emptyset \),

(ii) \( f^\mu(\bar{q}, \bar{s}) \subseteq \Lambda(q, s|\mu) \),

(iii) \( \overline{\Lambda} = \max \Lambda(\bar{q}, \bar{s}) \) and \( \lambda = f(\lambda) \) then \( \lambda = \max(\mu \cap \Lambda(q, s)) \).
Proof: (i) By its definition, if \( \Lambda(t,q) = \varnothing \) then \( f_{(t,q)} \) is cofinal into \( \nu \). Hence \( \text{ran}(f_{(t,q)}) \) is both cofinal in \( \mu \), and contained in \( \text{ran}(f_{(t,q)}) \) by Lemma 3.15(iv), thus \( \mu \cap \Lambda(q,\nu_0) = \varnothing \). This finishes (i). Note that By 3.20(iii) \( \Lambda(q,\nu_0) = \mu \cap \Lambda(q,\nu_0) \). Let \( f_0 = \text{red}(f) \).

(ii) Let \( \bar{\beta} = \lambda(f_{(\beta,q)}) \in \Lambda(\bar{\beta},q) \), and let \( f(\bar{\beta},\bar{\lambda}) = \beta, \lambda = f_0(\bar{\beta},\bar{\lambda}) \). Then \( f_{0}(\lambda(f_{(\beta,q)})) = \lambda(f_{(\beta,q,\mu)}) \in \Lambda(q,\nu_0) \).

(iii) Let \( \bar{\beta} = \text{sup}\{\gamma|f(\gamma,q) \leq \lambda\} \). Then \( \lambda(f_{(\beta,q)})) = \bar{\lambda} \), and by the assumed maximality of \( \bar{\beta} \) we have \( \lambda(f_{(\beta+1,q,s)}) = \nu \). Set \( \beta = f(\bar{\beta}) = f_0(\bar{\beta}) \), then by (IV)(2), \( \lambda = f_0(\bar{\beta}) = \lambda(f_{(\beta,q,\mu)}) \). However \( \lambda(f_{(\beta+1,q,\mu)}) \geq \mu \), since, again by Lemma 3.15(iv), \( \text{ran}(f_{(\beta+1,q,\mu)}) \subseteq \text{ran}(f_{(\beta+1,q,\mu)}) \). Thus \( \lambda = \text{max}(\Lambda(q,\nu_0)) = \text{max}(\mu \cap \Lambda(q,\nu_0)) \).

The p.r. definitions of \( \lambda(f), B(q,s), \Lambda(q,s) \), are uniform in the appropriate parameters. If \( s = (\mu,M_\mu) \in S^+ \), then if we may define \( F_s = \{f_{(\gamma,q,s)})|\nu \in S \cap \mu, q \in J_s, \gamma \leq \nu\} \), \( E_s = \{(\nu,M_\nu,p(s|\nu),h_s|\nu)|\nu \in S \cap \mu\} \), \( G_s = \{(q,s|s)|q \in J_s, \nu \in S \cap \mu\} \). Then we have:

**Lemma 3.22** (i) \( E_s, F_s, G_s \) are uniformly \( \Delta_1(J_s) \) for \( s \in S^+ \);

(ii) \( \mu' < \mu \implies E_{\mu'}, F_{\mu'}, G_{\mu'} \in J_s \).

**Lemma 3.23** Let \( f : s \relativeto s \) with \( \bar{t} \in J_s, f(\bar{q}) = q \). Then

(i) If \( f \) is cofinal then \( \text{ran}(f) : \langle J_s, \Lambda(\bar{q},s) \rangle \rightarrow \Sigma_1 \langle J_s, \Lambda(q,s) \rangle \);

(ii) Otherwise: \( \text{ran}(f) : \langle J_s, \Lambda(\bar{q},s) \rangle \rightarrow \Sigma_0 \langle J_s, \Lambda(q,s) \rangle \)

**Proof:** (i) It suffices to show that \( \text{ran}(f(\Lambda(\bar{q},s) \cap \tau) = \Lambda(q,s) \cap f(\tau) \) for arbitrarily large \( \tau < \nu_s \). However this follows from the last lemma and 3.21.

However, if \( \lambda \in \Lambda(q,s) \), then \( \Lambda(q,s) \cap \lambda = \Lambda(q,s) \) by Lemma 3.20, and by the last lemma, if \( f(\lambda) = \lambda \), we have \( f(\lambda, \bar{q},s|\lambda) = \Lambda(q,s) \) (with the latter equality by Lemma 3.20 again). If \( \Lambda(q,s) \) is unbounded in \( \nu_s \), this suffices; if it is empty or bounded, then the Lemma 3.21 takes care of these cases.

For non-cofinal maps (ii) we still have, if \( \lambda(f) = \mu \), that

\[ |f_0| : \langle J_s, \Lambda(\bar{q},s) \rangle \rightarrow \Sigma_1 \langle J_s, \Lambda(q,s) \rangle \]

where \( f_0 = \text{red}(f) \). But \( \Lambda(q,s|\mu) = \mu \cap \Lambda(q,s) \), and \( |f_0| = |f| \). Q.E.D.

The \( C_s \) sets may be decomposed into a finite sequence of sets of the form \( \Lambda(t^s_s,s) \).

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Definition 3.24 Let $s \in S^+, \eta \leq \nu_s$. $l^i_{\eta s} < \nu_s$ is defined for $i < m_{\eta s} \leq \omega$ by induction on $i$:

\[ l^0_{\eta s} = 0; \quad l^{i+1}_{\eta s} = \max(\eta \cap \Lambda(l^i_{\eta s}, s)). \]

We also write $l^i_s$ for $l^i_{\eta s}$ if the context is clear; also we set $l^i_s \simeq l^i_{\nu s}; m_s = m_{\nu s}$.

Some facts about this definition may be easily checked:

Fact 1. $l^{i+1}_{\eta s} \leq l^i_{\eta s}$ (i.e. $m_{\eta s}$ is monotone)
Fact 2. $i > 0 \implies l^i_{\eta s} \in \eta \cap C_s$.

Lemma 3.25 Let $f : \bar{s} \implies s$. (i) If $\lambda = \lambda(f)$ then $l^i_{\lambda s} \simeq f(l^i_s)$;
(ii) Let $\bar{\eta} < \nu_s, f(\bar{\eta}) = \eta$; then $l^i_{\bar{\eta} s} \simeq f(l^i_{\eta s})$.

Proof (i) By induction on $i$. If $i = 0$ this is trivial. Suppose $i = j + 1$.

1. Then, as inductive hypothesis $l^j_{\lambda s} = f(l^j_s)$, and thus $\frac{f}{l^j_s} : (J_s, \Lambda(l^j_s, s)) \rightarrow \Sigma_1$
i.e., $\Lambda(l^j_{\lambda s}, s) \lambda \implies \Lambda(l^j_{\lambda s}, s)$, by 3.20. Hence: $f(l^j_s) \simeq f(\max(\lambda \cap \Lambda(l^j_{\lambda s}, s))) \simeq l^i_{\lambda s}$

Corollary 3.26 (i) Let $f : \bar{s} \implies s$ cofinally. Then $l^i_s \simeq f(l^i_s)$.
(ii) Let $\lambda \in C_s$. Then $l^i_{\lambda s} \simeq l^i_{\lambda | s}$.

Proof (i) is immediate. For (ii) choose $f : \bar{s} \implies s$ with $\lambda = \lambda(f)$, and set $f_0 = \text{red}(f)$. Then $l^i_{\lambda s} \simeq f(l^i_s) \simeq f_0(l^i_s) \simeq l^i_{\lambda | s}$ with the last equality holding from (i).

Q.E.D.

Lemma 3.27 Let $\eta \leq \nu, \lambda = \min(C^+ \setminus \eta)$. Then $l^i_s \simeq l^i_{\lambda s} \simeq l^i_{\eta s}$ (for any $i < \omega$ for which either side is defined).

Proof Induction on $i$, again $i = 0$ is trivial. Suppose $l^i_s = l^i_{\eta s} = l^i_{\lambda s}$ and $i = j + 1$.

Set $l = l^i_{\eta s}$, then we have: $\Lambda(l, s) \cap \eta = \Lambda(l, s) \cap \lambda$, since $\Lambda(l, s) \subseteq C_s$ and $C_s \cap [\eta, \lambda) = \emptyset$. Suppose, without loss of generality that $l^i_{\eta s}$ is defined. Then $l^i_{\eta s} = \max(\eta \cap \Lambda(l, s)) = \max(\lambda \cap \Lambda(l, s)) = l^i_{\lambda s} = l^i_{\eta s}$. Q.E.D.

Lemma 3.28 Let $j \leq i < m_s$. Set $l = l^i_s$. Then $l^i_j \in \text{ran}(f_{0,1,i,s})$.

Proof Set $f = f_{0,1,i,s}$. Suppose $f : \bar{s} \implies s$, and $\lambda = \lambda(f)$. Then $l^i_{\lambda s} \simeq f(l^i_s)$ by Lemma 3.25(i). But $l^i_{\lambda s}$ exists, and $l^i_j < \lambda \leq \nu_s$. Hence $l^i_{\lambda s} = l^i_{\lambda s} \simeq f(l^i_{\lambda s})$. Q.E.D.

Importantly, the $l^i_{\lambda s}$ sequences are finite.
Lemma 3.29 Let $s \in S^+, \eta \leq \nu_s$. Then $m_{\eta s} < \omega$.

Proof Suppose this fails. Then for some $\eta \leq \nu_s$ we have that $l^i_{\eta s}$ is defined for $i < \omega$. Let $\lambda = \min(C^+_s \setminus \eta)$. then $l^i_{\lambda s} = l^i_{\eta s}$ by Lemma 3.27. Choose $f : \bar{s} \mapsto s$ with $\lambda = \lambda(f)$. Then $l^i_{\lambda s} = l^i_{\bar{s} \lambda} = f(l^i_{\bar{s}})$ for $i < \omega$ by Cor. 3.26(ii) & Lemma 3.25(i). Taking $\lambda$ for $\nu_s$, we assume, without loss of generality, that $l^i_{\nu_s}$ is defined for $i < \omega$ for some $s \in S$. We obtain an infinite descending chain of cardinals by showing that as $i$ increases, and with it $l^i_{\nu_s}$, the maximal $\beta^i$ that must be contained in the range of any $f : \bar{s} \mapsto s$ together with $l^i_{\nu_s}$ in order for $\text{ran}(f)$ to be unbounded in $s$ strictly decreases. This is absurd.

Set $l = l^i_{\nu_s}$. Define: $\beta^i = \beta^i_{\nu_s} = \max\{\beta | \lambda(f(\beta, l, s)) < \nu_s\}$. By the definition of $l^{i+1}_{\nu_s}$ we have that $\lambda(f(\beta, l, s)) < \nu_s \iff \lambda(f(\beta, l, s)) \leq l^{i+1}_{\nu_s}$. Furthermore, by the definition of $\beta^i$:

1. $\lambda(f(\beta, l, s)) \leq l^{i+1}_{\nu_s}$;
2. $\lambda(f(\beta, l, s)) = \nu_s$.

Claim $\beta^{i+1} < \beta^i$ for $i < \omega$.

Proof Set $f = f(\beta, l, s)$. Then $\lambda(f) = l^{i+2}$, dropping the subscript $\nu$. Let $f : s \mapsto s$. Then $l^i_s$ exists and $f(l^i_s) = l^i_{\nu_s}$ for $j \leq i + 1$ since $l^j < l^{i+1} < \nu_s$ (with the first equality from Lemma 3.25(i) and (1), the second from Lemma 3.27).

(3) $\beta^i \geq \beta^{i+1}$.

Proof of (3): Suppose not, then $(\beta^i + 1) \cup \{l^i\} \subseteq \text{ran}(f)$. Hence $\text{ran}(f(\beta, l, s), \nu_s) \subseteq \text{ran}(f)$, hence by (2), $\lambda(f) = \nu_s > l^{i+2}$. Contradiction!

(4) $\beta^i \neq \beta^{i+1}$.

Proof of (4): Suppose not. As $\beta^{i+1}$ is the first ordinal moved by $f$ we conclude that $f(\beta^i) > \beta^i$. Set $g = g(\beta, l, s), \bar{g} = g(\beta, l, s)$ where $\bar{l} = l^i_{\nu_s}$. Then $f = g\bar{g}$, since $f \upharpoonright \beta = \beta^i$ and $g(f(l)) = \bar{l}(= l^i_{\nu_s})$. Hence $l^{i+1} = \lambda(g) = \lambda(f) < l^{i+2} = \lambda(f)$.

Hence $\lambda(\bar{g}) < \nu_s$. Now we set: $g = f(\beta, l, s)$, $0 = \bar{g}(\beta, l, s)$. If further $f_0 = \text{red}(f)$, then we have also $y_0 = f_0(\beta, l, s)$. As $l^{i+1} = \lambda(g) < l^{i+2}$, Lemma 3.16(ii) applies and:

$f(\beta(\bar{g})) = f_0(\beta(\bar{g})) = \beta(g\bar{g}) = \beta^i$.

Hence $\beta^i \in \text{ran}(f)$ which is a contradiction. This proves the Claim and hence the Lemma.

Q.E.D.

We now set $l = l^m_{\eta s}$, where $m = m_{\nu_s}$. Again we write $l$ for $l_{\nu_s}$. Notice that then $\Lambda(l_{\eta s}, \nu_s) \cap \eta$ is either unbounded in $\eta$ or is empty. We first analyze the latter case.

Lemma 3.30 Suppose $\Lambda(l_{\eta s}, \nu_s) \cap \eta = \emptyset$. Set $l = l_{\eta s}$. Then:
(i) \( l = 0 \rightarrow C_s \cap \eta = \emptyset \),
(ii) \( l > 0 \rightarrow \liminf(C_s \cap \eta) \),
(iii) \( \eta \in C^+_s \rightarrow \eta = \lambda(f_{[0,l,s)}) \).

Proof Set \( \rho = \min(C^+_s \setminus (l+1)) \).

Proof: Set \( n = m_{\eta s} - 1 \). Then \( l = l^*_{\eta s} < l + 1 < \eta \). Hence (by Fact after 3) \( l = l^*_{\eta s} + 1 \). But \( A(l, s) \cap (l+1) = \emptyset \). Hence \( l^*_{\eta s} + 1 \) is undefined and \( l = l_{l+1,s} \). Hence \( l = l_{\rho s} \) by Lemma 3.27. Q.E.D. (1)

(2) \( \lambda(f_{[0,l,s)}) = \rho \).

Proof: Choose \( f : \bar{s} \rightarrow s \), with \( \lambda(f) = \rho \) witnessing that \( \rho \in C_s \). Then, by Lemma 3.25(i), \( f(l_\bar{s}) = l_{\rho s} = l \). Set \( l = l_{\bar{s}} \). Now note that we must have that \( \lambda(f_{[0,l,s)}) = \bar{s} \). For, if this failed then \( \lambda(f_{[0,l,s)}) = \lambda(f_{[0,l,s)}) < \rho \) by Lemma 3.16 and so the latter is in \( C^+_s \cap (l, \rho) \), which is absurd! Then \( \lambda(f_{[0,l,s)}) = \lambda(f_{[0,l,s)}) = \lambda(f) = \rho \). Q.E.D. (2)

From (2) and the definition of \( l \) as \( l_{\eta s} \) it follows that \( \rho \geq \eta \). There are thus three alternatives:

If \( l = 0 \) then (i) holds: \( \rho = \min(C^+_s \setminus 1) = \min(C^+_s) \geq \eta \). If \( l > 0 \) then \( \liminf(C_s \cap \eta) \) since \( (C_s \cap \eta) \setminus (l+1) \subseteq (C_s \cap \rho) \setminus (l+1) = \emptyset \) and thus we have (ii); finally for (iii) if \( \eta \in C^+_s \rightarrow \eta = \max(C^+_s \setminus (l+1) = \rho = \lambda(f_{[0,l,s)}) \). Q.E.D.

We now get a characterisation of the closed sets \( C^+_s \).

Lemma 3.31 Let \( \lambda \) be an element or a limit point of \( C^+_s \). Let \( l = l_{\lambda s} \). Then there is \( \beta \) such that \( \lambda = \lambda(f_{[\beta l,s)}) \). Hence \( C_s \) is closed in \( \nu_{\lambda} \), and \( C^+_s = \{ \lambda(f_{[\beta l,s)}) \mid \beta \leq \nu_{\lambda}, l < \nu_{\lambda} \} \).

Proof Case 1 \( \lambda \cap A(l, s) = \emptyset \)

Then \( C_s \cap \lambda = \emptyset \) or \( l = \max(C_s \cap \lambda) \) by the last lemma. Hence \( \lambda \) is not a limit point of \( C^+_s \). Hence \( \lambda \in C^+_s \), and thus \( \lambda = \lambda(f_{[\beta l,s)}) \) by (iii) of that lemma.

Case 2 \( \lambda \cap A(l, s) \) is unbounded in \( \lambda \).

Given \( \mu \in A(l, s) \cap \lambda \), let \( \beta_\mu \) be such that \( \lambda(f_{[\beta_\mu l,s)}) = \mu \). Then \( \lambda(f_{[\beta_\mu l,s)}) = \lambda \)

where \( \beta = \sup_{\mu} \beta_\mu \).

The last sentence is immediate from the previous one. Q.E.D.

We remark that we have just shown that the first conjunct of (i) of Theorem 3.2 holds. We move towards proving the other clauses. The following is (iii).

Lemma 3.32 \( \lambda \in C_s \rightarrow \lambda \cap C_s = C_s \mid \lambda \).
**Proof** Assume inductively the result proven for all \( \nu' \) with \( \nu' < \nu_s \) and \( s|\nu' \in S \), (that is, the lemma is proven with \( s|\nu' \) replacing \( s \)) and we prove the lemma for \( \nu_s \) by induction on \( \lambda \). Let \( l = l_{\lambda_s} \). Hence by Cor.3.26 \( l = l_{s|\lambda} \). By Lemma 3.31 \( \lambda \in \Lambda(l,s) \). Set \( \Lambda = \lambda \cap \Lambda(l,s) \). Then by Lemma 3.20(ii) \( \Lambda = \Lambda(l,s|\lambda) \).

**Case 1** \( \Lambda = \emptyset \).

If \( l = 0 \), then \( C_{s|\lambda} \subseteq \lambda \cap C_s = \emptyset \) (the latter by Lemma 3.30). If \( l > 0 \), then \( l = l_{s|\lambda} = \max(C_{s|\lambda} \cap \lambda) = \max(C_{s|\lambda}) = l_{\lambda_s} = \max(\lambda \cap C_s) \) by the same lemma. As \( \lambda < \lambda_s \), we use the inductive hypothesis on \( \lambda \): \( l \cap C_s = C_{s|l} = l \cap C_{s|\lambda} \) where the second equality is the inductive hypothesis taking \( \lambda = \nu' < \nu_s \). Hence \( C_{s|\lambda} = \lambda \cap C_s = C_{s|l} \cup \{l\} \).

**Case 2** \( \Lambda \) is unbounded in \( \lambda \).

Then \( \mu \in \Lambda \rightarrow \mu \in C_s \cap C_{s|\lambda} \). Hence by the overall inductive hypothesis \( C_{s|\mu} = \mu \cap C_{s|\lambda} \) and (as \( \mu < \lambda \)) \( C_{s|\mu} = \mu \cap C_s \). Hence \( C_{s|\lambda} = \lambda \cap C_s = \bigcup_{\mu \in \Lambda} C_{s|\mu} \).

Q.E.D.

Now (i) of the Theorem follows easily:

**Lemma 3.33** \( \sup(C_s) < \nu_s \rightarrow cf(\nu_s) = \omega \).

**Proof** Let \( l = \sup(C_s) = l_s \). Then \( ran(f_{(0,l,\nu_s)}) \) is countable, and cofinal in \( \nu_s \).

Q.E.D.

**Lemma 3.34** Let \( \bar{s} \rightarrow s \). Then \( \langle f \rangle: \langle J_{\bar{s}}, C_{\bar{s}} \rangle \rightarrow \Sigma_0 \langle J_s, C_s \rangle \).

**Proof:** It suffices to show that for arbitrarily large \( \tau < \nu_s \) that \( \langle f \rangle(C_{\bar{s}} \cap \tau) = C_s \cap \langle f \rangle(\tau) \). As usual we continue to write “\( f \)” for “\( \langle f \rangle \)”.

Set \( \bar{s} = \{l \} \).

**Case 1** \( \Lambda(\bar{l}, \nu_s) \) is unbounded in \( C_{\bar{s}} \).

If \( \bar{\lambda} \in C_{\bar{s}} \) and \( \lambda = f(\bar{\lambda}) \) then by 3.21 (and 3.20) \( \lambda \in \Lambda(\bar{l}, s) \subseteq C_s \). By Lemma 3.22 we have \( E_{s|\bar{\lambda}} \in J_s \) and \( f(E_{s|\bar{\lambda}}) = E_{s|\lambda} \). By Lemma 3.30 \( C_{s|\lambda} = \{\lambda(f_{(0,l,s)}) < \bar{\lambda}|l < \bar{\lambda}\} \in J_s \) and is uniformly \( \Sigma_0 \) from \( E_{s|\lambda} \) over \( J_s \). Consequently \( \langle f \rangle(C_{s|\lambda}) = C_s \cap \lambda \). By \( \Sigma_1 \)-elementarity of \( \langle f \rangle \). But \( C_{s|\lambda} \cap C_{s|\lambda} = \lambda \cap C_s \).

**Case 2** \( \Lambda(\bar{l}, \nu_s) = \emptyset \).

Let \( f(\bar{l}) = l \). Then \( l = l_{\lambda_s} \) where \( \lambda = \lambda(f) \). However \( \lambda(f_{(0,l,s)}) = \nu_s \) by our case hypothesis. Thus \( \lambda(f_{(0,l,s)}) = \lambda(f_{(0,l,s)}) = \lambda \). Hence \( \Lambda(l, \nu_s) \cap \lambda = \emptyset \).

By Lemma 3.30 we are reduced to the following two subcases:

**Case 2.1** \( l = 0 \). Then \( C_{s|\lambda} = C_s \cap C_s = \emptyset \), and the result is trivial.

**Case 2.2** \( l = \max(C_s) \). Then \( l > 0 \) and thus \( l = \max(C_s) \). Hence for sufficiently large \( \tau > \bar{l} \) \( f(\tau \cap C_{\bar{s}}) = f(C_{\bar{s}}) = f(C_s \cap \bar{l} \cup \{l\}) = C_s \cap C_s = f(\tau) \cap C_s \).

Q.E.D.
We now proceed towards calculating the order types of the $C_s$-sequences. This is
done (in a somewhat speedy manner) in [1], but the following comes from [9].
We first generalise the definition of $\beta^i$.

**Definition 3.35** For $\eta \leq \nu_s$ set : $\beta^i_{\eta s} \simeq \max\{\beta | \lambda(f(\beta, t_{\eta s}, s)) < \eta\}$.

In very close analogy to the $\beta^i = \beta^i_s$ we have parallel properties for the $\beta^i_{\eta s}$:
1. $\lambda(f(\beta, t_{\eta s}, s)) < \nu_s \iff \lambda(f(\beta, t_{\eta s}, s)) \leq t_{\eta s}^{i+1}$.
2. $\beta^{i+1}_{\eta s}$ is defined if and only if $t_{\eta s}^{i+1}$ is defined - i.e. $i + 1 < m_{\eta s}$.
3. $\beta^{i+1}_{\eta s} \simeq \beta^i_{\lambda s}$ if $\lambda = \min(C^+_{\eta s}) \setminus \eta$. $\lambda(f(\beta, t_{\eta s}, s)) < \eta \iff \lambda(f(\beta, t_{\eta s}, s)) < \lambda$.
4. $\beta^{i+1}_{\eta s} < \beta^i_{\eta s}$ when defined. (By the same argument as for $\beta^{i+1} < \beta^i$.)

Now we set $b_\eta = b_{\eta s} = \{\beta^i_{\eta s} \mid i + 1 < m_{\eta s}\}$. For $\eta \in C_s$ we then set $d_\eta = d_{\eta s} = \{b_{\eta s} \mid \eta^+ = \min(C^+_{\eta s} \setminus \eta+1)\}$.

Proof of 6: $l_{\eta s}^{i+1} = l_{\eta s}^i$ by 5. If $\eta \in \Lambda(l_{\eta s}^i, s)$ then $\eta$ is maximal in this set below $\eta^+$. So the first alternative holds. Note that $i \neq m_{\eta s} - 1$ (otherwise by Lemma 3.31 for some $\beta, \eta = \lambda(f(\beta, t_{\eta s}, s)) \in \Lambda(l_{\eta s}^i, s)$). Thus $l_{\eta s}^{i+1}$ is defined and $l_{\eta s}^{i+1}$ must equal this.

**Lemma 3.36** Let $\eta, \mu \in C_s$, with $\eta < \mu$. Then $d_\eta <^* d_\mu$.

**Proof** Let $\eta^+ = \min(C^+_{\eta} \setminus \eta + 1)$, $\mu^+ = \min(C^+_{\eta} \setminus \mu + 1)$. Let $i$ be maximal so that $l_{\mu s}^i = l_{\mu s}^{i+1}$. Then $\beta^{i+1}_{\mu s} \simeq \beta^i_{\eta s}$ for $j < i$. As $l_{\mu s}^j \leq \eta < \mu$, we have by 6. above that $l_{\mu s}^{i+1}$ is defined and $l_{\mu s}^{i+1} = \mu$ or $l_{\mu s}^{i+1}$. Moreover then $l_{\mu s}^{i+1}$ is defined, and by maximality of $i$, $l_{\mu s}^{i+1} \neq i_{\mu s}^{i+1}$.

Claim: $l_{\mu s}^{i+1} < l_{\mu s}^{i+1}$.

That $l_{\mu s}^{i+1} < \eta^+$ is ruled out: otherwise $l_{\mu s}^{i+1} = l_{\mu s}^{i+1}$ again). So $l_{\mu s}^{i+1} < \eta^+ \leq l_{\mu s}^{i+1}$.

Q.E.D. Claim.

As $\beta^{i+1} \simeq \mu^+$ is defined, if $\beta^i_{\eta s}$ is undefined, then we’d be finished. Set $l = l_{\mu s}^{i+1} = l_{\eta s}^{i+1}$.

Then $\lambda(f(\beta^{i+1}_{\eta s}, l, s)) = l_{\eta s}^{i+1}$ and $\lambda(f(\beta^{i+1}_{\mu s}, l, s)) = l_{\mu s}^{i+1}$ . Hence $\beta^{i+1}_{\eta s} < \beta^{i+1}_{\mu s}$ and thus $d_\eta <^* d_\mu$ as required.

Q.E.D.
Lemma 3.37 Let $\alpha$ be p.r. closed so that for some $\alpha_0 < \alpha \lambda(f(\alpha_0, \alpha)) = \nu$. Then $ot(C_\alpha) < \alpha$.

**Proof:** First note that $ot(\langle [\alpha]^{<\omega}, <^* \rangle) = \alpha$. Let $\alpha_0 < \alpha$ be such, with the property that $\lambda(f(\alpha_0, \alpha)) = \nu$. Then $\{ \beta^n_m \mid \eta \leq \nu, i + 1 < m \} \subseteq \alpha_0$. Thus $ot(\{ d_\eta \mid \eta \in C_\alpha \}, <^* \rangle) \leq ot(\langle [\alpha]^{<\omega}, <^* \rangle) < \alpha$. Thus $ot(C_\alpha) < \alpha$. Q.E.D.

To obtain the requisite $\langle C_\nu \mid \nu \in S \rangle$ for a Global sequence in $K$, we assign the appropriate level $K_{\beta(\nu)}$ over which $\nu$ is definably singularised. Then $s = \langle \nu, K_{\beta(\nu)} \rangle \in S^+$.

Q.E.D.(Global □)

4 Obtaining Inner Models with measurable cardinals

We assume that we have a Global □ sequence $\langle C_\nu \mid \nu \in S \rangle$ in $K$ constructed as in the last section. We have:

**Theorem 4.1** Assume $n > 3$ and $\{ \alpha < \omega_n \mid \alpha \in \text{Cof}(\omega_n) \cap K-Sing \}$ is, in $V$, stationary below $\omega_n$. Then

$$T_n =_{df} \{ \beta \in \text{Cof}(\omega_1) \cap \omega_n \mid ot(C_\beta) \geq \omega_n \}$$

is stationary in $\omega_n$.

**Proof** Let $C \subseteq \omega_n$ be an arbitrary closed and unbounded set in $\omega_n$. Take $\gamma \in C^* \cap \text{Cof}(\omega_n)$ with $\gamma$ a $K$-singular; in other words with $C_\gamma$ defined. As $cf(\gamma) > \omega$, $C_\gamma$ is cub in $\gamma$. Then $C \cap C_\gamma$ is closed unbounded in $\gamma$ of ordertype $\geq \omega_n$. Take $\beta \in (C \cap C_\gamma)^*$ such that $cf(\beta) = \omega_1$ and $ot(C \cap C_\gamma \cap \beta) \geq \omega_n$. By the coherency property 3.1(c), $C_\beta = C_\gamma \cap \beta$. Thus $\beta \in C \cap T_n \neq \emptyset$. □

Note that $(T_n)_{3 < n < \omega}$ as above would be a sequence of sets to which we could apply the $MS$-principle, if we knew that they were (in $V$) stationary beneath the relevant $\kappa_n$. This is what the assumption in the above theorem achieves. The following is essentially our main Theorem 1.4.

**Theorem 4.2** If $MS((\kappa_n)_{1 < n < \omega}, \omega_1)$ holds then there exists $k < \omega$ so that for all $n > k$, there is $D_n$, closed and unbounded in $\omega_n$, so that

$$D_n \cap \text{Cof}(\omega_n) \subseteq \{ \alpha < \omega_n \mid o^K(\alpha) \geq \omega_n \}.$$

**Proof:** We suppose not. Then for arbitrarily large $n < \omega \ S_n^{0} =_{df} \{ \alpha < \omega_n \mid \alpha \in \text{Cof}(\omega_n) \cap \text{Sing}^K(\alpha) \}$ is stationary in $\omega_n$ by appealing to Mitchell’s Weak Covering Lemma for $K$, 1.7.
We shall define a sequence \((S_n)_{1<n<\omega}\) of stationary sets. By Theorem 4.1, for arbitrarily large \(n < \omega\), \(T_n\) is stationary in \(\omega_n\); for such \(n\) (which we shall call relevant) let \(S_n = T_n\); for all other \(n > 1\) take \(S_n = \text{Cof}(\omega_1) \cap \omega_n\).

Define the first-order structure \(\mathfrak{A} = (H_{\omega_{\omega+1}}, K_{\omega_{\omega+1}}, \in, \prec, \langle f_n \rangle_{n<\omega}, \cdots)\) with a wellordering \(\prec\) of the domain of \(\mathfrak{A}\), and the sequence of finitary functions \(f_n\) including a complete set of skolem functions for \(\mathfrak{A}\). The mutual stationarity property yields some \(X < H_{\omega_{\omega+1}}\) such that

\[
\{\omega_n \mid n \leq \omega\} \subseteq X, \quad \forall n > 2 (\sup X \cap \omega_n) \in S_n, \text{ and } \omega_2 \subseteq X.
\]

(We may assume without loss of generality the latter clause, since a direct argument shows that all ordinals less than, say, \(\omega_2\) may be added to the hull \(X\) without increasing the \(\sup X \cap \omega_n\) for any \(n > k\). (This goes as follows: let \(X_0\) be a hull that satisfies the MS property and the first two requirements above: \(\{\omega_n \mid n \leq \omega\} \subseteq X_0, \forall n > 2 (\sup X_0 \cap \omega_n) \in S_n\). We now consider the enlarged hull of \(X =_{df} X_0 \cup \omega_k\) in \(\mathfrak{A}\). Let \(n > k\). Consider for each \(m\), and each \(x \in [X_0]^\omega\), \(\sup\{f_m(\xi, x) \cap \omega_n \mid \xi \in [\omega_k]^\omega\}\) where we have assumed that \(f_m\) is \(l + p\)-ary. But this is a supremum definable in \(X_0\) from \(f_n\), \(x\), \(\omega_n\), and \(\omega_k\). Hence it is less than \(\sup(X_0 \cap \omega_n)\). By choice of \(\langle f_n \rangle\), every \(y \in X\) is of the form \(f_m(\xi, x)\) so this suffices.)

Let \(\pi : (\mathcal{H}, \mathcal{K}, \ldots) \cong (X, K \cap X, \in, \ldots)\), be the inverse of the transitive collapse, and \(\beta_n =_{df} \pi^{-1}(\omega_n)\) for \(n \leq \omega\). For each \(2 < n < \omega : \beta_n > \aleph_2\) and \(\text{cof}(\beta_n) = \omega_1\). Let \(\beta^*_n =_{df} \sup(\pi'' \beta_n)\). We now consider the coiteration of \(K\) with \(\mathcal{K}\). Let \(((M_i, \pi_{i,j}, \nu_i)_{i \leq j \leq \theta}, (N_i, \sigma_{i,j}, \nu_i)_{i \leq j \leq \theta})\) be the resulting coiteration of \((K, \mathcal{K})\).

(1) \textbf{The first ultrapower on the }K\textbf{ side is taken after a truncation. In fact }\pi_0: M_0 \longrightarrow M_1, \text{ where } \pi \neq \text{id and } M_0^* \text{ is a proper initial segment of } K.\textbf{ Proof:} Note that \(\beta_3\) is a cardinal of \(\mathcal{H}\), whilst \(K_{\beta_3} = K_{\beta_3}\) as \(X \cap \omega_3\) is transitive. However \(\text{cf}(\beta_3) = \omega_1\) and is thus not a true cardinal of \(K\) (by the Covering Lemma for \(K\)). Hence the first action of the comparison will be a truncation on the \(K\) side to a structure \(M_0^*\) in which \(\beta_3\) is a cardinal., and thence the ultrapower map \(\pi_0,1\) as stated. \textbf{Q.E.D. (1)}

(2) \textbf{On the }\mathcal{K}\textbf{ side of the coiteration all the maps }\sigma_{i,j}\textbf{ are the identity: }\forall i \leq \theta N_i = \mathcal{K}.\textbf{ Proof:} Suppose this is false for a contradiction and let \(c\) be the least index where an ultrapower of \(N_c = \mathcal{K}\) is taken by some \(E_{\nu_c}^N\) with critical point \(\kappa_c\). On the \(K\) side let \(\zeta\) be least so that \(P(\kappa_i) \cap M_i || P(\kappa_i) \cap N_i\) Let us set \(M^*\) to be this \(M_i || \zeta\). (Note that no truncation is taken in the comparison on the \(\mathcal{K}\) side.) Note
that since $M_0^*$ was a truncate of $K$, we have that thereafter each $M_i$ is sound above $\kappa_i$ and that $\omega \rho_{M_i}^{n+1} \leq \kappa_i < \omega \rho_{M_i}^n$ for some $n = n(i)$.

As $E_N^i$ is a total measure on $N_i = \overrightarrow{K}$ we have that $E \equiv_{\pi} E_n^{\pi(\nu_i)} = \pi(E_N^i)$ is a total measure in $K$ with critical point $\tilde{\kappa} = \pi(\kappa_i)$.

We apply the measure $E_N^i$ to $M^*$ itself and form the fine structural ultrapower $\tilde{M} = \text{Ult}^* \left( M^*, E_N^i \right)$ with map $t : M^* \rightarrow \tilde{M}$. Note that by the weak amenability of $E_N^i$, $\tilde{M} \cap \mathcal{P}(\kappa_i) = M^* \cap \mathcal{P}(\kappa_i)$, and that $t$ is $\Sigma_0^{(n_i)}$ and cofinal.

We should like to compare $M^*$ with $\tilde{M}$ but for this we need the following Claim.

**Claim 1** $\tilde{M}$ is normally iterable above $\kappa_i$.

**Proof:** First note:

(i) $M^*$ and $\overrightarrow{K}$ agree up to $\nu_i$, hence if $E_i$ is the extender sequence on $M_i$ we have that $\pi \upharpoonright J_{E_i}^{E_i} : J_{E_i}^{E_i} \rightarrow J_{E_i}^{E_i^{\kappa_i}}$ cofinally for $\nu_i = \sup \pi_*(\nu_i)$.

(ii) $cf(\nu_i) > \omega$ and hence we have a canonical extension $\pi^* \supset \pi \upharpoonright J_{E_i}^{E_i}$ with $\pi^* : M^* \rightarrow M'$ with $\omega \rho_{M_i}^{n+1} \leq \kappa_i < \omega \rho_{M_i}^n$, implying that $\omega \rho_{M_i}^{n+1} \leq \tilde{\kappa} < \omega \rho_{M_i}^n$. $M'$ sound above $\tilde{\kappa}$, and $\pi^* \sum_{n_i}$ preserving.

Proof: Note that $cf(\nu_i) = cf(\nu_i^{M_i}) > \omega$ since otherwise we have that $\kappa_i^{M_i}$ is a $\overrightarrow{K}$ cardinal, which $H$ will think, by Weak Covering, has uncountable cofinality equal to some $\beta_i$. As $cf(\beta_i) = \omega_i$ it would be a contradiction to have $cf(\nu_i) = \omega$. By the definition of $\zeta$ we have that $\omega \rho_{M_i}^{n+1} \leq \kappa_i < \omega \rho_{M_i}^n$, for some $n$ and that $M^*$ is sound above $\kappa_i$. Consequently $\nu_i$ is definably singularize over $M^*$ and we have the right conditions to apply 2.11 with the other properties mentioned following from that.

Q.E.D.

(iii) $\tilde{\kappa}$ a $K$-cardinal, $\omega \rho_{M_i}^{n+1} \leq \tilde{\kappa}$, and $M'$ sound above $\kappa'$ imply that $M'$ is an initial segment of $\tilde{K}$.

Applying the full measure $E \equiv_{\pi} \tilde{E}$ yields $\sigma : K \rightarrow \tilde{K}$. Let $\tilde{M}' = \sigma(M')$, and this is also an initial segment of $\tilde{K}$. As $\pi^* \supset \pi \upharpoonright J_{E_i}^{E_i}$ we have:

(iv) $X \in E_N^{\kappa_i} \rightarrow \pi^*(X) = \pi(X) \in \tilde{E}$.

Defining $\mathcal{D}(M^*, E_N^i)$ the term model for the ultrapower we have:

(v) (a) The map $d([f]) = \sigma \circ \pi^*(f)(\tilde{\kappa})$ is a structure preserving map $d : \mathcal{D}(M^*, E_N^i) \rightarrow \tilde{M}'$. (a) The map $k : \tilde{M} \rightarrow \tilde{M}'$ is $\Sigma_0^{(n_i)}$-preserving with $k(\kappa_i) = \tilde{\kappa}$.

Proof: This is a standard computation for (a), and for (b) note that $\omega \rho_{M_i}^{n+1} \leq \sigma(\tilde{\kappa}) < \omega \rho_{M_i}^n$, by (ii) and the elementarity of $\sigma$.

Q.E.D.

By (v)(b) since $\tilde{M}'$ is normally iterable above $\tilde{\kappa}$ $\tilde{M}$ will be normally iterable above $\kappa_i$, as required.

Q.E.D. **Claim 1**.
Claim 2 $E_{\nu_i}^N = E_{\nu_i}^{M^*}$.

Proof: Since $M^*$ and $\tilde{M}$ agree up to $\nu_i$, the coiteration of these two is above $\kappa_i$. By Claim 1 this coiteration is successful with iterations $i : \tilde{M} \rightarrow \tilde{M}_i$ and $j : M^* \rightarrow M^*_i$ say.

(vi) The iteration $i$ of $\tilde{M}$ is above $(\kappa_i^+)\tilde{M} = (\kappa_i^+)M^*$.

Proof: $K, M^*, \tilde{M}$ all agree up to $\nu_i$ and forming $\tilde{W} = \text{Ult}(jE_{\nu_i}^{M^*}, E_{\nu_i}^N)$ we see therefore that it is an initial segment of $\tilde{M}$. From coherence of our extender sequences we know that

$$E_{\tilde{M}}^i | \nu_i = E_{\tilde{M}}^i \upharpoonright \nu_i = E_{\nu_i}^{M^*} \upharpoonright \nu_i \quad \text{and} \quad E_{\nu_i}^{\tilde{M}} = \emptyset = E_{\nu_i}^{\tilde{W}}.$$  

By the initial segment property of extender sequences we have that there are no further extenders on the $E_{\tilde{M}}^i$ sequence with critical point $\kappa_i$. Hence all critical points used in forming the iteration map $i$ are above $(\kappa_i^+)\tilde{M}$.

Q.E.D. (vi)

The rest of the argument is fairly standard.

(vii) $M_\theta = M^*_\theta$.

Proof: Let $A \in \Sigma_1^n(M^*)$ in $p_{M^*}$ be such that $A \cap \kappa_i \notin M^*$, and then note that $A \cap \kappa_i \notin \tilde{M}$ as they agree about subsets of $\kappa_i$. Hence if the iteration $j$ is simple, then $M^*_j$ is not a proper initial segment of $M_\theta$. But if $j$ is non-simple then we reach the same conclusion as no proper initial segment of $\tilde{M}_\theta$ can be unsound. Hence $\tilde{M}_\theta$ is an initial segment of $M^*_\theta$. But again we cannot have that it is a proper initial segment, since using the $\Sigma_0^n$ preservation property of $t$ we’d have $A \cap \kappa_i$ in $M^*_\theta$ a contradiction as before.

Q.E.D. (vii)

(viii) (i) $\omega \rho_{\tilde{M}}^{n+1} = \omega \rho_{M^*}^{n+1} = \omega \rho_{M^*_\theta}^{n+1}$.

(ii) If $p = p_{M^*} \setminus \omega \rho_{M^*}^{n+1}$ then $i \circ t(p) = p_{M^*_\theta}^{n+1}$.

(iii) $t$ is $\Sigma^*$-preserving.

Proof: These are standard arguments from the proof of solidity for mice - cf. [17] p153-4. In (ii) one first sees that $i \circ t(p) \in P_{M^*_\theta}^{n+1}$; a solidity argument on witnesses $W_{M^*_\theta}$ shows that in fact $i \circ t(p) = p_{M^*_\theta}^{n+1}$.

(ix) $j \upharpoonright \kappa = id = i \circ t \upharpoonright \kappa$; however $\text{crit}(j) = \kappa_i$.

Proof: As the first clause is immediate, we argue that $j(\kappa_i) > \kappa_i$. As $j$ is an iteration map $j(p) \in P_{M^*_\theta}^{n+1}$. By the Dodd-Jensen Lemma (cf. [17] Theorem 4.3.9) $j(p) \preceq i \circ t(p)$, and hence by (8)(ii) we have $j(p) = i \circ t(p)$.

By the soundness of above $\kappa_i$ we have that $\kappa = h_{M^*_\theta}^{n+1}(i, \xi, p)$ for some $i < \omega$, some $\xi < \kappa_i$. Hence $j(\kappa_i) = h_{M^*_\theta}^{n+1}(i, \xi, j(p))$. As $j(p) = i \circ t(p)$ we have
\[ j(\kappa_i) = i \circ t(\tilde{h}_M^{n+1}(i, \xi, p)) = i \circ t(\kappa_i) > \kappa_i. \]

Q.E.D. (ix)

Hence \( \kappa_i \) is the first point moved by \( j \) and thus some measure \( E_1^{M^*} \) is applied as the first ultrapower on the \( M^* \) side of the coiteration with \( \text{crit}(E_\gamma^{M^*}) = \kappa_i \) and \( \gamma \) least with \( E_\gamma^{M^*} \neq E_{\tilde{h}}^{M} \). As \( E_1^{M^*} \upharpoonright \nu_e = E_{\tilde{h}}^{M} \upharpoonright \nu_e \) and (see the proof of (vi)) \( E_{\nu_e}^{M} = \emptyset \) we must have \( \gamma = \nu_i \) here. But then

\[ X \in E_{\nu_i}^{M^*} \iff \kappa_i \in j(X) \iff i \circ t(X) \iff \kappa_i \in t(X) \iff X \in E_{\nu_i}^{N}. \]

Hence \( E_{\nu_i}^{N} = E_{\nu_i}^{M^*} \) which is our Claim 2.

Q.E.D. (2)

At the \( \theta \)th stage therefore, \( M_\theta \) is an end extension of \( K \). For \( n < \omega \), let \( i_n \) be the least stage \( i \) where \( \kappa_i \geq \beta_n \) if such an \( i \) exists, otherwise set \( i_n = \theta \). Let \( k_0 < \omega \) be the least \( k \) such that any truncations performed on the \( K \) iteration have been performed before stage \( i_k \). We may also assume that from this point \( i_k \) on, then, that the least \( m > 0 \) with \( \omega \rho_M^m < \kappa_i \) is fixed for all \( \tau \geq i_k \); for this \( m \) then, we set \( \rho = \omega \rho_M^m \) for any \( \iota \geq i_k \), and we shall have that any \( M_i \) is sound above \( \kappa_i \) for \( \iota \geq i_k \), and thus that \( M_i = \tilde{h}_M^m(\kappa_i \cup \{ p_M \}) \). Further by choice of \( m \) note that for \( n > k_0, \rho_M^{m+1} > \kappa_n \geq \beta_n \). As we have in the iteration that \( \pi_{i,j}(\langle d_M, p_M \rangle) = \langle d_M, p_M \rangle \), and parameters are finite sequences, we may further assume that \( k_0 \) has also been chosen sufficiently large so that for any \( n \geq k_0 \): (i) \( d_{M_n} \cap [\beta_n-1, \beta_n) = \emptyset \), (ii) \( k_0 \) is itself relevant.

(3) Suppose \( \langle \kappa_i \mid i < i_n \rangle \) is unbounded in \( \beta_n \), where \( n \) is relevant. Then for no \( i_0 < i_n \) do we have \( \pi_{i_0,i}(\kappa_{i_0}) = \kappa_j \) for unboundedly many \( \kappa_i < \kappa_{i_0} \).

Proof: If the conclusion failed then we should have \( \pi_{i,j}(\kappa_i) = \kappa_j \) for an \( \omega_1 \)-sub-sequence of the sequence of critical points \( \langle \kappa_i \mid i < i_n \rangle \); let us choose such an \( \omega_1 \)-sub-sequence, and call the set of its elements \( \overline{D} \) with the choice of \( \overline{D} \) ensuring that \( \overline{D} \) is closed below \( \beta_n \). These are all inaccessible in \( K \). Applying \( \pi \), if we set \( D = \pi^{\omega_1}\overline{D} \), then we have that \( D \) is a cub set of order type \( \omega_1 \) below \( \beta_n^\ast \) of \( K \)-inaccessibles. Note that \( \pi \) is continuous on \( \overline{D} \) since \( \overline{H} \) is correct about whether any ordinal \( \alpha \) has cofinality \( \omega \) or not, since all the \( \beta_n(n < \omega) \) have uncountable cofinality; hence, easily, if \( \kappa_\lambda \) is a limit point of \( \overline{D} \), then it has cofinality \( \omega \) in \( \overline{H} \). If \( f : \omega \rightarrow \kappa_\lambda \) is the least function in \( \overline{H} \) witnessing this, then \( \pi(\kappa_\lambda) = \pi(\sup\{ f^n(\omega) \}) = \sup\{ \pi(f(n)) \mid n \in \omega \} \). (We are using here that the MS property is formulated using all the \( S_n \)'s and not just a subsequence.) But \( n \) is relevant so \( \beta_n^\ast \) is singular in \( K \), but of uncountable cofinality. Thus the closed \( C_{\beta_n^\ast} \) sequence of \( K \) of \( K \)-singular ordinals, has non-empty intersection with \( D \), which is absurd.

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(4) If \( n \geq k_0 \) is relevant then \( \beta_n \) is \( \Sigma_1^{(m)} \) singularised over \( M_{i_n} \) and the latter is sound above \( \beta_n \).

Proof: The last conjunct follows from the definition of \( i_n : M_{i_n} \) is sound above \( \tilde{\kappa} = \delta \sup \langle \kappa_i | i < i_n \rangle \). Divide into the two cases of \( \tilde{\kappa} < \beta_n \) or \( \tilde{\kappa} = \beta_n \). In the first case then \( M_{i_n} = h_{M_{i_n}}(\tilde{\kappa} \cup \{ p_{M_{i_n}} \}) \) and hence \( \beta_n \) is so singularised over \( M_{i_n} \); in the second case take \( \delta < \beta_n \), \( \delta \geq \omega p_{M_{i_n}} \). Take \( i \) minimal such that \( \kappa_i \in [\delta, \beta_n) \).

Then \( M_i = h_{M_i}(\delta \cup \{ p_{M_i} \}) \) and in particular \( \kappa_i \in h_{M_i}(\delta \cup \{ p_{M_i} \}) \). By (3) take \( \gamma < \beta_n \) such that whenever \( \kappa_j \in (\gamma, \beta_n) \) then \( \kappa_j \neq \pi_{ij}(\kappa_i) \), and take some index \( j \) such that \( \kappa_j \in (\gamma, \beta_n) \). By elementarity, \( \pi_{ij}(\kappa_i) \in h_{M_j}(\delta \cup \{ p_{M_j} \}) \). Since \( \kappa_j > \pi_{ij}(\kappa_i) \), the point \( \pi_{ij}(\kappa_i) \) is not moved in the further iteration past stage \( j \), and so \( \pi_{ij}(\kappa_i) \in h_{M_m}(\delta \cup \{ p_{M_m} \}) \). We thus have that

\[
(*) \quad \rho > \alpha_{\beta_n} = \delta \max \{ \alpha | \sup(h_{M_{i_n}}(\alpha \cup \{ p_{M_{i_n}} \}) \cap \beta_n) = \alpha \}.
\]

But now there must be some \( \gamma < \beta_n \) with \( \sup(h_{M_{i_n}}(\gamma \cup \{ p_{M_{i_n}} \}) \cap \beta_n) = \beta_n \).

Because if this failed we could choose a sequence

\[
\gamma_0 = \rho, \gamma_{i+1} = \sup(h_{M_{i_n}}(\gamma_i \cup \{ p_{M_{i_n}} \}) \cap \beta_n) < \beta_n, \text{ and take } \gamma = \sup \gamma_i.
\]

As \( cf(\beta_n) > \omega, \gamma < \beta_n \). However we have then that

\[
\gamma = \sup(h_{M_{i_n}}(\gamma \cup \{ p_{M_{i_n}} \}) \cap \beta_n) < \beta_n
\]

and simultaneously \( \gamma > \alpha_{\beta_n} \). Contradiction! (4) is thus proven. Q.E.D.(4)

(5) If \( n \) is relevant, then in the notation of (4), if \( m > 1 \) then for no smaller \( m' < m \) is \( \beta_n \) \( \Sigma_1^{(m'-1)} \) singularised over \( M_{i_n} \).

Proof: Just note that as \( \rho_{M_{i_n}}^{m-1} \geq \rho_{M_{i_n}}^{m_n-1} > \beta_n \), any purported \( \Sigma_1^{(m'-1)} \) singularisation over \( M_{i_n} \) yields a cofinalising function in \( M_{i_n} \). This is absurd as \( \beta_n \) is regular in \( M_{i_n} \). Q.E.D.(5)

We thus have, by (4), that for relevant \( n \), \( s_n = \langle \beta_n, M_{i_n} \rangle \in S^+ \). We therefore have \( C_{s_n} \) sequences associated to such \( s_n \) as in the Global \( \Box \) proof of the previous section.

(6) For relevant \( n \geq k_0 \), we have \( cf(C_{s_n}) \leq \beta_\bar{n} \) where \( \beta_\bar{n} \) is the least p.r. closed ordinal above \( \beta_{k_0} \).

Proof: Set \( i = i_{k_0}; j = i_n \). Then by the usual property of ultrapowers \( \pi_{i,j}^{\omega p_{M_j}^{m-1}} \) is cofinal in \( \omega p_{M_j}^{m-1} \).
Set \( s = s_{k_0} \) and let \( \delta = \delta_{k_0} \) be least such that \( \lambda(f(\delta,0,s)) = \beta_{k_0}(= \nu_s) \) where \( f(\delta,0,s) \mapsto s \). Then \( \delta < \beta_{k_0} \). Let \( Y = \text{ran}(f(\delta,0,s)) \). As \( \text{ran}(f(\delta,0,s)) \) is a \( \Sigma^1_1 \) hull in \( M_\delta (= M_\delta) \) we have that \( Y \) is a \( \Sigma^1_1 \) hull in \( M_\delta (= M_{s_n}) \). We note that \( \alpha_s, \alpha_{s_n} \) (in the sense of Definition 3.15) are below \( \rho \) by \((*)\) of (4). Consequently if we define \( \tilde{Y} = \text{ran}(f(\beta_{k_0}+1,0,s_n)) \) then \( \tilde{Y} \) is a \( \Sigma^1_1 \) hull of \( M_j \). However \( \tilde{Y} \supseteq Y \), as \( \pi_{i,j}(p_s, d_s) = p_{s_n}, d_{s_n}, \pi_{i,j} \) is \( \Sigma^1_1 \)-preserving, and \( \pi_{i,j} \restriction \beta_{k_0} = \text{id} \). (We need Lemma 2.8 here on the preservation of the \( d_s \) parameters under iteration.)

By choice of \( \delta \) and Lemma 3.12 \( \rho(f(\delta,0,s)) = \omega \rho_s \). Hence \( Y \) is cofinal in \( \omega \rho_{s_n} \). However then \( \tilde{Y} \) is also so cofinal. That is \( \rho(f(\beta_{k_0}+1,0,s_n)) = \omega \rho_{s_n} \) which again by Lemma 3.12 implies \( \lambda(f(\beta_{k_0}+1,0,s_n)) = \nu_{s_n} = \beta_n \). By Lemma 3.37 this implies \( \text{ot}(C_{s_n}) \leq \beta_n \).

Q.E.D.(6)

For relevant \( n \) we form the “lift-up” map \( \pi^*_n : M_n \rightarrow M^*_n \) which extends \( \pi \restriction (\langle K \rangle | \beta^*_n) \) (where \( \beta^*_n = (\beta^*_n)^K \)). We obtain the structure \( M^*_n \) and the map \( \pi^*_n \) as a pseudo-ultrapower.

(7)(a) For relevant \( n, \pi^*_n \) is \( \Sigma^1_1 \)-preserving, and \( \beta^*_n \) is \( \Sigma^1_1 \)-singularised over \( M^*_n \); further, if \( m > 1 \), then for no smaller \( m' < m \), is \( \beta^*_n \) is \( \Sigma^1_{m-1} \)-singularised over \( M^*_n \).

(b) \( M^*_n \) is normally iterable above \( \beta^*_n \).

Proof : (a) The Pseudo-Ultrapower Theorem 2.11 (with \( k = m - 1 \)) shows the right degree of elementarity of \( \pi^*_n \), i.e. that it is \( \Sigma^1_0 \) preserving. It further states that the map is cofinal and thus \( \Sigma^1_{m-1} \)-preserving, and that it yields that \( \beta^*_n \) is \( \Sigma^1_{m-1} \)-singularised over \( M^*_n \), whilst \( \beta^*_n \) is \( \Sigma^1_{m-1} \)-regular over \( M^*_n \) for any \( m' < m \) (if \( m > 1 \)). For (b) this is a standard argument about canonical extensions defined from pseudo-ultrapowers using the fact that \( cf(\beta_n) = cf((\beta_n)^{M_n}) > \omega \). (Note \( cf((\beta_n)^K) = \omega_1 \), either because \( (\beta^*_n)^K = (\beta_n)^H \) and \( H \) is the weak covering lemma inside \( H \cdot H \mapsto \omega \), or otherwise by applying the Weak Covering Lemma inside \( H \cdot H \mapsto \omega \).) See [17] Lemma 5.6.5.

Q.E.D.(7)

(8) \( M^*_n \) is an initial segment of \( K \).

Proof: Note that by construction \( M^*_n \restriction \beta^*_n = K \restriction \beta^*_n \). By 7(i) \( \rho^m_{M^*_n} \leq \beta^*_n \); again the pseudo-ultrapower construction shows \( M^*_n \) is sound above \( \beta^*_n \) and hence is coded by a \( \Sigma^1_{m-1}(M^*_n) \) subset of \( \beta^*_n \), \( A \) say. An elementary iteration and comparison argument shows that, when \( K \) is compared with \( M^*_n \), to models \( N_\eta, M^*_n \)
then $A$ is $\Sigma^1_{m-1}$ definable over $N_{\eta}$, and thus is in $K$ itself. As $M_n^*$ is a mouse in $K$, its soundness above $\beta_n^*$ implies that after any supposedly necessary coiteration, we must have $N_1 = M_n^*$ and hence $\text{core}(N_1) = \text{core}(N_{\eta}) = \text{core}(M_n^*) = M_n^*$. Hence $M_n^*$ is an initial segment of $K$.

Q.E.D.(8)

(9)(a) $s_n^* = \langle \beta_n^*, M_n^* \rangle \in S^+$;

(b) $M_n^*$ is the assigned $K$-singularising structure for $\beta_n^*$; hence in $K$, $C_{\beta_n^*}$ is defined over $M_n^*$, that is $C_{\beta_n^*} = d C_{s_n^*}$.

Proof: For (a), by (7)(a) $M_n^*$ singularises appropriately, it is sound above $n$, and by (8) it is a mouse. For (b) we have shown that $M_n^*$ is an initial segment of $K$, and thus conforms to the definition of the segment chosen to define the canonical $C$-sequence associated to $\beta_n^*$ in $K$. Q.E.D.(9)

We thus conclude:

(10) For relevant $n \geq k_0$, $ot(C_{\beta_n^*}) \leq \pi(\beta) < \pi(\beta_{k_0+1}) = \omega_{k_0+1}$.

Proof: By (6) $ot(C_{s_n}) \leq \tilde{\beta}$ because $h_{s_n}(\beta_{k_0} + 1, p(s_n))$ is cofinal in $\omega_{s_n} = \omega_{M_n^*}^{m-1}$. Set $\beta' = \pi_n(\beta_{k_0} + 1)$. By the $\Sigma^1_{m-1}$-elementarity of $\pi_n^*$ we shall have that $\pi_n^* \cdot h_{s_n}(\beta_{k_0} + 1, p(s_n)) \subseteq h_{s_n^*}(\beta', p(s_n^*))$. As $\pi_n^* \cdot \omega_{s_n}$ is cofinal into $\omega_{s_n^*}$, we deduce that $\rho(f(\beta', 0, s_n^*)) = \omega_{s_n^*}$. By Lemma 3.12 this ensures that $\lambda(f(\beta', 0, s_n^*)) = \nu_n^* = \beta_n^*$. This in turn implies by Lemma 3.37, that $ot(C_{s_n^*})$ is less than the least p.r. closed ordinal greater than $\beta'$. However $\pi_n^* \cdot \beta_n^*$ extends $\pi \cdot \beta_n^*$, and thus this ordinal is $\pi(\beta)$. The final inequality is clear.

Now (10) yields the final contradiction, as for relevant $n$, $S_n$ was chosen to consist of points $\beta$ where $ot(C_{\beta}) \geq \omega_{n-3}$, whereas (10) establishes an ultimate bound on such order types of $\omega_{k_0+1}$. Q.E.D.(Theorem 4.2)

We finally remark that the Corollary 1.5 is immediate: after shifting our attention to cardinals above $\kappa_k$ we still use the same hypothesis concerning sufficient singular ordinals in $K$ in order to establish the stationarity of the $T_n$ now contained in $\text{Cof}(\omega_k)$. We take $\omega_k \subseteq X$ and now the analogues of the ordinals $\beta_n$ have cofinality $\omega_k$: $H$ is correct about the cofinality of any ordinal whose $V$-cofinality is less than $\omega_k$. The proof of (3) now shows that there is no closed $\omega_k$ subsequence of critical points $\kappa_i$ unbounded in such a $\beta_n$, as the map $\pi$ is now continuous at points of cofinality less than $\omega_k$. Hence we can deduce (4) that the iterates are indeed singularizing structures for the $\beta_n$ as required.
References


