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A Sequent Calculus for First-Order Logic[1]

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Summary. This article is part of a series of Mizar articles which constitute a formal proof (of a basic version) of Kurt Gödel's famous completeness theorem (K. Gödel, "Die Vollständigkeit der Axiome des logischen Funktionenkalküls", Monatshefte für Mathematik und Physik 37 (1930), 349–360). The completeness theorem provides the theoretical basis for a uniform formalization of mathematics as in the Mizar project. We formalize first-order logic up to the completeness theorem as in H. D. Ebbinghaus, J. Flum, and W. Thomas, Mathematical Logic, 1984, Springer Verlag New York Inc. The present article introduces a sequent calculus for first-order logic. The correctness of this calculus is shown and some important inferences are derived. The contents of this article correspond to Chapter IV of Ebbinghaus, Flum, Thomas.

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The notation and terminology used here are introduced in the following papers: [18], [11], [20], [4], [9], [14], [15], [3], [1], [2], [8], [23], [12], [21], [13], [24], [10], [17], [22], [16], [19], [6], [7], and [5].

1. Preliminaries

For simplicity, we adopt the following rules: a, b, c, d denote sets, i, j, m, n denote natural numbers, p, q, r denote elements of CQC-WFF, x, y denote bound variables, X denotes a subset of CQC-WFF, A denotes a non empty set,

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J denotes an interpretation of A, v, w denote elements of $\boldsymbol{V}(A)$, S_1 denotes a CQC-substitution, and f, g denote finite sequences of elements of CQC-WFF.

Let g be a finite sequence and let N be a set. Observe that $g{\restriction}N$ is finite subsequence-like.

Let D be a non empty set and let f be a finite sequence of elements of D. The functor $\mathrm{Ant}(f)$ yields a finite sequence of elements of D and is defined as follows:

(Def. 1)(i) For every i such that $\mathrm{len}\, f = i+1$ holds $\mathrm{Ant}(f) = f{\restriction}\mathrm{Seg}\, i$ if $\mathrm{len}\, f > 0$,

(ii) $\mathrm{Ant}(f) = \emptyset$, otherwise.

Let D be a non empty set and let f be a finite sequence of elements of D. Let us assume that $\mathrm{len}\, f > 0$. The functor $\mathrm{Suc}(f)$ yielding an element of D is defined as follows:

(Def. 2) $\mathrm{Suc}(f) = f(\mathrm{len}\, f)$.

Let D be a non empty set, let p be an element of D, and let f be a finite sequence of elements of D. We say that p is a tail of f if and only if:

(Def. 3) There exists i such that $i \in \mathrm{dom}\, f$ and $f(i) = p$.

Let us consider f, g. We say that f is a subsequence of g if and only if:

(Def. 4) There exists a subset N of $\mathbb{N}$ such that $f \subseteq \mathrm{Seq}(g{\restriction}N)$.

We now state several propositions:

(1) If f is a subsequence of g, then $\mathrm{rng}\, f \subseteq \mathrm{rng}\, g$ and there exists a subset N of $\mathbb{N}$ such that $\mathrm{rng}\, f \subseteq \mathrm{rng}(g{\restriction}N)$.

(2) If $\mathrm{len}\, f > 0$, then $\mathrm{len}\,\mathrm{Ant}(f) + 1 = \mathrm{len}\, f$ and $\mathrm{len}\,\mathrm{Ant}(f) < \mathrm{len}\, f$.

(3) If $\mathrm{len}\, f > 0$, then $f = (\mathrm{Ant}(f)) ^\frown \langle \mathrm{Suc}(f) \rangle$ and $\mathrm{rng}\, f = \mathrm{rng}\,\mathrm{Ant}(f) \cup \{\mathrm{Suc}(f)\}$.

(4) If $\mathrm{len}\, f > 1$, then $\mathrm{len}\,\mathrm{Ant}(f) > 0$.

(5) $\mathrm{Suc}(f ^\frown \langle p \rangle) = p$ and $\mathrm{Ant}(f ^\frown \langle p \rangle) = f$.

In the sequel f_1, f_2 are finite sequences.

We now state several propositions:

(6) $\mathrm{len}\, f_1 \leq \mathrm{len}(f_1 {}^\frown f_2)$ and $\mathrm{len}\, f_2 \leq \mathrm{len}(f_1 {}^\frown f_2)$ and if $f_1 \neq \emptyset$, then $1 \leq \mathrm{len}\, f_1$ and $\mathrm{len}\, f_2 < \mathrm{len}(f_2 ^\frown f_1)$.

(7) $\mathrm{Seq}((f ^\frown g){\restriction}\mathrm{dom}\, f) = (f ^\frown g){\restriction}\mathrm{dom}\, f$.

(8) f is a subsequence of $f ^\frown g$.

(9) $1 < \mathrm{len}(f_1 ^\frown \langle b \rangle ^\frown \langle c \rangle)$.

(10) $1 \leq \mathrm{len}(f_1 ^\frown \langle b \rangle)$ and $\mathrm{len}(f_1 ^\frown \langle b \rangle) \in \mathrm{dom}(f_1 ^\frown \langle b \rangle)$.

(11) If $0 < m$, then $\mathrm{len}\,\mathrm{Sgm}(\mathrm{Seg}\, n \cup \{n+m\}) = n+1$.

(12) If $0 < m$, then $\mathrm{dom}\,\mathrm{Sgm}(\mathrm{Seg}\, n \cup \{n+m\}) = \mathrm{Seg}(n+1)$.

(13) If $0 < \mathrm{len}\, f$, then f is a subsequence of $(\mathrm{Ant}(f)) ^\frown g ^\frown \langle \mathrm{Suc}(f) \rangle$.

(14) $1 \in \mathrm{dom}\langle c, d\rangle$ and $2 \in \mathrm{dom}\langle c, d\rangle$ and $(f \frown \langle c, d\rangle)(\mathrm{len}\, f + 1) = c$ and $(f \frown \langle c, d\rangle)(\mathrm{len}\, f + 2) = d$.

2. A Sequent Calculus

Let us consider f. The functor $\mathrm{snb}(f)$ yielding an element of 2^{BoundVar} is defined by:

(Def. 5) $a \in \mathrm{snb}(f)$ iff there exist i, p such that $i \in \mathrm{dom}\, f$ and $p = f(i)$ and $a \in \mathrm{snb}(p)$.

The set of CQC-WFF-sequences is defined as follows:

(Def. 6) $a \in$ the set of CQC-WFF-sequences iff a is a finite sequence of elements of CQC-WFF.

In the sequel P_1, P_2 denote finite sequences of elements of [: the set of CQC-WFF-sequences, $\mathbb{K}$:].

Let us consider P_1 and let n be a natural number. We say that step n in P_1 is correct if and only if:

(Def. 7)(i) There exists f such that $\mathrm{Suc}(f)$ is a tail of $\mathrm{Ant}(f)$ and $P_1(n)_{\mathbf{1}} = f$ if $P_1(n)_{\mathbf{2}} = 0$,

(ii) there exists f such that $P_1(n)_{\mathbf{1}} = f \frown \langle \mathrm{VERUM}\rangle$ if $P_1(n)_{\mathbf{2}} = 1$,

(iii) there exist i, f, g such that $1 \leq i$ and $i < n$ and $\mathrm{Ant}(f)$ is a subsequence of $\mathrm{Ant}(g)$ and $\mathrm{Suc}(f) = \mathrm{Suc}(g)$ and $P_1(i)_{\mathbf{1}} = f$ and $P_1(n)_{\mathbf{1}} = g$ if $P_1(n)_{\mathbf{2}} = 2$,

(iv) there exist i, j, f, g such that $1 \leq i$ and $i < n$ and $1 \leq j$ and $j < i$ and $\mathrm{len}\, f > 1$ and $\mathrm{len}\, g > 1$ and $\mathrm{Ant}(\mathrm{Ant}(f)) = \mathrm{Ant}(\mathrm{Ant}(g))$ and $\neg\,\mathrm{Suc}(\mathrm{Ant}(f)) = \mathrm{Suc}(\mathrm{Ant}(g))$ and $\mathrm{Suc}(f) = \mathrm{Suc}(g)$ and $f = P_1(j)_{\mathbf{1}}$ and $g = P_1(i)_{\mathbf{1}}$ and $(\mathrm{Ant}(\mathrm{Ant}(f))) \frown \langle \mathrm{Suc}(f)\rangle = P_1(n)_{\mathbf{1}}$ if $P_1(n)_{\mathbf{2}} = 3$,

(v) there exist i, j, f, g, p such that $1 \leq i$ and $i < n$ and $1 \leq j$ and $j < i$ and $\mathrm{len}\, f > 1$ and $\mathrm{Ant}(f) = \mathrm{Ant}(g)$ and $\mathrm{Suc}(\mathrm{Ant}(f)) = \neg p$ and $\neg\,\mathrm{Suc}(f) = \mathrm{Suc}(g)$ and $f = P_1(j)_{\mathbf{1}}$ and $g = P_1(i)_{\mathbf{1}}$ and $(\mathrm{Ant}(\mathrm{Ant}(f))) \frown \langle p\rangle = P_1(n)_{\mathbf{1}}$ if $P_1(n)_{\mathbf{2}} = 4$,

(vi) there exist i, j, f, g such that $1 \leq i$ and $i < n$ and $1 \leq j$ and $j < i$ and $\mathrm{Ant}(f) = \mathrm{Ant}(g)$ and $f = P_1(j)_{\mathbf{1}}$ and $g = P_1(i)_{\mathbf{1}}$ and $(\mathrm{Ant}(f)) \frown \langle \mathrm{Suc}(f) \wedge \mathrm{Suc}(g)\rangle = P_1(n)_{\mathbf{1}}$ if $P_1(n)_{\mathbf{2}} = 5$,

(vii) there exist i, f, p, q such that $1 \leq i$ and $i < n$ and $p \wedge q = \mathrm{Suc}(f)$ and $f = P_1(i)_{\mathbf{1}}$ and $(\mathrm{Ant}(f)) \frown \langle p\rangle = P_1(n)_{\mathbf{1}}$ if $P_1(n)_{\mathbf{2}} = 6$,

(viii) there exist i, f, p, q such that $1 \leq i$ and $i < n$ and $p \wedge q = \mathrm{Suc}(f)$ and $f = P_1(i)_{\mathbf{1}}$ and $(\mathrm{Ant}(f)) \frown \langle q\rangle = P_1(n)_{\mathbf{1}}$ if $P_1(n)_{\mathbf{2}} = 7$,

(ix) there exist i, f, p, x, y such that $1 \leq i$ and $i < n$ and $\mathrm{Suc}(f) = \forall_x p$ and $f = P_1(i)_{\mathbf{1}}$ and $(\mathrm{Ant}(f)) \frown \langle p(x, y)\rangle = P_1(n)_{\mathbf{1}}$ if $P_1(n)_{\mathbf{2}} = 8$,

(x) there exist i, f, p, x, y such that $1 \leq i$ and $i < n$ and $\mathrm{Suc}(f) = p(x, y)$ and $y \notin \mathrm{snb}(\mathrm{Ant}(f))$ and $y \notin \mathrm{snb}(\forall_x p)$ and $f = P_1(i)_{\mathbf{1}}$ and $(\mathrm{Ant}(f)) \frown \langle \forall_x p \rangle = P_1(n)_{\mathbf{1}}$ if $P_1(n)_{\mathbf{2}} = 9$.

Let us consider P_1. We say that P_1 is a formal proof if and only if:

(Def. 8) $P_1 \neq \emptyset$ and for every n such that $1 \leq n$ and $n \leq \mathrm{len}\, P_1$ holds step n in P_1 is correct.

Let us consider f. The predicate $\vdash f$ is defined by:

(Def. 9) There exists P_1 such that P_1 is a formal proof and $f = P_1(\mathrm{len}\, P_1)_{\mathbf{1}}$.

Let us consider p, X. We say that p is formally provable from X if and only if:

(Def. 10) There exists f such that $\mathrm{rng}\,\mathrm{Ant}(f) \subseteq X$ and $\mathrm{Suc}(f) = p$ and $\vdash f$.

Let us consider X, let us consider A, let us consider J, and let us consider v. The predicate $J, v \models X$ is defined as follows:

(Def. 11) If $p \in X$, then $J, v \models p$.

Let us consider X, p. The predicate $X \models p$ is defined as follows:

(Def. 12) If $J, v \models X$, then $J, v \models p$.

Let us consider p. The predicate $\vDash p$ is defined as follows:

(Def. 13) $\emptyset_{\text{CQC-WFF}} \models p$.

Let us consider f, A, J, v. The predicate $J, v \models f$ is defined as follows:

(Def. 14) $J, v \models \mathrm{rng}\, f$.

Let us consider f, p. The predicate $f \models p$ is defined by:

(Def. 15) If $J, v \models f$, then $J, v \models p$.

One can prove the following propositions:

(15) If $\mathrm{Suc}(f)$ is a tail of $\mathrm{Ant}(f)$, then $\mathrm{Ant}(f) \models \mathrm{Suc}(f)$.

(16) If $\mathrm{Ant}(f)$ is a subsequence of $\mathrm{Ant}(g)$ and $\mathrm{Suc}(f) = \mathrm{Suc}(g)$ and $\mathrm{Ant}(f) \models \mathrm{Suc}(f)$, then $\mathrm{Ant}(g) \models \mathrm{Suc}(g)$.

(17) If $\mathrm{len}\, f > 0$, then $J, v \models \mathrm{Ant}(f)$ and $J, v \models \mathrm{Suc}(f)$ iff $J, v \models f$.

(18) If $\mathrm{len}\, f > 1$ and $\mathrm{len}\, g > 1$ and $\mathrm{Ant}(\mathrm{Ant}(f)) = \mathrm{Ant}(\mathrm{Ant}(g))$ and $\neg\,\mathrm{Suc}(\mathrm{Ant}(f)) = \mathrm{Suc}(\mathrm{Ant}(g))$ and $\mathrm{Suc}(f) = \mathrm{Suc}(g)$ and $\mathrm{Ant}(f) \models \mathrm{Suc}(f)$ and $\mathrm{Ant}(g) \models \mathrm{Suc}(g)$, then $\mathrm{Ant}(\mathrm{Ant}(f)) \models \mathrm{Suc}(f)$.

(19) If $\mathrm{len}\, f > 1$ and $\mathrm{Ant}(f) = \mathrm{Ant}(g)$ and $\neg p = \mathrm{Suc}(\mathrm{Ant}(f))$ and $\neg\,\mathrm{Suc}(f) = \mathrm{Suc}(g)$ and $\mathrm{Ant}(f) \models \mathrm{Suc}(f)$ and $\mathrm{Ant}(g) \models \mathrm{Suc}(g)$, then $\mathrm{Ant}(\mathrm{Ant}(f)) \models p$.

(20) If $\mathrm{Ant}(f) = \mathrm{Ant}(g)$ and $\mathrm{Ant}(f) \models \mathrm{Suc}(f)$ and $\mathrm{Ant}(g) \models \mathrm{Suc}(g)$, then $\mathrm{Ant}(f) \models \mathrm{Suc}(f) \wedge \mathrm{Suc}(g)$.

(21) If $\mathrm{Suc}(f) = p \wedge q$ and $\mathrm{Ant}(f) \models p \wedge q$, then $\mathrm{Ant}(f) \models p$.

(22) If $\mathrm{Suc}(f) = p \wedge q$ and $\mathrm{Ant}(f) \models p \wedge q$, then $\mathrm{Ant}(f) \models q$.

(23) $J, v \models \langle p, S_1 \rangle$ iff $J, v \models p$.

In the sequel a is an element of A.

We now state several propositions:

(24) $J, v \models p(x, y)$ iff there exists a such that $v(y) = a$ and $J, v(x{\restriction}a) \models p$.

(25) If $\mathrm{Suc}(f) = \forall_x p$ and $\mathrm{Ant}(f) \models \mathrm{Suc}(f)$, then for every y holds $\mathrm{Ant}(f) \models p(x, y)$.

(26) For every set X such that $X \subseteq \mathrm{BoundVar}$ holds if $x \notin X$, then $v(x{\restriction}a){\restriction}X = v{\restriction}X$.

(27) For all v, w such that $v{\restriction}\mathrm{snb}(f) = w{\restriction}\mathrm{snb}(f)$ holds $J, v \models f$ iff $J, w \models f$.

(28) If $y \notin \mathrm{snb}(\forall_x p)$, then $v(y{\restriction}a)(x{\restriction}a){\restriction}\mathrm{snb}(p) = v(x{\restriction}a){\restriction}\mathrm{snb}(p)$.

(29) If $\mathrm{Suc}(f) = p(x, y)$ and $\mathrm{Ant}(f) \models \mathrm{Suc}(f)$ and $y \notin \mathrm{snb}(\mathrm{Ant}(f))$ and $y \notin \mathrm{snb}(\forall_x p)$, then $\mathrm{Ant}(f) \models \forall_x p$.

(30) $\mathrm{Ant}(f \frown \langle \mathrm{VERUM} \rangle) \models \mathrm{Suc}(f \frown \langle \mathrm{VERUM} \rangle)$.

(31) Suppose $1 \leq n$ and $n \leq \mathrm{len}\, P_1$. Then $P_1(n)_{\mathbf{2}} = 0$ or $P_1(n)_{\mathbf{2}} = 1$ or $P_1(n)_{\mathbf{2}} = 2$ or $P_1(n)_{\mathbf{2}} = 3$ or $P_1(n)_{\mathbf{2}} = 4$ or $P_1(n)_{\mathbf{2}} = 5$ or $P_1(n)_{\mathbf{2}} = 6$ or $P_1(n)_{\mathbf{2}} = 7$ or $P_1(n)_{\mathbf{2}} = 8$ or $P_1(n)_{\mathbf{2}} = 9$.

(32) If p is formally provable from X, then $X \models p$.

3. Derived Rules

Next we state a number of propositions:

(33) If $\mathrm{Suc}(f)$ is a tail of $\mathrm{Ant}(f)$, then $\vdash f$.

(34) If $1 \leq n$ and $n \leq \mathrm{len}\, P_1$, then step n in P_1 is correct iff step n in $P_1 \frown P_2$ is correct.

(35) If $1 \leq n$ and $n \leq \mathrm{len}\, P_2$ and step n in P_2 is correct, then step $n + \mathrm{len}\, P_1$ in $P_1 \frown P_2$ is correct.

(36) If $\mathrm{Ant}(f)$ is a subsequence of $\mathrm{Ant}(g)$ and $\mathrm{Suc}(f) = \mathrm{Suc}(g)$ and $\vdash f$, then $\vdash g$.

(37) If $1 < \mathrm{len}\, f$ and $1 < \mathrm{len}\, g$ and $\mathrm{Ant}(\mathrm{Ant}(f)) = \mathrm{Ant}(\mathrm{Ant}(g))$ and $\neg\, \mathrm{Suc}(\mathrm{Ant}(f)) = \mathrm{Suc}(\mathrm{Ant}(g))$ and $\mathrm{Suc}(f) = \mathrm{Suc}(g)$ and $\vdash f$ and $\vdash g$, then $\vdash (\mathrm{Ant}(\mathrm{Ant}(f))) \frown \langle \mathrm{Suc}(f) \rangle$.

(38) If $\mathrm{len}\, f > 1$ and $\mathrm{Ant}(f) = \mathrm{Ant}(g)$ and $\mathrm{Suc}(\mathrm{Ant}(f)) = \neg p$ and $\neg\, \mathrm{Suc}(f) = \mathrm{Suc}(g)$ and $\vdash f$ and $\vdash g$, then $\vdash (\mathrm{Ant}(\mathrm{Ant}(f))) \frown \langle p \rangle$.

(39) If $\mathrm{Ant}(f) = \mathrm{Ant}(g)$ and $\vdash f$ and $\vdash g$, then $\vdash (\mathrm{Ant}(f)) \frown \langle \mathrm{Suc}(f) \wedge \mathrm{Suc}(g) \rangle$.

(40) If $p \wedge q = \mathrm{Suc}(f)$ and $\vdash f$, then $\vdash (\mathrm{Ant}(f)) \frown \langle p \rangle$.

(41) If $p \wedge q = \mathrm{Suc}(f)$ and $\vdash f$, then $\vdash (\mathrm{Ant}(f)) \frown \langle q \rangle$.

(42) If $\mathrm{Suc}(f) = \forall_x p$ and $\vdash f$, then $\vdash (\mathrm{Ant}(f)) \frown \langle p(x, y) \rangle$.

(43) If $\mathrm{Suc}(f) = p(x, y)$ and $y \notin \mathrm{snb}(\mathrm{Ant}(f))$ and $y \notin \mathrm{snb}(\forall_x p)$ and $\vdash f$, then $\vdash (\mathrm{Ant}(f)) \frown \langle \forall_x p \rangle$.

(44) If $\vdash f$ and $\vdash (\mathrm{Ant}(f)) \frown \langle \neg \mathrm{Suc}(f) \rangle$, then $\vdash (\mathrm{Ant}(f)) \frown \langle p \rangle$.

(45) If $1 \leq \mathrm{len}\, f$ and $\vdash f$ and $\vdash f \frown \langle p \rangle$, then $\vdash (\mathrm{Ant}(f)) \frown \langle p \rangle$.

(46) If $\vdash f \frown \langle p \rangle \frown \langle q \rangle$, then $\vdash f \frown \langle \neg q \rangle \frown \langle \neg p \rangle$.

(47) If $\vdash f \frown \langle \neg p \rangle \frown \langle \neg q \rangle$, then $\vdash f \frown \langle q \rangle \frown \langle p \rangle$.

(48) If $\vdash f \frown \langle \neg p \rangle \frown \langle q \rangle$, then $\vdash f \frown \langle \neg q \rangle \frown \langle p \rangle$.

(49) If $\vdash f \frown \langle p \rangle \frown \langle \neg q \rangle$, then $\vdash f \frown \langle q \rangle \frown \langle \neg p \rangle$.

(50) If $\vdash f \frown \langle p \rangle \frown \langle r \rangle$ and $\vdash f \frown \langle q \rangle \frown \langle r \rangle$, then $\vdash f \frown \langle p \vee q \rangle \frown \langle r \rangle$.

(51) If $\vdash f \frown \langle p \rangle$, then $\vdash f \frown \langle p \vee q \rangle$.

(52) If $\vdash f \frown \langle q \rangle$, then $\vdash f \frown \langle p \vee q \rangle$.

(53) If $\vdash f \frown \langle p \rangle \frown \langle r \rangle$ and $\vdash f \frown \langle q \rangle \frown \langle r \rangle$, then $\vdash f \frown \langle p \vee q \rangle \frown \langle r \rangle$.

(54) If $\vdash f \frown \langle p \rangle$, then $\vdash f \frown \langle \neg\neg p \rangle$.

(55) If $\vdash f \frown \langle \neg\neg p \rangle$, then $\vdash f \frown \langle p \rangle$.

(56) If $\vdash f \frown \langle p \Rightarrow q \rangle$ and $\vdash f \frown \langle p \rangle$, then $\vdash f \frown \langle q \rangle$.

(57) $(\neg p)(x, y) = \neg p(x, y)$.

(58) If there exists y such that $\vdash f \frown \langle p(x, y) \rangle$, then $\vdash f \frown \langle \exists_x p \rangle$.

(59) $\mathrm{snb}(f \frown g) = \mathrm{snb}(f) \cup \mathrm{snb}(g)$.

(60) $\mathrm{snb}(\langle p \rangle) = \mathrm{snb}(p)$.

(61) If $\vdash f \frown \langle p(x, y) \rangle \frown \langle q \rangle$ and $y \notin \mathrm{snb}(f \frown \langle \exists_x p \rangle \frown \langle q \rangle)$, then $\vdash f \frown \langle \exists_x p \rangle \frown \langle q \rangle$.

(62) $\mathrm{snb}(f) = \bigcup \{\mathrm{snb}(p) : \bigvee_i \ (i \in \mathrm{dom}\, f \ \wedge \ p = f(i))\}$.

(63) $\mathrm{snb}(f)$ is finite.

(64) $\overline{\overline{\mathrm{BoundVar}}} = \aleph_0$ and BoundVar is not finite.

(65) There exists x such that $x \notin \mathrm{snb}(f)$.

(66) If $\vdash f \frown \langle \forall_x p \rangle$, then $\vdash f \frown \langle \forall_x \neg\neg p \rangle$.

(67) If $\vdash f \frown \langle \forall_x \neg\neg p \rangle$, then $\vdash f \frown \langle \forall_x p \rangle$.

(68) $\vdash f \frown \langle \forall_x p \rangle$ iff $\vdash f \frown \langle \neg \exists_x \neg p \rangle$.

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