# Degrees of subsets of the measurable cardinal in Prikry generic extensions

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**Abstract**

We prove that in a Prikry generic extension for a measure $U$ on $\kappa$ every subset of the measurable cardinal $\kappa$ in the Prikry extension is constructibly equivalent, over the ground model, to a subsequence of the Prikry sequence.

## Introduction

Generic extensions by Cohen or Solovay-random reals display a rather amorphous structure of the constructibility degrees over the ground universe. Some other extensions, notably Sacks forcing and its iterations, allow to control the structure of the constructibility degrees at least to some extent, see, for instance, [1]. In this note, we study the degrees of $\mathbf{V}$-constructibility in Prikry extensions of the ground model $\mathbf{V}$ with a measurable cardinal $\kappa$. The partial order of $\mathbf{V}$-constructibility is defined by: $X \leqslant_{\mathbf{V}} Y$ iff $X \in \mathbf{V}[Y]$; in terms of the Gödel constructibility, $X \leqslant_{\mathbf{V}} Y$ means that there is a set $z \in \mathbf{V}$, $z \subseteq \mathtt{Ord}$, such that $X \in \mathbf{L}[z, Y]$. The equivalence $X \equiv_{\mathbf{V}} Y$ means that both $X \leqslant_{\mathbf{V}} Y$ and $Y \leqslant_{\mathbf{V}} X$.

The Prikry forcing [3] produces a generic cofinal function $h : \omega \to \kappa$. Our main result says that every subset of $\kappa$ in the Prikry extension is $\mathbf{V}$-constructibly equivalent to a subsequence of $h$.

**Theorem 1** (the main theorem)**.** *Suppose that $h : \omega \to \kappa$ is Prikry-generic over the ground model $\mathbf{V}$. Then in the Prikry extension $\mathbf{V}[h]$ of $\mathbf{V}$ for every set $X \subseteq \kappa$ there exists a set $d \subseteq h$ satisfying $X \equiv_{\mathbf{V}} d$.*

*In addition, in $\mathbf{V}[h]$, if $c, c' \subseteq h$ then $c' \leqslant_{\mathbf{V}} c$ iff $c' \smallsetminus c$ is finite.*

We give two very different proofs of the main theorem. The first proof (sections 1 – 6) is combinatorial, based on indiscernible subsets of $\kappa$. It makes use of a representation of subsets of $\kappa$ in the Prikry extension by means of certain functions defined on $[\kappa]^{\text{fin}}$ in the ground universe. This is similar to some extent

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to the analysis of degrees of constructibility in iterated Sacks extensions, but the mechanism is absolutely different.

The second proof (sections 7 – 9) is model-theoretic, using iterated ultrapowers of the ground model.

## 1 Good indiscernible sets

A normal ultrafilter $U$ on a measurable cardinal $\kappa$ is fixed thoughout the paper.

Recall that a set $\varnothing \neq I \subseteq \kappa$ is an *indiscernible set* w.r.t. a family of sets $\mathscr{F}$ iff for any set $B \in \mathscr{F}$ and any $n \geq 1$ either every $s \in [I]^n$ belongs to $B$ or every $s \in [I]^n$ does not belong to $B$. It is known since the early 1960s that if $\kappa$ and $U$ are as above then for any family $\mathscr{F}$ of cardinality $< \kappa$ there exists a set $I \in U$ which is an indiscernible set w.r.t. $\mathscr{F}$. (The Rowbottom theorem.) We employ this basic result to find a slightly more convenient type of indiscernible sets. $\overleftarrow{\text{ref?}}$

**Proposition 2.** *Suppose that* $\mathscr{F}$ *is a family of cardinality* $< \kappa$.

*Then there exists a* **good set of indiscernibles** $I \in U$ w.r.t. $\mathscr{F}$, *that is, for any* $n \geq 1$, *any* $B \in \mathscr{F}$ *and any sets* $a \in [\kappa]^{\text{fin}}$ *and* $x, y \in [I]^n$, *if* $\mathtt{max}\, a < \mathtt{min}\, x$ *and* $\mathtt{max}\, a < \mathtt{min}\, y$ *then* $a \cup x \in B \Longleftrightarrow a \cup y \in B$.

Note that the ordinals in $a$ are not assumed to be members of the set $I$.

**Proof.** For any ordinal $\alpha < \kappa$ the family $\mathscr{F}_\alpha$ of all sets of the form

$$\{x \in [\kappa]^n : a \cup x \in B \wedge \mathtt{max}\, a \leqslant \alpha < \mathtt{min}\, x\}, \qquad \text{where } a \in [\alpha]^{\text{fin}} \text{ and } B \in \mathscr{F},$$

has cardinality $< \kappa$. Therefore there exists a set $I_\alpha \in U$, $I_\alpha \subseteq \kappa \smallsetminus \alpha$ that is an indiscernible set w.r.t. $\mathscr{F}_\alpha$. Consider the diagonal intersection $I = \mathop{\boldsymbol{\Delta}}_{\alpha<\kappa} I_\alpha$ of these sets $I_\alpha$. Thus $\xi \in I$ iff $\xi > 0$ and $\xi \in \bigcap_{\alpha<\xi} I_\alpha$. By the normality of $U$, $I$ still belongs to $U$. To check the good indiscernibility, let $n, B, a, x, y$ be as in Proposition 2. Then $\mu = \mathtt{max}\, a < \mathtt{min}\, x$, and hence $x \subseteq I_\mu$ by the definition of $I$. Similarly $y \subseteq I_\mu$. Then $a \cup x \in B \Longleftrightarrow a \cup y \in B$ for any $B \in \mathscr{F}$ by the choice of $I_\mu$. (If $a = \varnothing$ then we take $\mu = 0$ in this argument.) $\square$

## 2 Canonization of functions

The next theorem will be our main technical tool. For any sets $x, s \subseteq \mathtt{Ord}$ we define $x \,/\!\!/\, s \subseteq x$ as follows. Put elements of $x$ in the increasing order: $x = \{\xi_\gamma : \gamma < \delta\}$. Now define $x \,/\!\!/\, s = \{\xi_\gamma : \gamma \in s\}$. Note that if $y \subseteq x \in [\mathtt{Ord}]^n$ then there is a unique set $s \subseteq n$ such that $y = x \,/\!\!/\, s$.

**Theorem 3.** *Suppose that* $F$ *is a function defined on* $[\kappa]^{\text{fin}}$. *Then for any* $n \geq 1$ *and* $a \in [\kappa]^{\text{fin}}$ *there exist sets* $J_n(a) \in U$ *and* $\mathtt{bas}_n(a) \subseteq n$ *such that for all* $x, y \in [J_n(a)]^n$ *with* $\mathtt{max}\, a < \mathtt{min}\, x, \mathtt{min}\, y$ *we have* $F(a \cup x) = F(a \cup y)$ *if and only if* $x \,/\!\!/\, \mathtt{bas}_n(a) = y \,/\!\!/\, \mathtt{bas}_n(a)$.

**Proof.** Let $\vartheta$ be any cardinal bigger than $\kappa$ such that $\mathbf{V}_\vartheta$ (the $\vartheta$-th level of the von Neumann hierarchy) contains $F$. Let $\mathscr{F}$ be the collection of all sets $z \in \mathbf{V}_\vartheta$ definable in $\mathbf{V}_\vartheta$ by an $\in$-formula with $F$ as the only parameter; $\mathscr{F}$ is countable. Let $I \in U$ be given by Proposition 2 for such an $\mathscr{F}$.

We prove the theorem by induction on $n$.

Suppose that $n = 1$. Fix $a \in [\kappa]^{\text{fin}}$. Put $J_1(a) = I$. Take any $\xi \neq \eta \in I$ bigger than $\mathtt{max}\, a$. If $F(a \cup \{\xi\}) = F(a \cup \{\eta\})$ then by the choice of $I$ we have $F(a \cup \{\xi\}) = F(a \cup \{\eta\})$ for *every* pair of $\xi \neq \eta \in I$. (Indeed take the set

$$\{a \cup \{\xi, \eta\} : a \in [\kappa]^{\text{fin}} \wedge \mathtt{min}\, a < \xi, \eta < \kappa \wedge F(a \cup \{\xi\}) = F(a \cup \{\eta\})\}$$

as $B$ in Proposition 2.) Therefore $\mathtt{bas}_1(a) = \varnothing$ is as required. If $F(a \cup \{\xi\}) \neq F(a \cup \{\eta\})$ then similarly $\mathtt{bas}_1(a) = \{0\}$ works.

Now the induction step $n \to n + 1$. The idea is to reduce the level $n + 1$ to $n$ for bigger sets $a$. Fix $a \in [\kappa]^{\text{fin}}$. Take any $\xi < \kappa$, $\mathtt{max}\, a < \xi$. By the induction hypothesis there exist sets $J_n(a \cup \{\xi\}) \in U$ and $\mathtt{bas}_n(a \cup \{\xi\}) \subseteq n$ such that

$$F(a \cup \{\xi\} \cup x) = F(a \cup \{\xi\} \cup y) \quad \text{iff} \quad x \mathbin{/\!\!/} \mathtt{bas}_n(a \cup \{\xi\}) = y \mathbin{/\!\!/} \mathtt{bas}_n(a \cup \{\xi\})$$

holds for any pair of sets $x, y \in [J_n(a \cup \{\xi\})]^n$ with $\xi < \mathtt{min}\, x, \mathtt{min}\, y$.

Obviously there exist sets $J \in U$ and $s \subseteq n$ such that $\mathtt{max}\, a < \mathtt{min}\, J$ and $\mathtt{bas}_n(a \cup \{\xi\}) = s$ for all $\xi \in J$. The set $J' = I \cap J \cap \underset{\gamma \in J}{\mathbf{\Delta}} J_n(a \cup \{\gamma\})$ [1] belongs to $U$ since $U$ is a normal filter. Moreover we have

$$F(a \cup \{\xi\} \cup x) = F(a \cup \{\xi\} \cup y) \quad \text{iff} \quad x \mathbin{/\!\!/} s = y \mathbin{/\!\!/} s \tag{1}$$

for any $\xi \in J'$ and any pair of sets $x, y \in [J']^n$ with $\xi < \mathtt{min}\, x, \mathtt{min}\, y$.

We put $J_{n+1}(a) = J'$. To define $\mathtt{bas}_{n+1}(a)$, take any $\alpha \neq \gamma \in J'$ bigger than $\mathtt{max}\, a$. Also take any $z \in [J']^n$ with $\mathtt{min}\, z > \alpha, \gamma$.

*Case 1*: $F(a \cup \{\alpha\} \cup z) = F(a \cup \{\gamma\} \cup z)$. We show that $\mathtt{bas}_{n+1}(a) = 1 + s = \{1 + k : k \in s\}$ works. Take any $x', y' \in [J']^{n+1}$ with $\xi = \mathtt{min}\, x' > \mathtt{max}\, a$ and $\eta = \mathtt{min}\, y' > \mathtt{max}\, a$. Then $x = x' \smallsetminus \{\xi\}$ and $y = y' \smallsetminus \{\eta\}$ belong to $[J']^n$.

Suppose, for instance, that $\eta \leqslant \xi$. Then still $\eta < \mathtt{min}\, x$. Therefore, by the case assumption, the choice of $I$, and the fact that $J' \subseteq I$, the equality $F(a \cup x') = F(a \cup x'')$ holds, where $x'' = \{\eta\} \cup x$. Further by (1) $F(a \cup y') = F(a \cup x'')$ iff $x \mathbin{/\!\!/} s = y \mathbin{/\!\!/} s$. And finally the equality $x' \mathbin{/\!\!/} \mathtt{bas}_{n+1}(a) = y' \mathbin{/\!\!/} \mathtt{bas}_{n+1}(a)$ is equivalent to $x \mathbin{/\!\!/} s = y \mathbin{/\!\!/} s$.

*Case 2*: $F(a \cup \{\alpha\} \cup z) \neq F(a \cup \{\gamma\} \cup z)$. A pretty similar argument shows that setting $\mathtt{bas}_{n+1}(a) = \{0\} \cup (1 + s) = \{0\} \cup \{1 + k : k \in s\}$ works. □

The case when $m = 0$ and $a = \varnothing$ is of special interest. Define $J = \bigcap_n J_n(\varnothing)$ and $\mathtt{bas}_n = \mathtt{bas}_n(\varnothing)$. Then $J$ belongs to $U$ together with all sets $J_n(a)$, and $\mathtt{bas}_n \subseteq n$. Note that the construction of sets $J$ and $\mathtt{bas}_n$ depends also on $F$ and the choice of a cardinal $\vartheta$, and a set $I \in U$ in accordance with Proposition 2.

**Corollary 4.** *For all $n$ and $x, y \in [J]^n$ we have the equivalence $F(x) = F(y)$ iff $x \mathbin{/\!\!/} \mathtt{bas}_n = y \mathbin{/\!\!/} \mathtt{bas}_n$.* □

[1] Note that $\xi \in J'$ iff $\xi \in I \cap J$, $\xi > 0$, and $\xi \in J_n(a \cup \{\gamma\})$ for all $\gamma \in J$, $\gamma < \xi$.

## 3 Prikry extension

Recall that the Prikry forcing $\mathbb{P} = \mathbb{P}_\kappa(U)$ accociated to a normal ultrafilter $U$ on a measurable cardinal $\kappa$ consists of all pairs $p = \langle a_p, A_p \rangle$ of sets $a_p \in [\kappa]^{\text{fin}}$ and $A_p \in U$ (where $U$ is a fixed normal ultrafilter on $\kappa$) such that $\mathtt{max}\, a_p < \mathtt{min}\, A_p$. The order is as follows: $p \leqslant q$ (meaning that $p$ is stronger) iff $a_q \subseteq_{\mathtt{end}} a_p$ (meaning that $a_p$ is an end-extension of $a_q$, that is, $\mathtt{max}\, a_q < \mathtt{min}\,(a_p \smallsetminus a_q)$), $A_p \subseteq A_q$, and $a_p \smallsetminus a_q \subseteq A_q$. See [3] on the Prikry forcing.

$\mathbb{P}$-generic extensions are called *Prikry extensions*.

The proof of the following (well-known) result will be given in the next Section.

**Proposition 5.** *Suppose that* $p \in \mathbb{P}$ *and* $\varphi$ *is a closed formula of the* $\mathbb{P}$*-forcing language, possibly with* $\mathbb{P}$*-terms as parameters. Then there is a condition* $q \in \mathbb{P}$, $q \leqslant p$ *which decides* $\varphi$ *and satisfies* $a_p = a_q$. □

The following is an immediate corollary. (Use the fact that $U$ is $\vartheta$-complete for any cardinal $\vartheta < \kappa$.)

**Corollary 6.** *In a Prikry extension* $\mathbf{V}[G]$ *of the ground universe* $\mathbf{V}$, *if* $X \subseteq \alpha < \kappa$ *then* $X \in \mathbf{V}$. *In particular any* $\mathbf{V}$*-cardinal* $\vartheta < \kappa$ *remains a cardinal in the extension, with the same cofinality.*

*Moreover,* $\kappa$ *itself remains a cardinal, too, but its cofinality changes to* $\omega$.

*Finally all cardinals* $\vartheta > \kappa$ *remain cardinals, and the cofinality does not change provided it was* $> \kappa$. □

Given a set $G \subseteq \mathbb{P}$, we let $h_G = \bigcup_{p \in G} a_p$, a subset of $\kappa$. If $G$ is a generic then $h_G$ is a set of order type $\omega$, called the *Prikry sequence* associated to $G$.

By $\underline{G}$ we denote a name for the canonical generic subset of $\mathbb{P}$. Let $\underline{h}$ be a name for $h_G$. Then $\mathbb{P}$ forces that $\underline{h} \subseteq \check{\kappa}$ is a set of order type $\check{\omega}$ cofinal in $\check{\kappa}$, and $\underline{G} = \{p \in \check{\mathbb{P}} : a_p \subset_{\mathtt{end}} \underline{h} \wedge \underline{h} \smallsetminus a_p \subseteq A_p\}$. Thus $\underline{h}$ can be vieved as an increasing $\omega$-sequence cofinal in $\check{\kappa}$. Such sequences are called *Prikry sequences*.

## 4 Coding subsets of $\kappa$ in the Prikry extension

**Blanket agreement 7.** Let, in the notation of Section 3, $\underline{X}$ be a $\mathbb{P}$-name of a subset of $\check{\kappa}$. □

Our goal is to code $\underline{X}$ by a subset of the canonical Prikry sequence $\underline{h}$. We are going to find a set $I \in U$ and a $\mathbb{P}$-name $\underline{d}$ such that the condition $\langle \varnothing, I \rangle$ forces $\underline{d} \subseteq \underline{h} \wedge \underline{d} \equiv_{\mathbf{V}} \underline{X}$, where $\equiv_{\mathbf{V}}$ means equivalence over the ground model $\mathbf{V}$.

**Definition 8.** First of all define, for each $x \in [\kappa]^{\text{fin}}$,

$$F(x) = \{\xi < \kappa : \exists p \in \mathbb{P}\, (a_p = x \wedge p \Vdash \check{\xi} \in \underline{X})\}. \tag{2}$$

It follows from Corollary 4 that there exist a set $J \in U$ and a sequence $\{\mathtt{bas}_n\}_{n \in \omega}$ of sets $\mathtt{bas}_n \subseteq n$ such that the equivalence $F(x) = F(y)$ iff $x \mathbin{/\!\!/} \mathtt{bas}_n = y \mathbin{/\!\!/} \mathtt{bas}_n$ holds for all $n$ and $x, y \in [J]^n$.

Now fix a cardinal $\vartheta > \kappa$. Then $V = \mathbf{V}_\vartheta$ is a transitive set containing $\kappa$, $F$, $\mathbb{P}$, $J$, the ultrafilter $U$, the sequence $\{\mathtt{bas}_n\}_{n\in\omega}$, and the relation $p \mathrel{||\!\!-} \check\xi \in \underline{X}$ (of two arguments $p$ and $\xi$). Let $\mathscr{F}$ be the family of all subsets of $V$ definable in $V$ by an $\in$-formula with those seven sets involved as parameters. Let $I \in U$ satisfy Proposition 2 with these initial conditions. We can assume that $I \subseteq J$. $\square$

The following is the key technical instrument.

**Lemma 9.** *If* $\xi < \gamma < \kappa$, $p, q \in \mathbb{P}$, $a_p \cap \gamma = a_q \cap \gamma$, $(a_p \cup a_q) \smallsetminus \gamma \subseteq I$, $\gamma \leqslant \mathtt{min}\, A_p$, $\mathtt{min}\, A_q$ *, and* $A_p \cup A_q \subseteq I$. *Then* $p \mathrel{||\!\!-} \check\xi \in \underline{X}$ *iff* $q \mathrel{||\!\!-} \check\xi \in \underline{X}$.

*In particular if* $p$ *forces* $\check\xi \in \underline{X}$ *then so does* $q = \langle a_p \cap \gamma, I \smallsetminus \gamma\rangle$ *.*

**Proof.** Suppose this is not the case. Then (as $A_q \subseteq I$!) we can *w.l.o.g.* assume that $q$ forces $\check\xi \notin \underline{X}$. Put $a = a_p \cap \gamma = a_q \cap \gamma$. The remaining parts $y = a_q \smallsetminus a$ and $x = a_p \smallsetminus a$ are finite subsets of $I \smallsetminus \gamma$. We can assume that $|x| = |y|$ as otherwise the condition with the shorter part can be appropriately strengthened.

Now put $m = |a|$ and $n = |x| = |y|$. Consider the set $B$ of all unions of the form $\{\nu\} \cup u \cup v$ such that $\nu < \kappa$, $u \in [\kappa]^m$, $v \in [\kappa]^n$, and there is a condition $r \in \mathbb{P}$ such that $a_r = u \cup v$ and $r \mathrel{||\!\!-} \check\nu \in \underline{X}$. Then $B \in \mathscr{F}$. Moreover $\{\xi\} \cup a \cup x \in B$ is witnessed by $r = p$. It follows that $\{\xi\} \cup a \cup y \in B$ as well by the choice of $I$. (Note that $x \cup y \subseteq I$. But $a \subseteq I$ and $\xi \in I$ are not assumed.) Thus there is a condition $r \in \mathbb{P}$ with $a_r = a \cup y = a_q$ and $r \mathrel{||\!\!-} \check\xi \in \underline{X}$. Thus conditions $q, r$ with $s_q = s_r$ are incompatible. But this is a contradiction. $\square$

**Proof** (Prop. 5)**.** Define, in the Prikry extension, $\underline{X} = \{0\}$ if $\varphi$ is true, otherwise $\underline{X} = \varnothing$. Choose $I \in U$ for this particular $\underline{X}$ as in Definition 8. Consider any $p \in \mathbb{P}$. We may assume that $A_p \subseteq I$. There is a condition $q \in \mathbb{P}$, $q \leqslant p$ that decides $\varphi$, *i.e.*, either forces $0 \in \underline{X}$ or forces $0 \notin \underline{X}$. Let $\gamma = \mathtt{max}\, a_p + 1$. Then the condition $r = \langle a_q \cap \gamma, I \smallsetminus \gamma\rangle$ still decides $\varphi$ by Lemma 9. However $a_r = a_p$ . $\square$

**Lemma 10.** *For any* $\gamma < \kappa$, $p_0 = \langle \varnothing, I\rangle$ *forces* $\underline{X} \cap \check\gamma = \check F(\underline{h} \cap \check\gamma) \cap \check\gamma$ *.*

**Proof.** Fix any $\xi < \gamma$. Suppose that a condition $p \in \mathbb{P}$, $p \leqslant p_0$ forces $\check\xi \in \underline{X}$. We may *w.l.o.g.* assume that $\gamma \leqslant \mathtt{min}\, A_p$. Then $\underline{h} \cap \check\gamma$ is obviously forced by $p$ to be equal to $\check a$, where $a = a_p \cap \gamma$, and hence we have to show that $\xi \in F(a)$, that is, there exists a condition $q \in \mathbb{P}$ with $a_q = a$ which forces $\check\xi \in \underline{X}$. Yet $q = \langle a, I \smallsetminus \gamma\rangle$ is such a condition by Lemma 9.

Conversely suppose that a condition $p \in \mathbb{P}$, $p \leqslant p_0$ forces $\check\xi \notin \underline{X}$. Still assuming that $\gamma < \mathtt{min}\, A_p$ , we have to prove that $\xi \notin F(a)$, where $a = a_p \cap \gamma$. Otherwise there is a condition $q \in \mathbb{P}$ with $a_q = a$ such that $q \mathrel{||\!\!-} \check\xi \in \underline{X}$. It can be assumed that $\gamma \leqslant \mathtt{min}\, A_q$. Then Lemma 9 leads to contradiction. $\square$

For any $\alpha \in I$ let $\alpha^\dagger$ be the next element of $I$.

**Corollary 11.** *If* $p = \langle a, A\rangle \in \mathbb{P}$, $A \subseteq I$, *and* $\gamma = (\mathtt{max}\, a)^\dagger$ *then* $p$ *forces* $\underline{X} \cap \check\gamma = \check F(\check a) \cap \check\gamma$ *.* $\square$

It is interesting to figure out whether $\mathtt{bas}_n = \mathtt{bas}_k \cap n$ is true. But fortunately the result of the next lemma will suffice for our goals.

**Lemma 12.** *If* $n < k$ *then* $\mathtt{bas}_n \subseteq \mathtt{bas}_k$ .

**Proof.** It suffices to show that $F(x) = F(y)$ holds for any sets $x, y \subseteq I$ satisfying $|x| = |y| = n$ and $x /\!/ s = y /\!/ s$, where $s = \mathtt{bas}_{n+1} \cap n$. Suppose otherwise: $F(x) \neq F(y)$. Let say $\xi \in F(x) \smallsetminus F(y)$. Then there exists a condition $p \in \mathbb{P}$ with $a_p = x$ that forces $\check{\xi} \in \underline{X}$, and by Proposition 5 there is a condition $q \in \mathbb{P}$ with $a_q = y$ that forces $\check{\xi} \notin \underline{X}$. We may assume that $A_p = A_q \subseteq I$ and $\xi < \mu = \mathtt{min}\, A_p$. Then the sets $x' = x \cup \{\mu\}$ and $y' = y \cup \{\mu\}$ satisfy $|x'| = |y'| = n+1$ and $x' /\!/ \mathtt{bas}_{n+1} = y' /\!/ \mathtt{bas}_{n+1}$. It follows that $F(x') = F(y')$.

Consider conditions $p' = \langle x', A_p \smallsetminus \{\mu\}\rangle$ and $q' = \langle y', A_q \smallsetminus \{\mu\}\rangle$. Obviously $p' \leqslant p$ in $\mathbb{P}$, therefore $p'$ forces $\check{\xi} \in \underline{X}$, and then $\xi \in F(x') = F(y')$. It follows that there is a condition $r \in \mathbb{P}$ with $a_r = y'$ that still forces $\check{\xi} \in \underline{X}$. This is a contradiction because $q' \Vdash \check{\xi} \notin \underline{X}$ (indeed $q' \leqslant q$ ) and $a_{q'} = a_r = y'$. □

## 5 Getting the set from a subsequence

Put $S = \bigcup_n \mathtt{bas}_n$. It follows from Lemma 12 that for any $n$ there exists a number $k = k_n$ such that $S \cap n = \mathtt{bas}_k \cap n$ for all $k \geq k_n$, in particular $S \cap n = \mathtt{bas}_{k_n} \cap n$. Put $s_n = |\mathtt{bas}_n|$ and $\sigma_n = |S \cap n|$; these are numbers in $\omega$ and $s_n \leqslant \sigma_n$. There is a unique set $w_n \subseteq \sigma_n$ such that $\mathtt{bas}_n = (S \cap n) /\!/ w_n$. (If occasionally $\mathtt{bas}_n = S \cap n$ then $w_n = \sigma_n$, of course.)

Let $\underline{d}$ be a name for $\underline{h} /\!/ \check{S}$. This is a subsequence of $\underline{h}$ in the extension.

**Lemma 13.** *The condition* $\langle \varnothing, I\rangle$ *Prikry-forces* $\underline{X} \leqslant_{\mathbf{V}} \underline{d}$ .

**Proof.** *Arguing in the Prikry extension* $\mathbf{V}[G]$ of the ground universe $\mathbf{V}$, we define $h = h_G$, $X = \underline{X}[G]$, and $d = h /\!/ S$. We have to prove that $X \leqslant_{\mathbf{V}} d$. Say that a finite set $x \subseteq I$ is *compatible with* $d$, iff $x /\!/ S \subset_{\mathtt{end}} d$. In particular, if $x \subset_{\mathtt{end}} h$ is a finite initial segment of $h$ then $x$ is compatible with $d$ because $d = h /\!/ S$. However, *in the ground universe*, if $x, y \subseteq I$ are finite sets, $|x| = |y| = n$, and both of them are forced to be compatible with $\underline{d}$ by one and the same $p \in \mathbb{P}$ with $|a_p| \geq n$ then easily $x /\!/ S = y /\!/ S$, therefore $x /\!/ \mathtt{bas}_n = y /\!/ \mathtt{bas}_n$ because $\mathtt{bas}_n \subseteq S$, and finally $F(x) = F(y)$. Thus, by Lemma 10, $\underline{X}$ can be defined, *in the Prikry extension*, as $\bigcup_x (F(x) \cap \mathtt{max}\, x)$, where the union is taken over all finite sets $x \subseteq I$ compatible with $d$. And this witnesses $X \leqslant_{\mathbf{V}} d$ . □

The next simple lemma on subsets of $\underline{h}$ in the Prikry extension proves the additional claim of Theorem 1.

**Lemma 14.** *In the Prikry extension* $\mathbf{V}[G]$ *,*

(i) *for every* $c \subseteq h = h_G$ *there is a unique* $P \subseteq \omega$ *in* $\mathbf{V}$ *such that* $c = h /\!/ P$ *;*

(ii) *if* $c, c' \subseteq h$ *then* $c' \leqslant_{\mathbf{V}} c$ *iff* $c' \smallsetminus c$ *is finite.*

**Proof.** (i) Obviously in $\mathbf{V}[G]$ there is a unique set $P \subseteq \omega$ satisfying $c = h /\!/ P$. That it belongs to $\mathbf{V}$ follows from Corollary 6.

(ii) Let, by (i), $c = h /\!/ P$ and $c' = h /\!/ P'$, where $P, P' \subseteq \omega$ are sets in $\mathbf{V}$. Suppose on the contrary that $P' \smallsetminus P$ is infinite but a condition $p \in \mathbb{P}$ forces

$\underline{h} \,/\!\!/\, \check{P}' \leqslant_{\mathbf{V}} \underline{h} \,/\!\!/\, \check{P}$, and moreover, there exist a concrete absolute set theoretic function $f(\cdot,\cdot)$ and a set $x \in \mathbf{V}$ such that $p$ forces $\underline{h} \,/\!\!/\, \check{P}' = f(\check{x}, \underline{h} \,/\!\!/\, \check{P})$. Let $n \in P' \smallsetminus p$ be bigger than $|a_p|$. We can easily define a pair of conditions $q, r \in \mathbb{P}$, stronger than $p$ and such that $|a_q| = |a_r| > n$, $a_q \,/\!\!/\, P = a_r \,/\!\!/\, P$, but $\xi =$ the $n$th element of $a_q$ is strictly smaller than $\eta =$ the $n$th element of $a_r$. These conditions can be extended usual way to Prikry sequences with the same $\cdot \,/\!\!/\, P$ but different $\cdot \,/\!\!/\, P'$, leading to a contradiction. □

## 6 Getting the subsequence from the set

Here we prove the opposite reduction $\underline{d} \leqslant_{\mathbf{V}} \underline{X}$.

We consider a Prikry-generic extension of the form $\mathbf{V}[G]$, where $G \subseteq \mathbb{P}$ is a generic set containing $\langle \varnothing, I \rangle$.

In the extension, say that a finite set $x \subseteq I$ is *compatible* with a set $Y \subseteq \kappa$ iff $Y \cap \gamma = F(x) \cap \gamma$, where $\gamma = (\mathtt{max}\, x)^{\dagger}$. It follows from Corollary 11 that, in the Prikry extension, any finite initial segment of $h = h_G$ is compatible with the set $X = \underline{X}[G]$.

The next lemma is a warmup for a much more complicated Lemma 17 below.

**Lemma 15.** *Suppose that* $0 \in \mathtt{bas}_1$. *Then, in the extension,* $\{\mu\}$, *where* $\mu = h(0)$, *is the only* 1*-element set compatible with* $X = \underline{X}[G]$.

**Proof.** By Corollary 11 the existence of two compatible singletons leads us to the existence, in the ground universe, of a pair $\xi < \eta$ of elements of $I$ such that $F(\{\xi\}) \cap \xi^{\dagger} = F(\{\eta\}) \cap \xi^{\dagger}$.

*Case 1*: $\eta = \xi^{\dagger}$, so that in fact $F(\{\xi\}) \cap \eta = F(\{\eta\}) \cap \eta$. By the indiscernibility of $I$ this holds then for *every* pair of $\xi < \eta$ in $I$. It follows that $F(\{\xi\}) = F(\{\eta\})$ for all $\xi < \eta$ in $I$. (Indeed take any $\zeta \in I$ bigger than $\mathtt{max}\{\xi, \eta\}$. Then $F(\{\xi\}) \cap \zeta = F(\{\zeta\}) \cap \zeta = F(\{\eta\}) \cap \zeta$.) But this contradicts the assumption that $0 \in \mathtt{bas}_1$. Therefore, we have

*Case 2*: $\gamma = \xi^{\dagger} < \eta$. Still by the indiscernibility we have $F(\{\xi\}) \cap \gamma = F(\{\eta\}) \cap \gamma$ for all $\xi < \gamma < \eta$ in $I$. And once again we have $F(\{\xi\}) = F(\{\eta\})$ for all $\xi < \eta$ in $I$. (Indeed take any $\gamma < \zeta \in I$ with $\gamma > \mathtt{max}\{\xi, \eta\}$. Then $F(\{\xi\}) \cap \gamma = F(\{\zeta\}) \cap \gamma = F(\{\eta\}) \cap \gamma$.) □

**Lemma 16.** *Suppose that* $n \geq 2$, $x, y \in [I]^n$, $\mathtt{max}\, x = \mathtt{max}\, y$, *and* $F(x) \neq F(y)$. *Then* $F(x) \cap \gamma \neq F(y) \cap \gamma$ *where* $\gamma = (\mathtt{max}\, x)^{\dagger}$.

**Proof.** Otherwise by the indiscernibility of $I$ we would have $F(x) \cap \gamma = F(y) \cap \gamma$ for all $\gamma \in I$, $\gamma > \mathtt{max}\, x$. □

**Lemma 17.** *Suppose that* $n \geq 2$, $x, y \in [I]^n$, *and, in the extension,* $x, y$ *are compatible with* $X$. *Then* $x \,/\!\!/\, \mathtt{bas}_n = y \,/\!\!/\, \mathtt{bas}_n$.

**Proof.** If generally $\mathtt{bas}_n = \varnothing$ then $x \,/\!\!/\, \mathtt{bas}_n = y \,/\!\!/\, \mathtt{bas}_n$ is obvious. If $\mathtt{max}\, x = \mathtt{max}\, y = \mu$ then $F(x) \cap \mu^{\dagger} = F(y) \cap \mu^{\dagger}$ by the compatibility, hence $F(x) = F(y)$ by Lemma 16, and so we have $x \,/\!\!/\, \mathtt{bas}_n = y \,/\!\!/\, \mathtt{bas}_n$. Thus we shall assume that $\mathtt{bas}_n \neq \varnothing$ and $\mu = \mathtt{max}\, x < \mathtt{max}\, y = \nu$.

Then still $F(x) \cap \mu^\dagger = F(y) \cap \mu^\dagger$. By the indiscernibility we can assume that all elements of $x$ and $y$ have limit indices in the sense of the natural increasing order of $I$ — this allows us to move them, if necessary, without any change in their common configuration in the order of $I$. We have several cases.

*Case 1*: $\{n-1\} \notin \mathtt{bas}_n$. Then the sets $u = x \mathbin{/\!/} \mathtt{bas}_n$ and $v = y \mathbin{/\!/} \mathtt{bas}_n$ do not contain elements resp. $\mu$ and $\nu$. Thus $z = x \cup \{\nu\} \smallsetminus \{\mu\}$ satisfies $F(x) = F(z)$ but $\mathtt{max}\, z = \mathtt{max}\, y = \nu$. But then $F(y) = F(z)$ by the above, therefore $F(x) = F(y)$.

*Case 2*: $\{n-1\} \in \mathtt{bas}_n$. Then the sets $u$ and $v$ are different (since $\mu < \nu$). Here we have to obtain a contradiction. Our plan is to show that $F(x) = F(y)$.

*Case 2a*: $\mathtt{bas}_n = \{n-1\}$. Then $u = \{\mu\}$ and $v = \{\nu\}$. In this case the value of $F(y)$ does not depend on the values of ordinals in $y \smallsetminus \{\nu\}$, and hence we can assume that $\mu < \mathtt{min}\, y$. In this assumption, the same argument as in the proof of Lemma 15 shows that $F(x) = F(y)$, contrary to $u \neq v$.

*Case 2b*: the set $\mathtt{bas}_n$ contains both the number $n-1$ and at least one more element. Accordingly the sets $u$ and $v$ contain both resp. $\mu$ and $\nu$ and elements other than resp. $\mu$ and $\nu$.

*Case 2b1*: $u \smallsetminus \{\mu\} = v \smallsetminus \{\nu\}$. To prove $F(x) = F(y)$ fix any $\delta < \kappa$ and show that $F(x) \cap \delta = F(y) \cap \delta$. Let $\xi_1, \dots, \xi_k$ be the list of all elements $\xi \in y$ such that $\mu < \xi < \nu$, in the order of increase. (If there is no such $\xi$ then $k = 0$.) Note that none of $\xi_i$ is a member of $v$ by the case assumption. Put $\gamma = \mu^\dagger$. Then $F(x) \cap \gamma = F(y) \cap \gamma$ due to the compatibility of $x, y$ with $\underline{X}$. Fix any tuple $\gamma' \leqslant \xi'_1 < \dots < \xi'_k < \nu'$ of elements of $I$ such that $\mu < \gamma'$ and $\delta < \gamma'$, and in addition $\gamma' = \xi'_1$ iff $\gamma = \xi_1$. Let $y'$ be obtained from $y$ by changing of $\xi_1, \dots, \xi_k, \nu$ to $\xi'_1, \dots, \xi'_k, \nu'$. The order configuration of the complex $x, y', \gamma'$ is then similar to the configuration of the complex $x, y, \gamma$, and hence $F(x) \cap \gamma' = F(y') \cap \gamma'$ by the indiscernbility. On the other hand, using the Case 2b1 assumption, it is easy to see that the order configuration of $y, y', \gamma'$ is also similar to the order configuration of $x, y, \gamma$, and hence $F(y') \cap \gamma' = F(y) \cap \gamma'$. Thus $F(x) \cap \gamma' = F(y) \cap \gamma'$, as required.

*Case 2b2*: otherwise. Then there is $\alpha \in u \smallsetminus v$, $\alpha < \mu$. According to the assumption in the beginning of the proof, the ordinal $\alpha' = \alpha^\dagger$ does not occur in $x$ and/or $y$. Put $x' = x \cup \{\alpha'\} \smallsetminus \{\alpha\}$, and if $\alpha \in y$ then $y' = y \cup \{\alpha'\} \smallsetminus \{\alpha\}$ as well. Consider the pair of $x$ and $x'$. Obviously $x \mathbin{/\!/} \mathtt{bas}_n \neq x' \mathbin{/\!/} \mathtt{bas}_n$, hence $F(x) \neq F(x')$, moreover $F(x) \cap \gamma \neq F(x') \cap \gamma$ by Lemma 16. On the other hand, $y \mathbin{/\!/} \mathtt{bas}_n \neq y' \mathbin{/\!/} \mathtt{bas}_n$, because the substitution of $\alpha'$ for $\alpha$ does not alter the set $v = y \mathbin{/\!/} \mathtt{bas}_n$. Therefore $F(y) = F(y')$. And finally the order configuration of the complex $x, y, \gamma$ is clearly similar to the configuration of $x', y', \gamma$, and hence the equalities $F(x) \cap \gamma = F(y) \cap \gamma$ and $F(x') \cap \gamma = F(y') \cap \gamma$ hold or fail simultaneously, contradiction to the above. □

**Lemma 18.** *In the Prikry extension, $d \leqslant_{\mathbf{V}} X$.*

**Proof.** Fix a number $m \geq 1$ and, arguing in the Prikry extension, show how the set $D_m = d \mathbin{/\!/} m$ of $m$ first elements of $d = h \mathbin{/\!/} S$ can be recovered starting from

$X$. We assume that $d$ is infinite as otherwise there is nothing to prove. Then there is a least number $n = n_m \geq m$ such that $|S \cap n| \geq m$, and further there is a least number $k = k_m \geq n_m$ such that $s_k \cap n = S \cap n$.

Consider, still arguing in the Prikry extension, the set $C_k$ of all $k$-element sets $x \subseteq \kappa$ compatible with $X$. In particular the set $x_k = \{\underline{h}(i) : i < k\}$ of first $k$ elements of the whole Prikry sequence $h = h_G$ belongs to $C$. (See the beginning of this Section.) Suppose that $x, y \in C_k$. Then $x \mathbin{/\!\!/} \mathtt{bas}_k = y \mathbin{/\!\!/} \mathtt{bas}_k$ by Lemma 17, and hence the first $m$ elements of the sets $x \mathbin{/\!\!/} S$ and $y \mathbin{/\!\!/} S$ are the same by the choice of $k$ and $n$. In other words, for any $x \in C_k$ the first $m$ elements of the sets $x \mathbin{/\!\!/} S$ and $x_k \mathbin{/\!\!/} S$ are the same. But the first $m$ elements of the set $x_k = \{\underline{h}(i) : i < k\}$ are equal to the the set $D_m$ of the first $m$ elements of $\underline{d}$.

Thus the following plan of computing $D_m$ in the Prikry extension works: compute $n = n_m$ and $k = k_m$ as above, take any $x \in C_m$ and take the first $m$ elements of the set $x \mathbin{/\!\!/} S$. □

Lemmas 13 and 18 end the proof of Theorem 1.

## 7 Second proof: iterated ultrapowers

Beginning the alternative proof of Theorem 1, we suppose that $\kappa_0$ is a measurable cardinal in the ground model $\mathbf{V}$ and $U_0 \in \mathbf{V}$ is a normal ultrafilter on $\kappa_0$ in $\mathbf{V}$. (Note that $\kappa$ is used instead of $\kappa_0$ in the first proof of Theorem 1 above.)

Put $\boldsymbol{M}_0 = \mathbf{V}$. Following [2] we define the **iteration**

$$\{\boldsymbol{M}_\alpha, U_\alpha, \kappa_\alpha, \pi_{\alpha\beta}\}_{\alpha \leqslant \beta \in \mathtt{Ord}}$$

of $(\boldsymbol{M}_0, U_0)$ by recursion:

- $\pi_{00} = \mathrm{id}$;
- $\pi_{\alpha,\alpha+1} : \boldsymbol{M}_\alpha \to \boldsymbol{M}_{\alpha+1} = \mathtt{Ult}(\boldsymbol{M}_\alpha, U_\alpha)$ is the ultrapower of $\boldsymbol{M}_\alpha$ by $U_\alpha$;
- $\pi_{\delta,\alpha+1} = \begin{cases} \pi_{\alpha,\alpha+1} \circ \pi_{\delta\alpha} & \text{if } \delta \leqslant \alpha \\ \mathrm{id} & \text{if } \delta = \alpha + 1 \end{cases}$;
- $U_{\alpha+1} = \pi_{\alpha,\alpha+1}(U_\alpha)$, $\kappa_{\alpha+1} = \pi_{\alpha,\alpha+1}(\kappa_\alpha)$;
- for limit $\lambda$: $\boldsymbol{M}_\lambda, (\pi_{\alpha\lambda})_{\alpha \leqslant \lambda}$ is the **transitive** direct limit of the system $\{\boldsymbol{M}_\alpha, \pi_{\alpha\beta}\}_{\alpha \leqslant \beta < \lambda}$, and $U_\lambda = \pi_{0\lambda}(U_0)$, $\kappa_\lambda = \pi_{0\lambda}(\kappa_0)$.

Thus $\boldsymbol{M}_\alpha$, $\alpha \in \mathtt{Ord}$, is a system of transitive classes in the universe $\mathbf{V} = \boldsymbol{M}_0$. Moreover if $\alpha < \beta$ then $\boldsymbol{M}_\beta \subseteq \boldsymbol{M}_\alpha$ and $\pi_{\alpha\beta} : \boldsymbol{M}_\alpha \to \boldsymbol{M}_\beta$ is an elementary embedding, $\kappa_\beta = \pi_{\alpha\beta}(\kappa_\alpha) = \pi_{0\beta}(\kappa)$ is a measurable cardinal in $\boldsymbol{M}_\beta$ with $\kappa_\alpha < \kappa_\beta$, and $U_\beta = \pi_{\alpha\beta}(U_\alpha) = \pi_{0\beta}(U)$ is a normal ultrafilter on $\kappa_\beta$ in $\boldsymbol{M}_\beta$. This allows us to define $\mathbb{P}^\alpha = (\mathbb{P}_{\kappa_\alpha}(U_\alpha))^{\boldsymbol{M}_\alpha}$, the Prikry forcing in the universe $\boldsymbol{M}_\alpha$ associated to $\kappa_\alpha$ and $U_\alpha$ (see above).

The next proposition contains several more special but still well known (see, *e.g.*, [2]) facts regarding the iterated sequence.

**Proposition 19.** (i) $\pi_{\alpha\beta} \restriction \kappa_\alpha = \mathrm{id}$;

(ii) *for* $n < \omega$, $\boldsymbol{M}_n = \{\pi_{0n}(f)(\kappa_0, \ldots, \kappa_{n-1}) : f \in \boldsymbol{M}_0, f : \kappa_0^n \to \boldsymbol{M}_0\}$;

(iii) *if* $\lambda$ *is a limit ordinal then* $\kappa_\lambda = \sup_{\alpha<\lambda} \kappa_\alpha$;

(iv) *for* $A \in \boldsymbol{M}_\omega$, $A \subseteq \kappa_\omega$ : $A \in U_\omega$ *iff* $\{\kappa_m : m < \omega\} \setminus A$ *is finite*; ,

(v) $\{\kappa_m\}_{m<\omega}$ *is a Prikry sequence for* $U_\omega$ *and* $\boldsymbol{M}_\omega$, *that is, there exists a set* $G \subseteq \mathbb{P}^\omega$, $\mathbb{P}^\omega$*-generic over* $\boldsymbol{M}_\omega$ *and such that* $\{\kappa_m\}_{m<\omega}$ *is its Prikry sequence*;

(vi) *moreover, the Prikry extension* $\boldsymbol{M}_\omega[\{\kappa_m\}_{m<\omega}]$ *is equal to* $\bigcap_{n<\omega} \boldsymbol{M}_n$;

(vii) *for any* $\alpha$, *the sequences of* $\boldsymbol{M}_\beta, \kappa_\beta, U_\beta, \beta \geqslant \alpha$, *are definable in* $\boldsymbol{M}_\alpha$. □

Claim (v) is the key ingredient of the following proof of Theorem 1: it will allow is to infer properties of Prikry generic extensions of the ground universe $\mathbf{V}$ from properties of the sequence of iterated ultrapowers $\boldsymbol{M}_\alpha$. Claim (vi) (a really nontrivial one established in [2]) will not be used in the proof.

We continue with a couple of technical lemmas.

**Lemma 20.** *If* $x \in \boldsymbol{M}_\omega$ *then* $\pi_{m\omega}(x) = \pi_{\omega,\omega+\omega}(x)$ *for all but finitely many* $m < \omega$.

**Proof.** Let $x = \pi_{m_0\omega}(y)$, $m_0 < \omega$, $y \in \boldsymbol{M}_{m_0}$. Then for $m \in [m_0, \omega)$:

$$\begin{aligned} \pi_{m\omega}(x) &= \pi_{m\omega}(\pi_{m_0\omega}(y)) \\ &= \pi_{m\omega}(\pi_{m\omega}(\pi_{m_0m}(y))) & (3) \\ &= (\pi_{m\omega}(\pi_{m\omega}))(\pi_{m\omega}((\pi_{m_0m}(y)))) & (4) \\ &= \pi_{\omega,\omega+\omega}(\pi_{m_0\omega}(y)) & (5) \\ &= \pi_{\omega,\omega+\omega}(x). \end{aligned}$$

Line (4) arises from (3) by applying the map $\pi_{m\omega}$ to both terms in the functional application $\pi_{m\omega}(\pi_{m_0m}(y))$. For (5) note that $\pi_{m\omega}(\pi_{m\omega}) = \pi_{\omega,\omega+\omega}$ because $\pi_{m\omega}$ is the $\omega$-fold iteration starting from $\boldsymbol{M}_m$ whereas $\pi_{\omega,\omega+\omega}$ is the $\omega$-fold iteration starting from $\boldsymbol{M}_\omega$. □

Wellorder ascending finite sequences of ordinals $\langle \alpha_0 < \ldots < \alpha_{m-1} \rangle$ lexicographically from the top: $\langle \alpha_0, \ldots, \alpha_{m-1} \rangle \prec \langle \beta_0, \ldots, \beta_{n-1} \rangle$ iff there is some $i$ such that: $\alpha_{m-1} = \beta_{n-1}, \ldots, \alpha_{m-i} = \beta_{n-i}$, $\beta_{n-i-1}$ exists, and if $\alpha_{m-i-1}$ exists then $\alpha_{m-i-1} < \beta_{n-i-1}$.

**Lemma 21.** (i) *Suppose that* $u \in \boldsymbol{M}_n$. *Let* $\langle \alpha_0 < \ldots < \alpha_{m-1} \rangle$ *be a* $\prec$*-least tuple such that there is a function* $f \in \boldsymbol{M}_0$, $f : \kappa_0^m \to \boldsymbol{M}_0$, *satisfying*

$$u = \pi_{0n}(f)(\alpha_0, \ldots, \alpha_{m-1}).$$

(To see that such tuples exist, take $m = n$ and $\langle \alpha_0, \ldots, \alpha_{m-1} \rangle = \langle \kappa_0, \ldots, \kappa_{n-1} \rangle$ and apply Proposition 19(ii).) *Then* $\{\alpha_0, \ldots, \alpha_{m-1}\} \subseteq \{\kappa_0, \ldots, \kappa_{n-1}\}$.

(ii) *If* $\langle \alpha_0 < \ldots < \alpha_{m-1} \rangle$ *is* $\prec$*-minimal such that*

$$u = \pi_{0n}(f)(\alpha_0, \ldots, \alpha_{m-1})$$

*and if moreover* $u \subseteq \kappa_n$ *then* $\langle \alpha_0 < \ldots < \alpha_{m-1} \rangle$ *is* $\prec$*-minimal such that*

$$u = \pi_{0\omega}(f)(\alpha_0, \ldots, \alpha_{m-1}) \cap \kappa_n.$$

**Proof.** (i) Assume towards the contrary that $\{\alpha_0, \ldots, \alpha_{m-1}\} \not\subseteq \{\kappa_0, \ldots, \kappa_{n-1}\}$ and let $i$ be maximal such that $\alpha_i \notin \{\kappa_0, \ldots, \kappa_{n-1}\}$. Let $\kappa_\ell$ be minimal such that $\alpha_i < \kappa_\ell$. As obviously $\alpha_i \in \boldsymbol{M}_\ell$, by the representation theorem (Proposition 19(ii)) there is $g \in \boldsymbol{M}_0$, $g : \kappa_0^\ell \to \boldsymbol{M}_0$ such that

$$\alpha_i = \pi_{0\ell}(g)(\kappa_0, \ldots, \kappa_{\ell-1}).$$

Note that $\ell < n$, and hence applying $\pi_{\ell n}$ we obtain

$$\alpha_i = \pi_{0n}(g)(\kappa_0, \ldots, \kappa_{\ell-1}).$$

Let $\beta_0 < \ldots < \beta_{r-1}$ enumerate the set

$$\{\kappa_0, \ldots, \kappa_{\ell-1}\} \cup \{\alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{m-1}\}.$$

Note that $\langle \beta_0, \ldots, \beta_{r-1} \rangle \prec \langle \alpha_0, \ldots, \alpha_{m-1} \rangle$.

Let $\langle \kappa_0, \ldots, \kappa_{\ell-1} \rangle = \langle \beta_{j_0}, \ldots, \beta_{j_{l-1}} \rangle$ in this enumeration, and

$$\langle \alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{m-1} \rangle = \langle \beta_{k_0}, \ldots, \beta_{k_{i-1}}, \beta_{k_{i+1}}, \ldots, \beta_{k_{m-1}} \rangle.$$

Define $h : \kappa_0^r \to \boldsymbol{M}_0$ by

$$h(\xi_0, \ldots, \xi_{r-1}) = f(\xi_{k_0}, \ldots, \xi_{k_{i-1}}, g(\xi_{j_0}, \ldots, \xi_{j_{l-1}}), \xi_{k_{i+1}}, \ldots, \xi_{k_{m-1}}).$$

Then

$$\begin{aligned} u &= \pi_{0n}(f)(\alpha_0, \ldots, \alpha_{m-1}) \\ &= \pi_{0n}(f)(\alpha_0, \ldots, \alpha_{i-1}, \pi_{0n}(g)(\kappa_0, \ldots, \kappa_{l-1}), \alpha_{i+1}, \ldots, \alpha_{m-1}) \\ &= \pi_{0n}(f)(\beta_{k_0}, \ldots, \beta_{k_{i-1}}, \pi_{0n}(g)(\beta_{j_0}, \ldots, \beta_{j_{l-1}}), \beta_{k_{i+1}}, \ldots, \beta_{k_{m-1}}) \\ &= \pi_{0n}(h)(\beta_0, \ldots, \beta_{r-1}) \end{aligned}$$

contradicting the minimality of $\langle \alpha_0, \ldots, \alpha_{m-1} \rangle$.

(ii) Apply $\pi_{n\omega}$. □

## 8 Second proof: auxiliary model

To prove Theorem 1 means to establish a certain property of Prikry-generic extensions of the ground set universe $\mathbf{V} = \boldsymbol{M}_0$ with a measurable cardinal $\kappa = \kappa_0$. As an auxiliary result, we prove the same fact with respect to the extension of $\boldsymbol{M}_\omega$, as the ground set universe, by the sequence $h = \{\kappa_m : m < \omega\}$ of successive images of $\kappa$ in the iteration scheme. That this indeed leads to the proof of Theorem 1 see below.

**Theorem 22.** *Let* $h = \{\kappa_m : m < \omega\}$. *For every* $Z \in \boldsymbol{M}_\omega[h]$, $Z \subseteq \kappa_\omega$, *there exists* $h' \in \boldsymbol{M}_\omega[h]$ *such that* $h' \subseteq h$ *and* $\boldsymbol{M}_\omega[Z] = \boldsymbol{M}_\omega[h']$.

**Proof.** For $Z \in \boldsymbol{M}_\omega$ the theorem is obvious. Thus assume that $Z \notin \boldsymbol{M}_\omega$. We prove two auxiliary lemmas.

**Lemma 23.** $\kappa_\omega$ *is singular in* $\boldsymbol{M}_\omega[Z]$.

**Proof.** Assume not. For $m < \omega$ let

$$Z = \pi_{0m}(f_m)(\kappa_0, \ldots, \kappa_{m-1}) \in \boldsymbol{M}_m.$$

Then $Z \cap \kappa_m = \pi_{0m}(f_m)(\kappa_0, \ldots, \kappa_{m-1}) \cap \kappa_m$ and

$$Z \cap \kappa_m = \pi_{0\omega}(f_m)(\kappa_0, \ldots, \kappa_{m-1}) \cap \kappa_m.$$

So in the model $\boldsymbol{M}_\omega[Z]$,

$$\forall \zeta < \kappa_\omega \, \exists m < \omega \, \exists \xi_0, \ldots, \xi_{m-1} < \zeta \, (Z \cap \zeta = \pi_{0\omega}(f_m)(\xi_0, \ldots, \xi_{m-1}) \cap \zeta).$$

This defines **regressive** functions, and there are values $m_0$ and $\eta_0, \ldots, \eta_{m_0-1}$ such that for a stationary set $S \subseteq \kappa_\omega$

$$\forall \zeta \in S \, (Z \cap \zeta = \pi_{0\omega}(f_{m_0})(\eta_0, \ldots, \eta_{m_0-1}) \cap \zeta).$$

But then

$$Z = \pi_{0\omega}(f_{m_0})(\eta_0, \ldots, \eta_{m_0-1}) \cap \kappa_\omega \in \boldsymbol{M}_\omega.$$

Contradiction. □

**Lemma 24.** *In* $\boldsymbol{M}_\omega[Z]$, *there is an infinite subset* $h_0 \subseteq h$.

Note that any such set $h_0$ is cofinal in $\kappa_\omega$.

**Proof.** Let $\{\alpha_\nu : \nu < \gamma\} \in \boldsymbol{M}_\omega[Z]$ be cofinal in $\kappa_\omega$ where $\gamma < \kappa_\omega$. Without loss of generality, $\gamma < \kappa_0$.

Work in $\boldsymbol{M}_0$. For $\nu < \gamma$ consider the minimal $m_\nu$ such that $\alpha_\nu < \kappa_{m_\nu}$ and a $\prec$-minimal finite sequence $\vec{\kappa}_\nu \subseteq h$ such that for some $f_\nu$

$$\alpha_\nu = \pi_{0m_\nu}(f_\nu)(\vec{\kappa}_\nu).$$

Since $\gamma < \kappa_0$, we have

$$\{\pi_{0\omega}(f_\nu)\}_{\nu<\gamma} = \pi_{0\omega}(\{f_\nu\}_{\nu<\gamma}) \in \boldsymbol{M}_\omega.$$

By Lemma 21 we can, in $\boldsymbol{M}_\omega[Z]$, define $\vec{\kappa}_\nu$ as the $\prec$-minimal sequence such that

$$\alpha_\nu = \pi_{0\omega}(f_\nu)(\vec{\kappa}_\nu).$$

Let $h_0 = \bigcup_{\nu<\gamma} \vec{\kappa}_\nu$, so that $h_0 \in \boldsymbol{M}_\omega[Z]$ and $h_0 \subseteq h$. If $h_0$ were finite then

$$\{\alpha_\nu : \nu < \gamma\} \subseteq \{\pi_{0\omega}(f_\nu)(\vec{\kappa}) : \nu < \gamma \wedge \vec{\kappa} \subseteq h_0\} \in \boldsymbol{M}_\omega$$

would make $\kappa_\omega$ singular in $\boldsymbol{M}_\omega$, contradiction to the measurability of $\kappa_\omega$ in $\boldsymbol{M}_\omega$. □

To prove Theorem 22, let $\lambda_0 < \lambda_1 < \dots$ enumerate $h_0$. For $m < \omega$ let $\vec{\kappa}_m \subseteq D$ be the $\prec$-minimal tuple such that there is a function $f_m \in \boldsymbol{M}_0$, $f_m : \kappa_0^{\mathrm{length}(\vec{\kappa}_m)} \to \boldsymbol{M}_0$ satisfying

$$Z \cap \lambda_m = \pi_{0\omega}(f_m)(\vec{\kappa}_m) \cap \lambda_m. \tag{6}$$

Let $h' = h_0 \cup \bigcup_{m<\omega} \vec{\kappa}_m \subseteq h$. Observe that

$$\{\pi_{0\omega}(f_m)\}_{m<\omega} = \pi_{0\omega}(\{f_m\}_{m<\omega}) \in \boldsymbol{M}_\omega. \tag{7}$$

By (6) and (7), $Z \in \boldsymbol{M}_\omega[h']$.

Conversely, $h_0 \in \boldsymbol{M}_\omega[Z]$, and $\{\vec{\kappa}_m\}_{m<\omega}$ can be defined in $\boldsymbol{M}_\omega[Z]$ by: $\vec{\kappa}_m$ is $\prec$-minimal such that

$$Z \cap \lambda_m = \pi_{0\omega}(f_m)(\vec{\kappa}_m) \cap \lambda_m.$$

Hence $h' \in \boldsymbol{M}_\omega[Z]$.

Thus $\boldsymbol{M}_\omega[Z] = \boldsymbol{M}_\omega[h']$ as required. $\square$ (*Theorem 22*)

## 9 Second proof: finalization

Let $\Phi(h, \kappa)$ be the formula

$$h \subseteq \kappa \in \mathtt{Ord} \wedge \forall Z \subseteq \kappa \, \exists h' \subseteq h \, (\mathbf{V}[h'] = \mathbf{V}[Z]).$$

We want to show that the top condition $\langle \varnothing, \kappa_0 \rangle$ in $\mathbb{P} = \mathbb{P}_{\kappa_0}(U_0)$ forces $\Phi(\underline{h}, \kappa_0)$, where $\underline{h}$ is a canonical name for the Prikry sequence $h_G$. Assume not, and let a condition $\langle a, A \rangle \in \mathbb{P}$ force $\neg\, \Phi(\underline{h}, \kappa_0)$, in $\boldsymbol{M}_0$ as the ground model.

Then by elementarity, it is true in $\boldsymbol{M}_\omega$ that the condition $\langle \pi_{0\omega}(a), \pi_{0\omega}(A) \rangle$ in $\mathbb{P}_{\kappa_\omega}(U_\omega)$ forces $\neg\, \Phi(\underline{h}, \kappa_\omega)$ in $\boldsymbol{M}_\omega$. Clearly $\pi_{0\omega}(a) = a$, a finite subset of $\kappa_0$ by Proposition 19(i), while $A' = \pi_{0\omega}(A) \in \boldsymbol{M}_\omega$ belongs to $U_\omega$.

Let, by Proposition 19(iv), $n < \omega$ satisfy $\{\kappa_m : n \leqslant m < \omega\} \subseteq A'$. Recall that $h = \{\kappa_n : n < \omega\}$ is a Prikry sequence by Proposition 19(v), and hence so is $\widetilde{h} = a \cup \{\kappa_m : n \leqslant m < \omega\}$ since the notion of a Prikry sequence is invariant under finite changes. And the condition $\langle a, A' \rangle$ is obviously compatible with $\widetilde{h}$. Therefore we have $\neg\, \Phi(\widetilde{h}, \kappa_\omega)$ in $\boldsymbol{M}_\omega[\widetilde{h}]$.

Finally $\boldsymbol{M}_\omega[\widetilde{h}] = \boldsymbol{M}_\omega[h]$ and the formula $\Phi(h, \kappa_\omega)$ is obviously invariant under finite changes in $h$. We conclude that $\neg\, \Phi(h, \kappa_\omega)$ holds in $\boldsymbol{M}_\omega[h]$. But this contradicts Theorem 22. $\square$ (*Theorem 1, alternative proof*)

## 10 Larger sets

We don't know whether Theorem 1 remains true for arbitrary sets $X$ in the Prikry extension (that is, not necessarily subsets of $\kappa$), or at least for sets $X \subseteq \kappa^+$. This is an interesting open problem. The following theorem can be a first step in this direction.

**Theorem 25.** *Suppose that $h : \omega \to \kappa$ is Prikry-generic over the ground model $\mathbf{V}$. Then in the Prikry extension $\mathbf{V}[h]$ of $\mathbf{V}$, every set $X \subseteq \kappa^+$ satisfying $X \cap \xi \in \mathbf{V}$ for all $\xi < \kappa^+$ belongs to $\mathbf{V}$.*

**Proof.** The result can be obtained by a rather direct (but lengthy) argument. We prefer to follow the basic plan of the 2nd proof of Theorem 1, that yields a comparably shorter proof. The result is a corollary of the next lemma, and the derivation of the theorem from the lemma (similar to the argument in Section 9) is left to the reader.

**Lemma 26.** *Suppose that* $Z \in \boldsymbol{M}_\omega[h]$, $Z \subseteq ({\kappa_\omega}^+)^{\boldsymbol{M}_\omega[h]}$, *and* $Z \cap \xi \in \boldsymbol{M}_\omega$ *for all* $\xi < ({\kappa_\omega}^+)^{\boldsymbol{M}_\omega[h]}$. *Then* $Z \in \boldsymbol{M}_\omega$.

**Proof** (Lemma)**.** We have $\boldsymbol{M}_\omega[h] \subseteq \boldsymbol{M}_m$ for all $m < \omega$ by Proposition 19(vii). Therefore by the representation property of Proposition 19(ii) there exists a sequence of functions $\{f_m\}_{m\in\omega} \in \boldsymbol{M}_0$ such that

$$Z = \pi_{0m}(f_m)(\kappa_0, \dots, \kappa_{m-1}).$$

for $m < \omega$. If $\xi < ({\kappa_\omega}^+)^{\boldsymbol{M}_\omega[h]}$, then

$$\begin{aligned} Z \cap \xi &= \pi_{0m}(f_m)(\kappa_0, \dots, \kappa_{m-1}) \cap \xi, \\ \pi_{m\omega}(Z \cap \xi) &= \pi_{0\omega}(f_m)(\kappa_0, \dots, \kappa_{m-1}) \cap \pi_{m\omega}(\xi). \end{aligned}$$

Noting that $Z \cap \xi \in \boldsymbol{M}_\omega$ we may use Proposition 20 to replace $\pi_{m\omega}(Z \cap \xi)$ by $\pi_{\omega\omega+\omega}(Z \cap \xi)$ for almost all (except for finitely many) indices $m$. Similarly for $\pi_{m\omega}(\xi)$. So for almost all $m < \omega$,

$$\pi_{\omega\omega+\omega}(Z \cap \xi) = \pi_{0\omega}(f_m)(\kappa_0, \dots, \kappa_{m-1}) \cap \pi_{\omega\omega+\omega}(\xi).$$

So for every $\xi < (\kappa^+)^{\boldsymbol{M}_\omega[h]}$ we can find $m < \omega$ and $\alpha_0, \dots, \alpha_{m-1} < \kappa$ such that

$$\pi_{\omega\omega+\omega}(Z \cap \xi) = \pi_{0\omega}(f_m)(\alpha_0, \dots, \alpha_{m-1}) \cap \pi_{\omega\omega+\omega}(\xi).$$

This defines regressive functions on $({\kappa_\omega}^+)^{\boldsymbol{M}_\omega[h]}$ inside $\boldsymbol{M}_\omega[Z]$ and so there are a cofinal set $S \subseteq ({\kappa_\omega}^+)^{\boldsymbol{M}_\omega[h]}$ and $m_0 < \omega$ and ordinals $\beta_0, \dots, \beta_{m_0-1} < \kappa_\omega$ such that for all $\xi \in S$

$$\pi_{\omega\omega+\omega}(Z \cap \xi) = \pi_{0\omega}(f_{m_0})(\beta_0, \dots, \beta_{m_0-1}) \cap \pi_{\omega\omega+\omega}(\xi).$$

Then

$$\begin{aligned} Z &= \{\zeta < ({\kappa_\omega}^+)^{\boldsymbol{M}_\omega} : \exists \xi < ({\kappa_\omega}^+)^{\boldsymbol{M}_\omega} \big(\pi_{\omega\omega+\omega}(\zeta) \in \pi_{\omega\omega+\omega}(Z \cap \xi)\big)\} \\ &= \{\zeta < ({\kappa_\omega}^+)^{\boldsymbol{M}_\omega} : \pi_{\omega\omega+\omega}(\zeta) \in \pi_{0\omega}(f_{m_0})(\beta_0, \dots, \beta_{m_0-1})\} \end{aligned}$$

is a definition of $Z$ in $\boldsymbol{M}_\omega$. □ (*Lemma*)

□ (*Theorem 25*)

## References

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