An Iteration Model violating the Singular Cardinals Hypothesis

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1 Introduction

Models of Set Theory showing exotic behaviour at singular cardinals are usually constructed via forcing. The archetypical method is Prikry-Forcing [Pr1970], which has been generalized in various ways, as for example by Gitik and Magidor [GiMa1992]. It was observed early that Prikry generic sequences can be obtained as successive critical points in an iteration of the universe V by a normal ultrafilter ([Ka1994], see [De1978] for an exhausting analysis). In this paper iterations by stronger extenders are studied similarly and yield the following theorem:

Main Theorem:

Assume there is an elementary embedding $\pi\colon V \to M$, $V \models GCH$, M transitive, $\pi \restriction \kappa = id$, $\pi(\kappa) \geq \kappa^{++}$, ${}^{\kappa}M \subseteq M$. Then there is an inner model N of

$$\mathbf{ZF} \wedge \neg\mathbf{AC} \wedge \forall \nu < \lambda\ 2^{\nu} = \nu^{+} \wedge \neg 2^{\lambda} = \lambda^{+} \wedge \lambda \text{ has cofinality } \omega.$$

This says that N violates, in a choiceless way, the **S**ingular **C**ardinals **H**ypothesis (**SCH**), since **SCH** implies that the generalized continuum hypothesis is true at singular strong limit cardinals. The model N will roughly be defined as the intersection of all models obtained by finitely iterating the embedding π.

The proof of the Main Theorem stretches over the rest of this paper. In section 2 we investigate iterations of elementary embeddings. In section 3 the intersection model N is defined and shown to be a model of **ZF**. Sections 4 and 5 are used to establish the cardinality properties and the negative result about choice in N, respectively.

From now on let us assume that $\pi\colon V \to M$ is as above. We may also assume that π is $\in$-definable from some parameters.

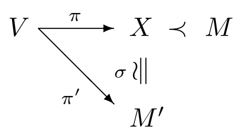
2 Iterations

To analyze the intersection model it is advantageous to have efficient representations of the elements of M and further iterates. Therefore we may have to modify π a bit:

Lemma 1 *There is an elementary map $\pi'\colon V \to M'$, M' transitive, $\pi' \restriction \kappa = id$, $\pi'(\kappa) = \pi(\kappa) \geq \kappa^{++}$, ${}^{\kappa}M' \subseteq M'$ with the added property:*

$$M' = \{\pi'(f)(x) | f\colon (\kappa)^{<\omega} \to V,\ x \in (\pi'(\kappa))^{<\omega}\}$$

Proof: Let $X = \{\pi(f)(x) | f\colon \big(\kappa\big)^{<\omega} \to V,\, x \in \big(\pi(\kappa)\big)^{<\omega}\}$. Since the functions $f\colon \big(\kappa\big)^{<\omega} \to V$ can be used as Skolem functions for V, X is an elementary submodel of M: $X \prec M$. Let $\sigma\colon X \simeq M'$, M' transitive, $\pi' = \sigma \circ \pi$:



We show that M' is κ-closed, the other properties are easily verified for $\pi'\colon V \to M'$. It suffices to show:
Claim: ${}^{\kappa}X \subseteq X$
For $i < \kappa$ consider $\pi(f_i)(x_i) \in X$ as above. Let $g\colon \kappa \leftrightarrow H_\kappa$ be a bijection, then $(x_i)_{i<\kappa} \in H^M_{\pi(\kappa)} = \pi(g)''\pi(\kappa)$. Let $\xi_0 < \pi(\kappa)$, such that $(x_i)_{i<\kappa} = \pi(g)(\xi_0)$ and define a function $h\colon \kappa \to V$ by cases: if $g(\xi)\colon \kappa_\xi \to V$ for some $\kappa_\xi < \kappa$, then

$$h(\xi)\colon \kappa_\xi \to V,\; h(\xi)(i) = f_i(g(\xi)(i)).$$

Otherwise set $h(\xi) = \emptyset$. Then $\pi(f_i)(x_i) = (\pi(f_i))((\pi(g)(\xi_0))(i)) = (\pi(h)(\xi_0))(i)$ and so $(\pi(f_i)(x_i))_{i<\kappa} = \pi(h)(\xi_0) \in X$ as required. □

By the Lemma we may assume that π already satisfies the

Assumption: $M = \{\pi(f)(x) | f\colon \big(\kappa\big)^{<\omega} \to V,\, x \in \big(\pi(\kappa)\big)^{<\omega}\}$.

A definable elementary embedding of V may be applied to its own definition and thus be iterated. This process can be iterated transfinitely along the ordinals. All iterates will be transitive inner models. For A a definable class define its image under π as $\pi(A) = \bigcup\{\pi(A \cap V_\alpha) | \alpha \in \mathrm{On}\}$. Then $\pi(A)$ is definable in M just like A is definable in V with all parameters mapped by π.

Definition 1 *The* iteration $(M_i, \pi_{ij})_{i\le j<\theta}$, $\theta \le \infty$ *of* V *by* π *is defined recursively until breakdown:* $M_0 = V$, $\pi_{0,0} = id$;
$M_{i+1} = \pi_{0,i}(M)$, $\pi_{i,i+1} = \pi_{0,i}(\pi)$, $\pi_{j,i+1} = \pi_{i,i+1} \circ \pi_{j,i}$ *for* $j < i$, $\pi_{i+1,i+1} = id \restriction M_{i+1}$;
if j *is a limit ordinal then* $(M_j, \pi_{i,j})_{i<j}$ *is a direct limit of* $(M_i, \pi_{i,i'})_{i\le i'<j}$, *and* $\pi_{j,j} = id \restriction M_j$.
If any of these M_i *is wellfounded we also require it to be transitive. If there exists a minimal* i *with* M_i *non-wellfounded set* $\theta = i + 1$ *and stop the construction; otherwise let* $\theta = \infty$. *The* M_i *for* $i \le \theta$ *are the* iterates *of* V *by* π.

Indeed this construction does not break down:

Theorem 1 *The embedding* π *is* iterable, *i.e., every iterate of* V *by* π *is transitive, and* $\theta = \infty$.

Proof: Assume not. Then there is a unique last iterate $M_j = (M_j, \in')$ of V that is illfounded. By the construction j cannot be a successor ordinal. The image range$(\pi_{0,j})$ lies $\in'$-cofinally in M_j. Let $\alpha \in \mathrm{On}$ be minimal such that $\pi_{0,j}(\alpha)$ is in the illfounded part of $(M_j, \in')$. There is $\eta \in' \pi_{0,j}(\alpha)$ such that η is still in the illfounded part of $(M_j, \in')$. Let

$i < j$ and $\beta \in \mathrm{On}$ be the preimage of $\pi_{i,j}(\beta) = \eta \in' \pi_{0,j}(\alpha)$. In M_i, M_j is the unique illfounded iterate of M_i by $\pi_{i,i+1}$. By absoluteness properties of iterations as defined above, β witnesses the existential statement:

$$M_i \models \quad \exists\gamma < \pi_{0,i}(\alpha): \quad \text{"}\pi_{i,j}(\gamma) \text{ is in the illfounded part of the unique non-wellfounded iterate of } M_i \text{ by } \pi_{i,i+1}\text{"}$$

Since $\pi_{0,i}\colon V \mapsto M_i$ is elementary, the corresponding statement holds in V: $\exists\gamma < \alpha$: "$\pi_{0,j}(\gamma)$ is in the illfounded part of the unique non-wellfounded iterate of V by $\pi_{0,1} = \pi$". This contradicts the minimality of α. □

The critical points of the maps $\pi_{i,i+1}$ are given by $\kappa_i = \pi_{0,i}(\kappa)$. The following facts are proved by a straightforward induction along the iteration (see also [Je1978]):

Lemma 2 *For $i, j \in \mathrm{On}, i < j$:*

(a) $\kappa_i < \kappa_j$

(b) $\pi_{i,j} \restriction \kappa_i = id$, $\pi_{i,j}(\kappa_i) = \kappa_j$.

(c) $V_{\kappa_i} \cap M_i = V_{\kappa_i} \cap M_j$.

(d) $\mathcal{P}(\kappa_i) \cap M_i = \mathcal{P}(\kappa_i) \cap M_j$.

(e) $M_i \supseteq M_j$, $M_i \neq M_j$.

(f) *κ_i is a cardinal in M_j.*

(g) *If j is a limit ordinal, then $\kappa_j = \lim_{i<j} \kappa_i$.*

(h) *If $i < \omega$, then M_i is κ-closed: ${}^{\kappa}M_i \subseteq M_i$.*

The representation property of Lemma 1 can be generalized to all iterates.

Lemma 3 *For all $i < \infty$:*
$M_i = \{\pi_{0,i}(f)(x) | f\colon (\kappa)^{<\omega} \to V,\ x \in (\kappa_i)^{<\omega}\}$

Proof: By induction. The initial cases $i = 0, 1$ are trivial by our Assumption. The limit case is easy because M_i is a direct limit of earlier iterates.
For the succesor step assume the claim for i and let $z \in M_{i+1} = \pi_{0,i}(M)$. By the Assumption and the elementarity of $\pi_{0,i}$ we may assume $z = \pi_{i,i+1}(g)(y)$ for some $g \in M_i$, $g\colon (\kappa_i)^{<\omega} \to M_i$, $y \in (\kappa_{i+1})^{<\omega}$. By the induction hypothesis $g = \pi_{0,i}(h)(z)$ for some $h\colon (\kappa)^{<\omega} \to V$, $z \in (\kappa_i)^{<\omega}$. Hence $z = \pi_{i,i+1}(g)(y) = \pi_{i,i+1}(\pi_{0,i}(h)(z))(y) = (\pi_{0,i+1}(h)(z))(y) = \pi_{0,i+1}(f)(x)$ with $x = z^\frown y$ and $f\colon (\kappa)^{<\omega} \to V$ defined by $f(v^\frown u) := (h(v))(u)$ if this is welldefined and $\mathrm{length}(v) = \mathrm{length}(z)$, $\mathrm{length}(u) = \mathrm{length}(y)$, $f(v^\frown u) = \emptyset$ otherwise. □

We shall need the following "algebraic" facts about the system of iteration maps (see [De1978] for more general statements of this kind):

Lemma 4 **(a)** *If $i < \omega$ then $\pi_{i,\omega}(\pi_{i,\omega}) = \pi_{\omega,\omega+\omega}$.*

(b) *If $x \in M_\omega$ then $\pi_{i,\omega}(x) = \pi_{\omega,\omega+\omega}(x)$ for almost all [1] $i < \omega$.*

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[1] = all but finitely many

Proof: (a) $M_i \models$ "$\pi_{i,\omega}$ is the iteration map from V [which is M_i here] to its ω-th iterate." By $\pi_{i,\omega}$ this is mapped elementarily to $M_\omega \models$ "$\pi_{i,\omega}(\pi_{i,\omega})$ is the iteration map from V [which is M_ω now] to its ω-th iterate." By the absoluteness properties of iterations this map is just $\pi_{\omega,\omega+\omega}\colon M_\omega \to M_{\omega,\omega+\omega}$.
(b) Let $x \in M_\omega, x = \pi_{j,\omega}(y)$ for some $j < \omega$, $y \in M_j$. For $j < i < \omega$ we see:

$$\begin{aligned}
\pi_{i,\omega}(x) &= \pi_{i,\omega}(\pi_{j,\omega}(y)) \\
&= \pi_{i,\omega}(\pi_{i,\omega}(\pi_{j,i}(y))) \\
&= (\pi_{i,\omega}(\pi_{i,\omega}))(\pi_{i,\omega}(\pi_{j,i}(y))) \qquad (*) \\
&= \pi_{\omega,\omega+\omega}(\pi_{j,\omega}(y)) \\
&= \pi_{\omega,\omega+\omega}(x),
\end{aligned}$$

in $(*)$, $\pi_{i,\omega}$ is applied to the term "$\pi_{i,\omega}$ evaluated at the argument $\pi_{j,i}(y)$" □

3 The Intersection Model

From the iteration of V by π we can define the intersection model $N := \bigcap_{i<\omega} M_i$.

Lemma 5 *For $i < \omega$: $M_\omega \subseteq N \subseteq M_i$ and N is uniformly definable in M_i from π_i as the intersection of the finite iterates of M_i.*

Theorem 2 *N is an inner model of Zermelo-Fraenkel set theory* **ZF**.

Proof: N is transitive and contains the class On, which implies extensionality and foundation in N. For the other axioms the existence of certain abstraction terms $t = \{x \in N | \varphi^N(x, \bar{a})\}$ for $\bar{a} \in N$ has to be shown in N. For all $i < \omega$, N is definable in M_i. Hence t exists in all M_i and $t \in N = \bigcap_{i<\omega} M_i$. □

The status of the Axiom of Choice (**AC**) will be discussed later. N and its inner model M_ω are in some close relationship reminiscent of Prikry- or Gitik-Magidor generic extensions.

Set $\lambda = \kappa_\omega$. Then

Lemma 6 **(a)** *$N \cap V_\lambda = M_\omega \cap V_\lambda$.*

(b) *$N \models \lambda$ is a strong limit cardinal, $N \models \forall \nu < \lambda\ 2^\nu = \nu^+$.*

Proof: (a) $\supseteq$ is clear by Lemma 5. Let $x \in N \cap V_\lambda$. For some $i < \omega$: $x \in N \cap V_{\kappa_i}$. Then by Lemma 2(c), $x \in M_i \cap V_{\kappa_i} = M_\omega \cap V_{\kappa_i}$.
(b) is true since the corresponding statements hold in M_ω and are absolute between M_ω and N by (a). □

Every $z \in M_\omega$ is the limit of a *thread* $\pi^{-1}_{0,\omega}(z), \pi^{-1}_{1,\omega}(z), \pi^{-1}_{2,\omega}(z), \ldots$. These threads provide us with natural Prikry sequences for M_ω; N can see the system of these sequences modulo finite changes.

Definition 2 *Let $k < \omega$. For $\alpha < \kappa_{\omega+1}$ set $c^k_\alpha := \{\pi^{-1}_{i,\omega}(\alpha) | k \le i < \omega$ and $\alpha \in \mathrm{rge}(\pi_{i,\omega})\}$. Define $C^k := (c^k_\alpha | \alpha < \kappa_{\omega+1})$. This definition can be carried out inside M_k, hence $C^k \in M_k$. $\pi^{-1}_{i,\omega}(\alpha) < \kappa_{i+1} < \lambda$ and so $c^k_\alpha \subseteq \lambda$. For any $x \subseteq \lambda$ define $\tilde{x} = \{y \subseteq \lambda | x \triangle y$ is finite$\}$. We call $\tilde{C} := (\tilde{c}^0_\alpha | \alpha < \kappa_{\omega+1})$ the* Prikry-System *derived from iterating V by π.*

Obviously $\tilde{c}^0_\alpha = \tilde{c}^k_\alpha$ and $\tilde{C} := (\tilde{c}^k_\alpha | \alpha < \kappa_{\omega+1})$, so $\tilde{C}$ can be defined from $C^k \in M_k$ for all k and we obtain $\tilde{C} \in N = \bigcap_{k<\omega} M_k$.

Lemma 7 **(a)** *If $\alpha < \beta < \kappa_{\omega+1}$, then $\tilde{c}^0_\alpha \neq \tilde{c}^0_\beta$*

(b) *There is a surjective map $s\colon \mathcal{P}(\lambda) \to \kappa_{\omega+1}$, $s \in N$.*

(c) *$N \models \lambda$ is singular of cofinality ω.*

Proof: (a) The threads c^k_α and c^k_β differ on an endsegment.
(b) For $x \subseteq \lambda$ let $s(x)$ be the unique α such that $x \in \tilde{c}^0_\alpha$, if this exists, and 0 otherwise.
(c) The sequence $(\kappa_i | i < \omega)$ is cofinal in κ_ω (by Lemma 2(g)) and $(\kappa_i | i < \omega) \in \tilde{c}^0_\lambda \in N$. □

4 Cardinal Preservation

Our assumptions on π imply that $\kappa_{\omega+1} \geq (\lambda^{++})^{M_\omega}$. If $(\lambda^{++})^{M_\omega} = (\lambda^{++})^N$ then Lemma 7(b) provides us with the desired negation of **SCH**. Therefore we show cardinal preservation between M_ω and N. The proof of the following "covering theorem" is based on "naming" elements of N by the normal form given in Lemma 3 and counting "names".

Lemma 8 *Let $f\colon \eta \to \theta$, $\eta, \theta \in \mathrm{On}$, $f \in N$. Then there is a function $F\colon \eta \to M_\omega$, $F \in M_\omega$ such that*

(a) $\forall \xi < \eta\colon f(\xi) \in F(\xi)$.

(b) $\forall \xi < \eta\colon$ *card*$^{M_\omega}(F(\xi)) \leq \lambda$.

Proof: By Lemma 3, f can be represented in the various M_i, $i < \omega$, as: $f = \pi_{0,i}(f_i)(x_i)$ with $f_i\colon [\kappa]^{<\omega} \to V$, $x_i \in [\kappa_i]^{<\omega}$. For $\xi < \eta$, $\zeta < \theta$ we have

$$\begin{aligned}(\xi,\zeta) \in f \quad &\leftrightarrow \quad \pi_{i,\omega}(\xi,\zeta) \in \pi_{i,\omega}(f) = \pi_{0,\omega}(f_i)(x_i)\\ &\leftrightarrow \quad \pi_{\omega,\omega+\omega}(\xi,\zeta) \in \pi_{0,\omega}(f_i)(x_i),\ \text{for almost all } i < \omega.\end{aligned}$$

Define $F\colon \eta \to V$ by
$F(\xi) = \{\zeta < \theta | \exists i < \omega, x \in [\lambda]^{<\omega}\colon \pi_{0,\omega}(f_i)(x)$ is a function and $\pi_{\omega,\omega+\omega}(\xi,\zeta) \in \pi_{0,\omega}(f_i)(x)\}$.
Then $F \in M_\omega$ since it is definable in M_ω using the parameters η, θ, λ and $(\pi_{0,\omega}(f_i))_{i<\omega} = \pi_{0,\omega}((f_i)_{i<\omega})$. Property (a) holds by the preceding equivalences, (b) is immediate from the definition of F. □

Theorem 3 *card*$^{M_\omega}$ = *card*^{N}.

Proof: The inclusion $\supseteq$ is clear since $M_\omega \subseteq N$.
If $\theta \leq \lambda$ is a cardinal in M_ω, then it is a cardinal in N by Lemma 6(a). If $\theta > \lambda$ is not a cardinal in N, there is $f\colon \eta \to \theta$ onto, $\eta < \theta$, $f \in N$. Take $F\colon \eta \to M_\omega$, $F \in M_\omega$ as in Lemma 8. By Lemma 8(a) $\theta \subseteq \bigcup_{\xi<\eta} F(\xi)$, and by (b) M_ω satisfies $\mathrm{card}(\bigcup_{\xi<\eta} F(\xi)) \leq \eta \cdot \lambda < \theta \cdot \theta = \theta$. So θ is not a cardinal in M_ω. □

Concerning the proof of our main theorem this yields

Theorem 4 $N \models \neg 2^\lambda = \lambda^+$

Proof: Assume $N \models 2^\lambda = \lambda^+$ instead. In N, there is a surjective map $\lambda^+ \overset{\text{onto}}{\longrightarrow} \mathcal{P}(\lambda)$ and by Lemma 7(b) there is a surjective map $\mathcal{P}(\lambda) \overset{\text{onto}}{\longrightarrow} \kappa_{\omega+1}$. So $N \models \mathrm{card}(\kappa_{\omega+1}) \leq \lambda^+$ and by the preceding cardinal absoluteness $M_\omega \models \mathrm{card}(\kappa_{\omega+1}) \leq \lambda^+$. This can be pulled back by $\pi_{0,\omega}$ to V where we get $\mathrm{card}(\kappa_1) = \mathrm{card}(\pi(\kappa)) \leq \kappa^+$ contradicting our assumptions on π. □

5 No Choice

Theorem 5 *In N, $\tilde{C} \upharpoonright \lambda^{++}$ has no choice function, hence in N the Axiom of Choice fails for sequences of length λ^{++}.*

Proof: Assume for a contradiction, that $h\colon \lambda^{++} \to \mathcal{P}(\lambda)$, $h \in N$ is a choice function for $\tilde{C} \upharpoonright \lambda^{++}$. Then $\forall \xi < \lambda^{++}\colon h(\xi) \in \tilde{C}(\xi) = \tilde{c}^0_\xi$. We use λ^{++} to denote $(\lambda^{++})^N = (\lambda^{++})^{M_\omega}$. Since $h \in M_i$ for all i we get by our normal form result of section 2: $h = \pi_{0,i}(f_i)(x_i)$ for $i < \omega$, with $f_i\colon (\kappa)^{<\omega} \to V$, $x_i \in (\kappa_i)^{<\omega}$. Consider $\xi < \xi' < \lambda^{++}$. By Lemma 7(a) $h(\xi) \neq h(\xi')$. There is $i < \omega$ such that $\kappa_i \cap h(\xi) \neq \kappa_i \cap h(\xi')$ and by Lemma 4(b) we can assume (increasing i if necessary) that $\pi_{i,\omega}(\xi) = \pi_{\omega,\omega+\omega}(\xi)$ and $\pi_{i,\omega}(\xi') = \pi_{\omega,\omega+\omega}(\xi')$. Then $\kappa_i \cap (\pi_{0,i}(f_i)(x_i))(\xi) \neq \kappa_i \cap (\pi_{0,i}(f_i)(x_i))(\xi')$. Applying $\pi_{i,\omega}$ we get

$$(*) \qquad \kappa_i \cap (\pi_{0,\omega}(f_i)(x_i))(\pi_{\omega,\omega+\omega}(\xi)) \neq \kappa_i \cap (\pi_{0,\omega}(f_i)(x_i))(\pi_{\omega,\omega+\omega}(\xi')).$$

In M_ω, we define the function $H\colon \lambda^{++} \to M_\omega$, $H(\xi) = (h^\xi_i | i < \omega)$, where $h^\xi_i\colon (\lambda)^{<\omega} \to \mathcal{P}(\lambda)$ is defined by $h^\xi_i(x) = (\pi_{0,\omega}(f_i)(x))(\pi_{\omega,\omega+\omega}(\xi))$, if this is a subset of λ, and $h^\xi_i(x) = \emptyset$ else. By $(*)$ H is injective and its domain is λ^{++}. But this contradicts $M_\omega \models \mathrm{card}(\mathrm{rge} H) \leq ((2^\lambda)^\lambda)^\omega = 2^\lambda = \lambda^+$, using **GCH** inside M_ω □

6 Further Aspects

1. More detailed studies of the choice situation show $N \models \lambda^+ - \mathbf{AC}$, the axiom of choice for λ^+-sequences and $N \models < \lambda^+ - \mathbf{DC}$, the axiom of dependent choice for sequences shorter than λ^+. This is not true for sequences of length λ^+, i.e. $N \models \neg\lambda^+ - \mathbf{DC}$. Indeed it is not possible to force over N with partial choice functions for $\tilde{C}$ of size $\leq \lambda$ without collapsing λ^+ and thus destroying the $\neg$**SCH**-situation.

2. $N = M_\omega[\tilde{C}]$, i.e., N is the smallest transitive model of **ZF** containing M_ω and $\tilde{C}$.

3. Gitik-Magidor forcing over M_ω with the canonical extender at λ derived from $\pi_{\omega,\omega+1}$ yields a model $N^* = M_\omega[(c_\alpha | \alpha < \kappa_{\omega+1})]$, where each c_α is an ω-sequence cofinal in κ. It is possible in the context of countable ground models to find N^* such that $\forall \alpha < \kappa_{\omega+1}\colon c_\alpha \in \tilde{C}(\alpha)$. Then N is a natural submodel of N^* and the generic object for Gitik-Magidor forcing is basically a choice function for the Prikry system $\tilde{C}$. We shall discuss this in a subsequent article.

4. Ideas from this paper can be applied to other "Prikry-like" forcings as e.g. Magidor forcing [Ma1975] and Radin forcing.

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