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Consequences of the Sequent Calculus[1]

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Summary. This article is part of a series of Mizar articles which constitute a formal proof (of a basic version) of Kurt Gödel's famous completeness theorem (K. Gödel, "Die Vollständigkeit der Axiome des logischen Funktionenkalküls", Monatshefte für Mathematik und Physik 37 (1930), 349-360). The completeness theorem provides the theoretical basis for a uniform formalization of mathematics as in the Mizar project. We formalize first-order logic up to the completeness theorem as in H. D. Ebbinghaus, J. Flum, and W. Thomas, Mathematical Logic, 1984, Springer Verlag New York Inc. The first main result of the present article is that the derivablility of a sequent doesn't depend on the ordering of the antecedent. The second main result says: if a sequent is derivable, then the formulas in the antecendent only need to occur once.

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The articles [15], [16], [3], [14], [4], [1], [2], [17], [10], [6], [8], [13], [12], [9], [18], [11], [5], and [7] provide the terminology and notation for this paper.

1. f is a Subsequence of g^f

For simplicity, we adopt the following convention: p, q denote elements of CQC-WFF, k, m, n, i denote natural numbers, f, g denote finite sequences of elements of CQC-WFF, and a, b, b_1, b_2, c denote natural numbers.

Let m, n be natural numbers. The functor $\text{seq}(m, n)$ yielding a set is defined as follows:

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(Def. 1) $\mathrm{seq}(m, n) = \{k : 1 + m \leq k \ \wedge \ k \leq n + m\}$.

Let m, n be natural numbers. Then $\mathrm{seq}(m, n)$ is a subset of $\mathbb{N}$.

One can prove the following propositions:

(1) $c \in \mathrm{seq}(a, b)$ iff $1 + a \leq c$ and $c \leq b + a$.

(2) $\mathrm{seq}(a, 0) = \emptyset$.

(3) $b = 0$ or $b + a \in \mathrm{seq}(a, b)$.

(4) $b_1 \leq b_2$ iff $\mathrm{seq}(a, b_1) \subseteq \mathrm{seq}(a, b_2)$.

(5) $\mathrm{seq}(a, b) \cup \{a + b + 1\} = \mathrm{seq}(a, b + 1)$.

(6) $\mathrm{seq}(m, n) \approx n$.

Let us consider m, n. Observe that $\mathrm{seq}(m, n)$ is finite.

Let us consider f. Observe that $\mathrm{len}\, f$ is finite.

Next we state a number of propositions:

(7) $\mathrm{seq}(m, n) \subseteq \mathrm{Seg}(m + n)$.

(8) $\mathrm{Seg}\, n$ misses $\mathrm{seq}(n, m)$.

(9) For all finite sequences f, g holds $\mathrm{Seg}\,\mathrm{len}(f \frown g) = \mathrm{Seg}\,\mathrm{len}\, f \cup \mathrm{seq}(\mathrm{len}\, f, \mathrm{len}\, g)$.

(10) $\mathrm{len}\,\mathrm{Sgm}\,\mathrm{seq}(\mathrm{len}\, g, \mathrm{len}\, f) = \mathrm{len}\, f$.

(11) $\mathrm{dom}\,\mathrm{Sgm}\,\mathrm{seq}(\mathrm{len}\, g, \mathrm{len}\, f) = \mathrm{dom}\, f$.

(12) $\mathrm{rng}\,\mathrm{Sgm}\,\mathrm{seq}(\mathrm{len}\, g, \mathrm{len}\, f) = \mathrm{seq}(\mathrm{len}\, g, \mathrm{len}\, f)$.

(13) If $i \in \mathrm{dom}\,\mathrm{Sgm}\,\mathrm{seq}(\mathrm{len}\, g, \mathrm{len}\, f)$, then $(\mathrm{Sgm}\,\mathrm{seq}(\mathrm{len}\, g, \mathrm{len}\, f))(i) = \mathrm{len}\, g + i$.

(14) $\mathrm{seq}(\mathrm{len}\, g, \mathrm{len}\, f) \subseteq \mathrm{dom}(g \frown f)$.

(15) $\mathrm{dom}((g \frown f) \restriction \mathrm{seq}(\mathrm{len}\, g, \mathrm{len}\, f)) = \mathrm{seq}(\mathrm{len}\, g, \mathrm{len}\, f)$.

(16) $\mathrm{Seq}((g \frown f) \restriction \mathrm{seq}(\mathrm{len}\, g, \mathrm{len}\, f)) = \mathrm{Sgm}\,\mathrm{seq}(\mathrm{len}\, g, \mathrm{len}\, f) \cdot (g \frown f)$.

(17) $\mathrm{dom}\,\mathrm{Seq}((g \frown f) \restriction \mathrm{seq}(\mathrm{len}\, g, \mathrm{len}\, f)) = \mathrm{dom}\, f$.

(18) f is a subsequence of $g \frown f$.

Let D be a non empty set, let f be a finite sequence of elements of D, and let P be a permutation of $\mathrm{dom}\, f$. The functor $\mathrm{Per}(f, P)$ yielding a finite sequence of elements of D is defined as follows:

(Def. 2) $\mathrm{Per}(f, P) = P \cdot f$.

In the sequel P denotes a permutation of $\mathrm{dom}\, f$.

The following propositions are true:

(19) $\mathrm{dom}\,\mathrm{Per}(f, P) = \mathrm{dom}\, f$.

(20) If $\vdash f \frown \langle p \rangle$, then $\vdash g \frown f \frown \langle p \rangle$.

2. The Ordering of the Antecedent is Irrelevant

Let us consider f. The functor $\mathrm{Begin}(f)$ yielding an element of CQC-WFF is defined by:

(Def. 3) $\mathrm{Begin}(f) = \begin{cases} f(1), & \text{if } 1 \leq \mathrm{len}\, f, \\ \mathrm{VERUM}, & \text{otherwise.} \end{cases}$

Let us consider f. Let us assume that $1 \leq \mathrm{len}\, f$. The functor $\mathrm{Impl}(f)$ yields an element of CQC-WFF and is defined by the condition (Def. 4).

(Def. 4) There exists a finite sequence F of elements of CQC-WFF such that

(i) $\mathrm{Impl}(f) = F(\mathrm{len}\, f)$,

(ii) $\mathrm{len}\, F = \mathrm{len}\, f$,

(iii) $F(1) = \mathrm{Begin}(f)$ or $\mathrm{len}\, f = 0$, and

(iv) for every n such that $1 \leq n$ and $n < \mathrm{len}\, f$ there exist p, q such that $p = f(n+1)$ and $q = F(n)$ and $F(n+1) = p \Rightarrow q$.

We now state a number of propositions:

(21) $\vdash f \frown \langle p \rangle \frown \langle p \rangle$.

(22) If $\vdash f \frown \langle p \wedge q \rangle$, then $\vdash f \frown \langle p \rangle$.

(23) If $\vdash f \frown \langle p \wedge q \rangle$, then $\vdash f \frown \langle q \rangle$.

(24) If $\vdash f \frown \langle p \rangle$ and $\vdash f \frown \langle p \rangle \frown \langle q \rangle$, then $\vdash f \frown \langle q \rangle$.

(25) If $\vdash f \frown \langle p \rangle$ and $\vdash f \frown \langle \neg p \rangle$, then $\vdash f \frown \langle q \rangle$.

(26) If $\vdash f \frown \langle p \rangle \frown \langle q \rangle$ and $\vdash f \frown \langle \neg p \rangle \frown \langle q \rangle$, then $\vdash f \frown \langle q \rangle$.

(27) If $\vdash f \frown \langle p \rangle \frown \langle q \rangle$, then $\vdash f \frown \langle p \Rightarrow q \rangle$.

(28) If $1 \leq \mathrm{len}\, g$ and $\vdash f \frown g$, then $\vdash f \frown \langle \mathrm{Impl}(\mathrm{Rev}(g)) \rangle$.

(29) If $\vdash (\mathrm{Per}(f, P)) \frown \langle \mathrm{Impl}(\mathrm{Rev}(f \frown \langle p \rangle)) \rangle$, then $\vdash (\mathrm{Per}(f, P)) \frown \langle p \rangle$.

(30) If $\vdash f \frown \langle p \rangle$, then $\vdash (\mathrm{Per}(f, P)) \frown \langle p \rangle$.

3. Multiple Occurrence in the Antecedent is Irrelevant

Let us consider n and let c be a set. We introduce $\mathrm{IdFinS}(c, n)$ as a synonym of $n \mapsto c$.

We now state the proposition

(31) For every set c such that $1 \leq n$ holds $\mathrm{rng}\, \mathrm{IdFinS}(c, n) = \mathrm{rng} \langle c \rangle$.

Let D be a non empty set, let n be a natural number, and let p be an element of D. Then $\mathrm{IdFinS}(p, n)$ is a finite sequence of elements of D.

The following proposition is true

(32) If $1 \leq n$ and $\vdash f \frown \mathrm{IdFinS}(p, n) \frown \langle q \rangle$, then $\vdash f \frown \langle p \rangle \frown \langle q \rangle$.

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