l-ADIC PERVERSE SHEAVES

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0.1. **Introduction.** The purpose of this expository note is to outline the construction of the abelian category of perverse sheaves in the l-adic setting and to illustrate Deligne's theory of weights in this context. But even though we get to this at the very end, the bulk of the paper is devoted to prerequisite constructions leading to it. We begin by constructing the abelian category of $\overline{\mathbb{Q}}_l$-sheaves on a scheme X (see section 0.3 for assumptions on X that are valid throughout), as well as the corresponding triangulated "derived" category $D_c^b(X, \overline{\mathbb{Q}}_l)$. We then discuss the yoga of Grothendieck's six operations in this setting and introduce the standard and perverse t-structures which lead to l-adic perverse sheaves on X. Finally, we move on to the theory of weights and state the existence of a functorial weight filtration for mixed sheaves both in the usual and perverse contexts.

To keep the exposition brief we were forced to omit most proofs but have attempted to give references in such instances and hope that the reader seeking more detail will have no trouble locating it in the literature.

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0.3. **Conventions.** Throughout X (or Y, etc.) will denote a finite type separated scheme over a finite field $\mathbb{F}_q$ or over its fixed algebraic closure $\mathbb{F} = \overline{\mathbb{F}}_q$. Here q is a power of the characteristic p. All sheaves that we are going to consider will be sheaves of abelian groups on the étale site of X (or Y, etc.). We will fix a prime $l \neq p$ and denote by $\mathbb{Q}_l$ be the field of l-adic numbers. E (or F, F') will stand for a finite extension $E/\mathbb{Q}_l$ inside a *fixed* algebraic closure $\overline{\mathbb{Q}}_l$. We will denote by $\mathfrak{o}$ the ring of integers of E with a uniformizer $\pi \in \mathfrak{o}$. Sometimes we will be dealing with a tower of (always finite) extensions $F'/F/E$ in which case the corresponding rings of integers will be $\mathfrak{O}'/\mathfrak{O}/\mathfrak{o}$ with uniformizers Π', Π and π, respectively. We will write $\mathfrak{o}_i$ for $\mathfrak{o}/\pi^i\mathfrak{o}$, $i \geqslant 1$, and similarly $\mathfrak{O}_i$ for $\mathfrak{O}/\Pi^i\mathfrak{O}$. When dealing with categories $\mathcal{A}$ we often slip and write $\mathscr{F} \in \mathcal{A}$ instead of $\mathscr{F} \in \mathrm{Ob}(\mathcal{A})$ to indicate that $\mathscr{F}$ is an object of $\mathcal{A}$.

It should be noted that some constructions that we carry out can be made in a less stringent setting than the one described in the previous paragraph. To be consistent we chose these conventions to be valid throughout but the reader should have no trouble in locating instances where we do not make full use of these assumptions.

1. $\overline{\mathbb{Q}}_l$-SHEAVES

1.1. **Constructible sheaves.** Recall that a sheaf of sets $\mathscr{F}$ on X is called constructible if any of the following equivalent conditions hold:

1. X can be written as a finite union of locally closed subschemes X_i such that each $\mathscr{F}|_{X_i}$ is finite locally constant, i.e., for each X_i there is an étale cover $\{U_{ij} \to X_i\}_{j \in J_i}$, $\# J_i < \infty$ such that $\mathscr{F}|_{U_{ij}}$ are constant and $\mathscr{F}(U_{ij})$ are finite.
2. $\mathscr{F}$ is a noetherian object in the abelian category of torsion sheaves on X. (Recall that a sheaf $\mathscr{G}$ is called *torsion* if each element of $\mathscr{G}(U)$ is torsion; an object of an abelian category is called *noetherian* if every increasing chain of its subobjects stabilizes.)

The proof of the equivalence can be found in [FK88, Proposition I.4.8]. Constructible sheaves form an abelian subcategory of torsion sheaves: the zero sheaf, subobjects and quotients of constructible sheaves are constructible using **2.**, a direct sum of two constructible sheaves is constructible using **1.** (more generally, the full subcategory of noetherian objects in a locally noetherian abelian category is an abelian subcategory).

1.2. **Artin-Rees categories.** A *projective system* of sheaves is a collection $\mathscr{F} = (\mathscr{F}_i)_{i \geqslant 1}$ of constructible sheaves of torsion $\mathfrak{o}$-modules together with structure morphisms $\mathscr{F}_{i+1} \to \mathscr{F}_i$. A morphism of projective systems $\mathscr{F} \to \mathscr{G}$ is a collection of morphisms $\mathscr{F}_i \to \mathscr{G}_i$ which together with the structure morphisms of $\mathscr{F}$ and $\mathscr{G}$ form an infinite commutative ladder. The category of all projective systems is abelian, we will call it $\mathcal{P}$. Given a projective system $\mathscr{F}$ one can form its shifts $\mathscr{F}[n]$, $n \geqslant 0$ by setting $(\mathscr{F}[n])_i = \mathscr{F}_{n+i}$ together with obvious structure morphisms. Shifts $\mathscr{F}[n]$ come equipped with morphisms $\mathscr{F}[n] \to \mathscr{F}$ obtained using the structure morphisms of $\mathscr{F}$. A *null system* is a projective system $\mathscr{N}$ such that $\mathscr{N}[n] \to \mathscr{N}$ is zero for big enough n; this means that there is an n such that all composites of structure morphisms $\mathscr{N}_{i+n} \to \mathscr{N}_i$ vanish. The full subcategory $\mathcal{N}$ of null systems is a Serre subcategory of the abelian category $\mathcal{P}$ of projective systems: indeed, if $0 \to \mathscr{G} \to \mathscr{F} \to \mathscr{H} \to 0$ is exact and $\mathscr{G}[n] \to \mathscr{G}$, $\mathscr{H}[m] \to \mathscr{H}$ vanish then so does $\mathscr{F}[n+m] \to \mathscr{F}$. The *Artin-Rees category* (A-R category for short) is the quotient abelian category $\mathcal{P}/\mathcal{N}$. We remind the reader that the objects of $\mathcal{P}/\mathcal{N}$ are those of $\mathcal{P}$, while the set of morphisms between $\mathscr{F}, \mathscr{G} \in \mathrm{Ob}(\mathcal{P}/\mathcal{N})$ is $\mathrm{Hom}_{\mathcal{P}/\mathcal{N}}(\mathscr{F}, \mathscr{G}) = \varinjlim \mathrm{Hom}_{\mathcal{P}}(\mathscr{F}', \mathscr{G}/\mathscr{G}')$ where the colimit is taken over all subobjects $\mathscr{F}'$ of $\mathscr{F}$ such that $\mathscr{F}/\mathscr{F}' \in \mathcal{N}$ and all subobjects $\mathscr{G}'$ of $\mathscr{G}$ such that $\mathscr{G}' \in \mathcal{N}$.

There is another description of the A-R category which is more useful in practice. Namely, one can show that the class $\mathfrak{N}$ of all morphisms of the form $\mathscr{F}[n] \to \mathscr{F}$ is both a left and a right Ore system (in other terminology these morphisms admit a calculus of fractions); it is easy but somewhat tedious to check the axioms, this is done in [SGA_5, Exposé V, Proposition 2.4.1]. Therefore, the localized category $\mathcal{P}[\mathfrak{N}^{-1}]$ admits a particularly simple description: it's objects are those of $\mathcal{P}$, while the set[1] of morphisms between $\mathscr{F}, \mathscr{G} \in \mathrm{Ob}(\mathcal{P}[\mathfrak{N}^{-1}])$ is $\mathrm{Hom}_{\mathcal{P}[\mathfrak{N}^{-1}]} = \varinjlim \mathrm{Hom}_{\mathcal{P}}(\mathscr{F}[n], \mathscr{G})$. Note that since $\mathscr{F}/\mathscr{F}[n]$ is a null system we have a canonical functor $F\colon \mathcal{P}[\mathfrak{N}^{-1}] \to \mathcal{P}/\mathcal{N}$ which is identity on objects.

Proposition 1.2.1. *The functor F induces an* isomorphism *of categories $\mathcal{P}[\mathfrak{N}^{-1}] \cong \mathcal{P}/\mathcal{N}$. In particular, the morphisms in the A-R category between $\mathscr{F}$ and $\mathscr{G}$ are equivalence classes of morphisms $\mathscr{F}[n] \to \mathscr{G}$, so that $\mathscr{N} \cong 0$ in the A-R category if and only if $\mathscr{N} \in \mathcal{N}$.*

Proof. Consider the first functor in $\mathcal{P} \to \mathcal{P}[\mathfrak{N}^{-1}] \xrightarrow{F} \mathcal{P}/\mathcal{N}$. For $\mathscr{N} \in \mathcal{N}$ it sends each $\mathscr{N}[n] \to \mathscr{N}$ to an isomorphism. But for big enough n these morphisms are 0. In an additive category a zero morphism can be an isomorphism only between zero objects. Thus each $\mathscr{N} \cong 0$ in $\mathcal{P}[\mathfrak{N}^{-1}]$ and the universal property of $\mathcal{P}/\mathcal{N}$ gives the functor $G\colon \mathcal{P}/\mathcal{N} \to \mathcal{P}[\mathfrak{N}^{-1}]$ which using the universal properties of $\mathcal{P}/\mathcal{N}$ and $\mathcal{P}[\mathfrak{N}^{-1}]$ is seen to be a strict inverse to F. (Alternatively, the same argument shows

[1] Here and in the sequel we do not worry about set theoretic issues. The usual yadda yadda applies: for instance, one can use universes to avoid headaches.

that $\mathcal{P}/\mathcal{N}$ satisfies the universal property of $\mathcal{P}[\mathfrak{N}^{-1}]$ so that F is an equivalence; being identity on objects it must be an isomorphism.) $\square$

As all morphisms $\mathscr{F}[n] \to \mathscr{F}$ become isomorphisms in the A-R category $\mathcal{P}/\mathcal{N}$ one sees that the (co)kernel of a morphism $\mathscr{F} \to \mathscr{G}$ in $\mathcal{P}/\mathcal{N}$ can be computed by taking the (co)kernel of a representing morphism $\mathscr{F}[n] \to \mathscr{G}$.

1.3. **π-adic sheaves.** A *π-adic sheaf* is a projective system $\mathscr{F} = (\mathscr{F}_i)_{i\geqslant 1}$ such that $\pi^i \mathscr{F}_i = 0$ (effectively, each $\mathscr{F}_i$ is a sheaf of $\mathfrak{o}_i$-modules) and the morphisms $\mathscr{F}_{i+1} \to \mathscr{F}_i$ are induced from $\mathscr{F}_{i+1}/\pi^i\mathscr{F}_{i+1} \cong \mathscr{F}_i$. Unfortunately, the category of π-adic sheaves is not abelian if one defines it in the naïve way as a full subcategory of the category of projective systems because there is no reason for the kernel of a morphism of π-adic sheaves to be π-adic. This can be fixed by declaring an *Artin-Rees (or A-R) π-adic sheaf* to be a projective system $\mathscr{G}$ which is isomorphic to a π-adic sheaf in the A-R category. In the view of 1.2.1 this means that there is a π-adic sheaf $\mathscr{F}$ and a morphism $\mathscr{F}[n] \to \mathscr{G}$ whose kernel and cokernel are null systems. It is proved in [FK88, Proposition I.12.11] that A-R π-adic sheaves form an abelian subcategory $\mathrm{Sh}_{\text{A-R}}(X,\mathfrak{o})$ of the A-R category (strictly speaking, in loc. cit. the claim is proved for A-R l-adic sheaves but the same argument applies to the more general situation). The nontrivial part is in showing that kernels of morphisms of A-R π-adic sheaves are again A-R π-adic; in loc. cit. this is done by reducing to the case of stalks and then using Artin-Rees lemma.

Let us denote by $\mathrm{Sh}(X,\mathfrak{o})$ the full subcategory of $\mathrm{Sh}_{\text{A-R}}(X,\mathfrak{o})$ spanned by π-adic sheaves. In fact, as the notation might suggest, this subcategory is nothing else but the category of π-adic sheaves:

Proposition 1.3.1. *The natural functor F from the category of π-adic sheaves to* $\mathrm{Sh}(X,\mathfrak{o})$ *is an* isomorphism *of categories.*

Proof. As F is identity on objects, in the view of 1.2.1 all we need to show is that for π-adic sheaves $\mathscr{F}$ and $\mathscr{G}$ any morphism $\mathscr{F}[n] \to \mathscr{G}$ factors uniquely through a morphism $\mathscr{F} \to \mathscr{G}$. But indeed, as $\mathscr{G}_m$ is annihilated by π^m any morphism $\mathscr{F}_{m+n} \to \mathscr{G}_m$ factors uniquely through $\mathscr{F}_{m+n}/\pi^m\mathscr{F}_{m+n} \cong \mathscr{F}_m \to \mathscr{G}_m$. $\square$

Essentially by definition the category $\mathrm{Sh}_{\text{A-R}}(X,\mathfrak{o})$ is equivalent to its full subcategory $\mathrm{Sh}(X,\mathfrak{o})$. Construct a fully faithful surjective functor $E_{\mathfrak{o}}\colon \mathrm{Sh}_{\text{A-R}}(X,\mathfrak{o}) \to \mathrm{Sh}(X,\mathfrak{o})$ by choosing for each $\mathscr{G} \in \mathrm{Sh}_{\text{A-R}}(X,\mathfrak{o})$ an isomorphism $\mathscr{G} \xrightarrow{\sim} \mathscr{F} \in \mathrm{Sh}(X,\mathfrak{o})$ in such a way that for each $\mathscr{G} \in \mathrm{Sh}(X,\mathfrak{o})$ the identity is chosen. Since $\mathrm{Hom}_{\mathrm{Sh}(X,\mathfrak{o})}(\mathscr{F},\mathscr{G})$ is an $\mathfrak{o}$-module for any $\mathscr{F},\mathscr{G} \in \mathrm{Sh}(X,\mathfrak{o})$ (because in $\varprojlim \mathrm{Hom}(\mathscr{F}_i,\mathscr{G}_i)$ each $\mathrm{Hom}(\mathscr{F}_i,\mathscr{G}_i)$ is an $\mathfrak{o}_i$-module), we can use functors $E_{\mathfrak{o}}$ to view morphism abelian groups in $\mathrm{Sh}_{\text{A-R}}(X,\mathfrak{o})$ as $\mathfrak{o}$-modules.

Given a π-adic sheaf $\mathscr{F}$ no information is lost if we only consider every e^{th} sheaf $\mathscr{G}_i := \mathscr{F}_{ie}$ with transition morphisms for the projective system $\mathscr{G} = (\mathscr{G}_i)_{i\geqslant 1}$ induced from $\mathscr{G}_{i+1}/\pi^{ie}\mathscr{G}_{i+1} \cong \mathscr{G}_i$; such $\mathscr{G}$ will be called π^e-adic sheaves. In more precise terms, the category of π^e-adic sheaves $\mathscr{G}$ is equivalent to the category of π-adic sheaves $\mathscr{F}$. In particular, we can associate to a π-adic sheaf $\mathscr{F}$ the Π^e-adic sheaf $\mathscr{G} = \{\mathscr{F}_i \otimes_{\mathfrak{o}} \mathfrak{O}\}_{i\geqslant 1}$ and hence a Π-adic sheaf $\widetilde{S}_{\mathfrak{O}/\mathfrak{o}}(\mathscr{F})$ (cf. section 0.3 for the setup). It is clear that $\widetilde{S}_{\mathfrak{O}/\mathfrak{o}}\colon \mathrm{Sh}(X,\mathfrak{o}) \to \mathrm{Sh}(X,\mathfrak{O})$ becomes an additive functor and that there are natural isomorphisms $\widetilde{S}_{\mathfrak{o}/\mathfrak{o}} \cong \mathrm{Id}$, $\widetilde{S}_{\mathfrak{O}'/\mathfrak{O}} \circ \widetilde{S}_{\mathfrak{O}/\mathfrak{o}} \cong \widetilde{S}_{\mathfrak{O}'/\mathfrak{o}}$.

Using the functors $\widetilde{S}_{\mathfrak{O}/\mathfrak{o}}$ and $E_{\mathfrak{o}}$ we get functors $S_{\mathfrak{O}/\mathfrak{o}}\colon \mathrm{Sh}_{\text{A-R}}(X,\mathfrak{o}) \to \mathrm{Sh}_{\text{A-R}}(X,\mathfrak{O})$ and the compatibilities for $\widetilde{S}_{\mathfrak{O}/\mathfrak{o}}$ translate into analogous compatibilities for $S_{\mathfrak{O}/\mathfrak{o}}$.

1.4. **$\overline{\mathbb{Q}}_l$-sheaves.** The category $\mathrm{Sh}(X, E)$ of E-sheaves is constructed as follows: it has the same objects as $\mathrm{Sh}_{\text{A-R}}(X, \mathfrak{o})$, and $\mathscr{F} \in \mathrm{Sh}_{\text{A-R}}(X, \mathfrak{o})$ is denoted by $\mathscr{F} \otimes E$ when viewed as an object of $\mathrm{Sh}(X, E)$; its morphism sets are $\mathrm{Hom}_{\mathrm{Sh}(X,E)}(\mathscr{F} \otimes E, \mathscr{G} \otimes E) = \mathrm{Hom}_{\mathrm{Sh}_{\text{A-R}}(X,\mathfrak{o})}(\mathscr{F}, \mathscr{G}) \otimes_{\mathfrak{o}} E$ (recall from section 1.3 that $\mathrm{Hom}_{\mathrm{Sh}_{\text{A-R}}(X,\mathfrak{o})}(\mathscr{F}, \mathscr{G})$ are $\mathfrak{o}$-modules). Composition of morphisms is defined by composition in $\mathrm{Sh}(X, \mathfrak{o})$ on the first factor and by multiplication in E on the second. From functors $S_{\mathfrak{O}/\mathfrak{o}}$ and inclusions $E \hookrightarrow F$ we get functors $S_{F/E} \colon \mathrm{Sh}(X, E) \to \mathrm{Sh}(X, F)$ which are functorial in the sense that there are natural isomorphisms $S_{E/E} \cong \mathrm{Id}_{\mathrm{Sh}(X,E)}$, $S_{F'/F} \circ S_{F/E} \cong S_{F'/E}$.

We are now ready to define the category $\mathrm{Sh}(X, \overline{\mathbb{Q}}_l)$ of $\overline{\mathbb{Q}}_l$-sheaves as a "direct limit" of categories $\mathrm{Sh}(X, E)$ (and functors $S_{F/E}$) in the following way: the objects of $\mathrm{Sh}(X, \overline{\mathbb{Q}}_l)$ are $\mathscr{F} \otimes E \in \mathrm{Sh}(X, E)$ with the understanding that $\mathscr{F} \otimes E$ and its images $S_{F/E}(\mathscr{F} \otimes E)$ represent the same object. Thus any two objects of $\mathrm{Sh}(X, \overline{\mathbb{Q}}_l)$ can be represented by $\mathscr{F} \otimes E, \mathscr{G} \otimes E \in \mathrm{Sh}(X, E)$ for some E, and the set of morphisms between the corresponding objects of $\mathrm{Sh}(X, \overline{\mathbb{Q}}_l)$ is declared to be $\varinjlim_{F/E} \mathrm{Hom}_{\mathrm{Sh}(X,F)}(S_{F/E}(\mathscr{F} \otimes E), S_{F/E}(\mathscr{G} \otimes E))$ where the colimit is taken over all finite extensions F/E and is independent of the choice of representatives.

2. Bounded derived category of constructible $\overline{\mathbb{Q}}_l$-sheaves

2.1. **Perfect complexes.** Consider the bounded derived category $D^b(X, \mathfrak{o}_i)$ of the category $\mathrm{Sh}(X, \mathfrak{o}_i)$ of sheaves of $\mathfrak{o}_i$-modules. As any derived category of an abelian category, it is triangulated. Its full additive subcategory $D^b_c(X, \mathfrak{o}_i)$ consisting of complexes with constructible cohomology sheaves is a triangulated subcategory. Indeed, to see this one only needs to argue that whenever two vertices of a distinguished triangle are in $D^b_c(X, \mathfrak{o}_i)$, so is the third. This can be done using the associated long exact cohomology sequence because extensions of constructible sheaves are constructible (the latter can be seen using **2.** above coupled with Mitchell's embedding theorem because an extension of noetherian R-modules is noetherian for any ring R).

A sheaf $\mathscr{K}^n$ of $\mathfrak{o}_i$-modules is called *flat* if for each geometric point $\overline{x} \to X$ the stalk $\mathscr{K}^n_{\overline{x}}$ is a free $\mathfrak{o}_i$-module. A *perfect complex* is a complex $\mathscr{K}^\bullet \in D^b_c(X, \mathfrak{o}_i)$ which is *bounded*, meaning $\mathscr{K}^n = 0$ for n big enough or small enough (note that this condition is stronger than *cohomologically bounded* which means that the same holds for cohomology sheaves $H^n(\mathscr{K}^\bullet)$), and all $\mathscr{K}^n$ are constructible flat sheaves of $\mathfrak{o}_i$-modules. Let $D^b_{ctf}(X, \mathfrak{o}_i)$ be the full subcategory of $D^b_c(X, \mathfrak{o}_i)$ of complexes isomorphic to a perfect complex[2]. It is a triangulated subcategory of $D^b_c(X, \mathfrak{o}_i)$; to see this one only needs to argue that the cone construction can be done within $D^b_{ctf}(X, \mathfrak{o}_i)$ and this follows by noticing that the standard explicit construction of the mapping cone on the level of complexes preserves boundedness, flatness, and constructibility.

2.2. **Construction of $D^b_c(X, \mathfrak{o})$.** As $\mathfrak{o} = \varprojlim \mathfrak{o}_i$, we want to define the bounded derived category of cohomologically constructible "sheaves of $\mathfrak{o}$-modules" $D^b_c(X, \mathfrak{o})$ as a "projective limit" of categories $D^b_c(X, \mathfrak{o}_i)$. For technical reasons, however, we will restrict our attention to $D^b_{ctf}(X, \mathfrak{o}_i)$ rather than working with all of $D^b_c(X, \mathfrak{o}_i)$. The precise construction is this: the objects of $D^b_c(X, \mathfrak{o})$ are the sequences of complexes $\mathscr{K}^\bullet = (\mathscr{K}^\bullet_i)_{i \geqslant 1}$, $\mathscr{K}^\bullet_i \in D^b_{ctf}(X, \mathfrak{o}_i)$ together with isomorphisms $\phi^{\mathscr{K}}_i : \mathscr{K}^\bullet_{i+1} \otimes^{\mathbf{L}}_{\mathfrak{o}_{i+1}} \mathfrak{o}_i \xrightarrow{\sim} \mathscr{K}^\bullet_i$ (we explain why $- \otimes^{\mathbf{L}}_{\mathfrak{o}_{i+1}} \mathfrak{o}_i$ makes sense in the next paragraph). The morphisms in $D^b_c(X, \mathfrak{o})$ between $\mathscr{K}^\bullet$ and $\mathscr{L}^\bullet$ are given by systems $\{\psi_i \colon \mathscr{K}^\bullet_i \to \mathscr{L}^\bullet_i\}_{i \geqslant 1}$ of morphisms

[2] The reader may be wondering what 'ctf' stands for in $D^b_{ctf}(X, \mathfrak{o}_i)$. 'c' = constructible, 'tf' = finite Tor-dimension (see [SGA_4 III, Exposé XVII, Définition 4.1.9] for the latter).

ψ_i in $D^b_{ctf}(X,\mathfrak{o}_i)$ which are compatible in the sense that they render all the diagrams

$$\begin{tikzcd}
\mathscr{K}^\bullet_{i+1}\otimes^{\mathbf{L}}_{\mathfrak{o}_{i+1}}\mathfrak{o}_i \arrow[r, "\psi_{i+1}\otimes^{\mathbf{L}}_{\mathfrak{o}_{i+1}}\mathfrak{o}_i"] \arrow[d, "\phi_i^{\mathscr{K}}"] & \mathscr{L}^\bullet_{i+1}\otimes^{\mathbf{L}}_{\mathfrak{o}_{i+1}}\mathfrak{o}_i \arrow[d, "\phi_i^{\mathscr{L}}"] \\
\mathscr{K}^\bullet_i \arrow[r, "\psi_i"] & \mathscr{L}^\bullet_i
\end{tikzcd}$$

commutative.

The functor $-\otimes_{\mathfrak{o}_{i+1}}\mathfrak{o}_i\colon \mathrm{Sh}(X,\mathfrak{o}_{i+1})\to\mathrm{Sh}(X,\mathfrak{o}_i)$ is right exact and the class of flat sheaves of $\mathfrak{o}_{i+1}$-modules is easily seen to be an acyclic class for it in the sense that it satisfies the dual of the conditions of [Hai06, Proposition 5.6]. What this means is that it makes sense to talk of the left derived functor $-\otimes^{\mathbf{L}}_{\mathfrak{o}_{i+1}}\mathfrak{o}_i\colon D^b(X,\mathfrak{o}_{i+1})\to D^b(X,\mathfrak{o}_i)$ and that resolutions of objects of $D^b(X,\mathfrak{o}_{i+1})$ by complexes of flat sheaves can be used to compute its values. In particular, if $\mathscr{K}^\bullet_{i+1}\in D^b_{ctf}(X,\mathfrak{o}_{i+1})$ then $\mathscr{K}^\bullet_{i+1}\otimes^{\mathbf{L}}_{\mathfrak{o}_{i+1}}\mathfrak{o}_i\cong\mathscr{K}^\bullet_{i+1}\otimes_{\mathfrak{o}_{i+1}}\mathfrak{o}_i\in D^b_{ctf}(X,\mathfrak{o}_i)$, so that the $\phi_i^{\mathscr{K}}$ in the previous paragraph are interpreted as morphisms in $D^b_{ctf}(X,\mathfrak{o}_i)$. Also, if $\mathscr{F}^\bullet\in D^b(X,\mathfrak{o}_{i+1})$ then the natural morphism $\mathscr{F}^\bullet\to\mathscr{F}^\bullet\otimes_{\mathfrak{o}_{i+1}}\mathfrak{o}_i$ gives rise to a morphism $H^n(\mathscr{F}^\bullet)\to H^n(\mathscr{F}^\bullet\otimes_{\mathfrak{o}_{i+1}}\mathfrak{o}_i)$ of n^{th} cohomology sheaves. Using this for a flat resolution (which exists because of loc. cit.) we get a morphism $H^n(\mathscr{F}^\bullet)\to H^n(\mathscr{F}^\bullet\otimes^{\mathbf{L}}_{\mathfrak{o}_{i+1}}\mathfrak{o}_i)$ for $\mathscr{F}^\bullet\in D^b(X,\mathfrak{o}_{i+1})$. To see that the morphism obtained is independent of the chosen resolution and is functorial one considers all flat resolutions of $\mathscr{F}^\bullet$ simultaneously, notes that they form a cofinal pro-system among all quasi-isomorphisms $\mathscr{G}^\bullet\xrightarrow{\sim}\mathscr{F}^\bullet$ (loc. cit.) and invokes Deligne's definition of derived functors [SGA$_4$ III, Exposé XVII, Définition 1.2.1].

In the view of the discussion above, if we consider an object $\mathscr{K}^\bullet\in D^b_c(X,\mathfrak{o})$ then for each $n\in\mathbb{Z}$ we get a functorially associated projective system $\{H^n(\mathscr{K}^\bullet_i)\}_{i\geqslant 1}$ which we call the n^{th} cohomology sheaf of $\mathscr{K}^\bullet$. Indeed, the $\mathscr{K}^\bullet_i\in D^b_c(X,\mathfrak{o}_i)$ have constructible cohomology sheaves which are torsion because each $\mathscr{K}^\bullet_i$ is a complex of sheaves of $\mathfrak{o}_i$-modules (cf. section 1.2 for the definition of a projective system). Actually, more is true:

Claim 2.2.1. *Each $\{H^n(\mathscr{K}^\bullet_i)\}_{i\geqslant 1}$ is an A-R π-adic sheaf. Moreover, it is zero for n big enough or small enough.*

The second part of the claim provides justification for the superscript ^{b} in $D^b_c(X,\mathfrak{o})$; also, it is in showing this result that the advantages of using $D^b_{ctf}(X,\mathfrak{o}_i)$ instead of $D^b_c(X,\mathfrak{o}_i)$ when defining $D^b_c(X,\mathfrak{o})$ surface. The proof of the claim is technical and we will not carry it out here, see [KW01, Lemma II.5.5] and the discussion preceding it for the argument. There is another important technical condition that makes the proof work which we have suppressed throughout: one has to assume that if the scheme X is over a field k and k'/k is a finite separable extension with the absolute Galois group $G=\mathrm{Gal}((k')^s/k')$ then all Galois cohomology groups $H^n(G,\mathbb{Z}/l\mathbb{Z})$ are finite. In our case k is either finite or algebraically closed and thus so is k'. The condition is clearly verified in the algebraically closed case; in the finite case one uses that finite fields are C_1 (Chevalley-Warning theorem) and hence of cohomological dimension $\leqslant 1$, moreover, $H^1(G,A)$ is finite for finite A, see [Ser02, II.§3 and III.§4 Proposition 8].

The reader may wonder what is wrong with the naïve definition of $D^b_c(X,\mathfrak{o})$ as some sort of subcategory of the derived category of the abelian category $\mathrm{Sh}_{\text{A-R}}(X,\mathfrak{o})$ of A-R π-adic sheaves. A major disadvantage of such approach is that $\mathrm{Sh}_{\text{A-R}}(X,\mathfrak{o})$ does not have enough injectives which hinders the development of a good theory of derived functors.

2.3. **Construction of $D_c^b(X,\overline{\mathbb{Q}}_l)$.** Recall that $E/\mathbb{Q}_l$ is a finite extension with the ring of integers $\mathfrak{o}$ and a uniformizer $\pi\in\mathfrak{o}$. Let us define $D_c^b(X,E)$ by declaring its objects to be those of $D_c^b(X,\mathfrak{o})$. We will write $\mathscr{K}^\bullet\otimes E$ when we view $\mathscr{K}^\bullet\in D_c^b(X,\mathfrak{o})$ as an object of $D_c^b(X,E)$. Note that for $\mathscr{K}^\bullet,\mathscr{L}^\bullet\in D_c^b(X,\mathfrak{o})$ the set of morphisms $\mathrm{Hom}_{D_c^b(X,\mathfrak{o})}(\mathscr{K}^\bullet,\mathscr{L}^\bullet)$ is naturally an $\mathfrak{o}$-module (this is because each $\mathrm{Hom}_{D_c^b(X,\mathfrak{o}_i)}(\mathscr{K}_i^\bullet,\mathscr{L}_i^\bullet)$ is an $\mathfrak{o}_i$-module). We let the morphism sets be the localizations $\mathrm{Hom}_{D_c^b(X,E)}(\mathscr{K}^\bullet\otimes E,\mathscr{L}^\bullet\otimes E)=\mathrm{Hom}_{D_c^b(X,\mathfrak{o})}(\mathscr{K}^\bullet,\mathscr{L}^\bullet)\otimes_{\mathfrak{o}}E$. The composition is defined by composition in $D_c^b(X,\mathfrak{o})$ on the first factor and multiplication on the second. Effectively, what is achieved with this definition is that in $D_c^b(X,E)$ all morphisms f of $D_c^b(X,\mathfrak{o})$ such that $\pi^n f=0$ for some $n\geqslant 1$ become zero.

For F/E (cf. section 0.3) we wish to construct a functor $T_{F/E}\colon D_c^b(X,E)\to D_c^b(X,F)$. To begin with we construct a functor $D_c^b(X,\mathfrak{o})\to D_c^b(X,\mathfrak{O})^e$, where $D_c^b(X,\mathfrak{O})^e$ is defined in an analogous manner to $D_c^b(X,\mathfrak{O})$ except that for $\mathscr{L}^\bullet\in D_c^b(X,\mathfrak{O})^e$ each $\mathscr{L}_i^\bullet\in D_{ctf}^b(X,\mathfrak{O}_{ie})$ and one uses $-\otimes^{\mathbf{L}}_{\mathfrak{O}_{(i+1)e}}\mathfrak{O}_{ie}$ to define the $\phi_i^{\mathscr{L}}$. To an object $\mathscr{K}^\bullet\in D_c^b(X,\mathfrak{o})$ one associates $\mathscr{L}^\bullet\in D_c^b(X,\mathfrak{O})^e$ with[3] $\mathscr{L}_i^\bullet=\mathscr{K}_i^\bullet\otimes^{\mathbf{L}}_{\mathfrak{o}_i}\mathfrak{O}_{ie}$, and the structure morphisms $\phi_i^{\mathscr{L}}\colon(\mathscr{K}_{i+1}^\bullet\otimes^{\mathbf{L}}_{\mathfrak{o}_{i+1}}\mathfrak{O}_{(i+1)e})\otimes^{\mathbf{L}}_{\mathfrak{O}_{(i+1)e}}\mathfrak{O}_{ie}\to\mathscr{K}_i^\bullet\otimes^{\mathbf{L}}_{\mathfrak{o}_i}\mathfrak{O}_{ie}$ obtained from $\phi_i^{\mathscr{K}}\otimes^{\mathbf{L}}_{\mathfrak{o}_i}\mathfrak{O}_{ie}$ after taking into account natural isomorphisms $(-\otimes^{\mathbf{L}}_{\mathfrak{o}_{i+1}}\mathfrak{O}_{(i+1)e})\otimes^{\mathbf{L}}_{\mathfrak{O}_{(i+1)e}}\mathfrak{O}_{ie}\cong-\otimes^{\mathbf{L}}_{\mathfrak{o}_{i+1}}\mathfrak{O}_{ie}\cong(-\otimes^{\mathbf{L}}_{\mathfrak{o}_{i+1}}\mathfrak{o}_i)\otimes^{\mathbf{L}}_{\mathfrak{o}_i}\mathfrak{O}_{ie}$ which are "composition of derived functors vs. derived functor of the composition" isomorphisms that can be argued by computing both sides for a flat resolution (in which case one uses that, e.g., $-\otimes_{\mathfrak{o}_{i+1}}\mathfrak{O}_{(i+1)e}$ brings flat sheaves of $\mathfrak{o}_{i+1}$-modules to flat sheaves of $\mathfrak{O}_{(i+1)e}$-modules). The effect of this construction on morphisms being clear, we get a functor $D_c^b(X,\mathfrak{o})\to D_c^b(X,\mathfrak{O})^e$ and in fact this gives rise to a functor $D_c^b(X,\mathfrak{o})\to D_c^b(X,\mathfrak{O})$ by observing that given $\mathscr{L}^\bullet\in D_c^b(X,\mathfrak{O})^e$ one can "fill in the gaps between multiples of e" by using the functors $-\otimes^{\mathbf{L}}_{\mathfrak{O}_{ie}}\mathfrak{O}_{ie-1},-\otimes^{\mathbf{L}}_{\mathfrak{O}_{ie-1}}\mathfrak{O}_{ie-2},\ldots$ (the new $\phi_j^{\mathscr{L}}$ will be identities for j not a multiple of e) and arguing a natural isomorphism $(\cdots(-\otimes^{\mathbf{L}}_{\mathfrak{O}_{ie}}\mathfrak{O}_{ie-1})\otimes^{\mathbf{L}}_{\mathfrak{O}_{ie-1}}\cdots)\otimes^{\mathbf{L}}_{\mathfrak{O}_{(i-1)e+1}}\mathfrak{O}_{(i-1)e}\cong-\otimes^{\mathbf{L}}_{\mathfrak{O}_{ie}}\mathfrak{O}_{(i-1)e}$ similarly to how we did before. The fact that this operation of "filling in the gaps" is functorial as well as full and faithfull is clear; it is also essentially surjective but this is far less obvious. What one needs to show is that the $\mathscr{K}_i^\bullet$ (for i not divisible by e, say, though this doesn't matter) in $\mathscr{K}^\bullet\in D_c^b(X,\mathfrak{O})$ can be replaced by isomorphic ones in a manner that gives an isomorphic $\mathscr{L}^\bullet\cong\mathscr{K}^\bullet$. This can indeed be done but the proof uses the finiteness assumptions that we discussed at the end of section 2.2, see [KW01, Note at the bottom of p. 96]. To sum up, we get a functor $T_{\mathfrak{O}/\mathfrak{o}}\colon D_c^b(X,\mathfrak{o})\to D_c^b(X,\mathfrak{O})$ which is seen to be functorial in the sense that there are natural isomorphisms $T_{\mathfrak{o}/\mathfrak{o}}\cong\mathrm{Id}_{D_c^b(X,\mathfrak{o})}$, $T_{\mathfrak{O}'/\mathfrak{O}}\circ T_{\mathfrak{O}/\mathfrak{o}}\cong T_{\mathfrak{O}'/\mathfrak{o}}$.

Having defined $T_{\mathfrak{O}/\mathfrak{o}}$ we get $T_{F/E}\colon D_c^b(X,E)\to D_c^b(X,F)$ by setting $T_{F/E}(\mathscr{K}^\bullet\otimes E)=T_{\mathfrak{O}/\mathfrak{o}}(\mathscr{K}^\bullet)\otimes F$ and letting $T_{F/E}\colon\mathrm{Hom}_{D_c^b(X,\mathfrak{o})}(\mathscr{K}^\bullet,\mathscr{L}^\bullet)\otimes_{\mathfrak{o}}E\to\mathrm{Hom}_{D_c^b(X,\mathfrak{O})}(T_{\mathfrak{O}/\mathfrak{o}}(\mathscr{K}^\bullet),T_{\mathfrak{O}/\mathfrak{o}}(\mathscr{L}^\bullet))\otimes_{\mathfrak{O}}F$ be $T_{\mathfrak{O}/\mathfrak{o}}$ on the first factor and the inclusion $E\hookrightarrow F$ on the second (here we are using that $T_{\mathfrak{O}/\mathfrak{o}}$ on morphisms is compatible with multiplication by elements of $\mathfrak{o}$). From the compatibility of the functors $T_{\mathfrak{O}/\mathfrak{o}}$ we immediately get the compatibility of $T_{F/E}$ in the sense that there are natural isomorphisms $T_{E/E}\cong\mathrm{Id}_{D_c^b(X,E)}$, $T_{F'/F}\circ T_{F/E}\cong T_{F'/E}$.

Now we are ready to define $D_c^b(X,\overline{\mathbb{Q}}_l)$ as a "direct limit" of categories $D_c^b(X,E)$ (and functors $T_{F/E}$). In fact, the construction is very similar to what we have already seen when constructing $\overline{\mathbb{Q}}_l$-sheaves in section 1.4. Namely, the objects of $D_c^b(X,\overline{\mathbb{Q}}_l)$ are the objects $\mathscr{K}^\bullet\otimes E\in D_c^b(X,E)$ for varying E with the identification of $\mathscr{K}^\bullet\otimes E$ and all $T_{F/E}(\mathscr{K}^\bullet\otimes E)$. The morphisms in $D_c^b(X,\overline{\mathbb{Q}}_l)$

[3]Notice that the functor $-\otimes_{\mathfrak{o}_i}\mathfrak{O}_{ie}$ is exact (because $\mathfrak{O}$ is a free $\mathfrak{o}$-module and hence $\mathfrak{O}_{ie}$ is a free $\mathfrak{o}_i$-module) so that we could write $-\otimes_{\mathfrak{o}_i}\mathfrak{O}_{ie}$ instead of $-\otimes^{\mathbf{L}}_{\mathfrak{o}_i}\mathfrak{O}_{ie}$. We stick to the latter, however, mostly for consistency reasons because in other situations that we often find ourselves in, like that of $-\otimes^{\mathbf{L}}_{\mathfrak{o}_{i+1}}\mathfrak{o}_i$, one cannot drop the $\mathbf{L}$.

between two objects represented by $\mathscr{K}^\bullet \otimes E, \mathscr{L}^\bullet \otimes E \in D^b_c(X, E)$ are $\varinjlim_{F/E} \mathrm{Hom}_{D^b_c(X,F)}(T_{F/E}(\mathscr{K}^\bullet \otimes E), T_{F/E}(\mathscr{L}^\bullet \otimes E))$ where the colimit is taken over all finite extensions F/E and the definition is independent of the choice of representatives $\mathscr{K}^\bullet \otimes E$ and $\mathscr{L}^\bullet \otimes E$. We sometimes write $\mathscr{K}^\bullet \otimes \overline{\mathbb{Q}}_l$ when we view $\mathscr{K}^\bullet \otimes E$ as an object of $D^b_c(X, \overline{\mathbb{Q}}_l)$; other times we simply write $\mathscr{K}^\bullet$ to denote an object of $D^b_c(X, \overline{\mathbb{Q}}_l)$.

2.4. **Cohomology objects.** We have already seen in 2.2.1 that any $\mathscr{K}^\bullet \in D^b_c(X, \mathfrak{o})$ gives rise to cohomology sheaves $\{H^n(\mathscr{K}^\bullet_i)\}_{i \geqslant 1} \in \mathrm{Sh}_{\text{A-R}}(X, \mathfrak{o})$ which vanish for almost all n. In this section we want to extend this construction to $D^b_c(X, \overline{\mathbb{Q}}_l)$.

First of all, we will find it much more convenient to assume that the cohomology sheaves of $\mathscr{K}^\bullet \in D^b_c(X, \mathfrak{o})$ are actual π-adic sheaves. We will therefore modify them up to an isomorphism by postcomposing the cohomology functors with the functor $E_\mathfrak{o}$ constructed in section 1.3. This will be implicitly assumed from now on when we talk of cohomology and the resulting cohomology functors will still be denoted $H^n(-)$. With this caveat we observe the natural isomorphism of functors $H^n(T_{\mathfrak{O}/\mathfrak{o}}(-)) \cong S_{\mathfrak{O}/\mathfrak{o}}(H^n(-))$ which results from the fact that $- \otimes_{\mathfrak{o}_i} \mathfrak{O}_{ie}$ is exact (see the footnote on the previous page) so that $- \otimes_{\mathfrak{o}_i} \mathfrak{O}_{ie}$ commutes with taking cohomology. Passage to E-sheaves involves applying $- \otimes_\mathfrak{o} E$ to morphisms which gives us $H^n(-)\colon D^b_c(X, E) \to \mathrm{Sh}(X, E)$ and the natural isomorphism above translates to

$$H^n(T_{F/E}(-)) \cong S_{F/E}(H^n(-)). \tag{2.4.1}$$

This allows us to define the desired cohomology functors $H^n(-)\colon D^b_c(X, \overline{\mathbb{Q}}_l) \to \mathrm{Sh}(X, \overline{\mathbb{Q}}_l)$ and in this context the second part of 2.2.1 says that for each $\mathscr{K}^\bullet \otimes \overline{\mathbb{Q}}_l \in D^b_c(X, \overline{\mathbb{Q}}_l)$ almost all cohomology sheaves $H^n(\mathscr{K}^\bullet \otimes \overline{\mathbb{Q}}_l) \in \mathrm{Sh}(X, \overline{\mathbb{Q}}_l)$ are zero.

2.5. **The six operations.** Having constructed $D^b_c(X, \overline{\mathbb{Q}}_l)$ we would like to have the standard yoga of Grothendieck's six operations in this context to be able to work with it. The good news is that there are constructions fulfilling this desideratum. The bad news is that those constructions, when carried out in detail, are lengthy and we cannot do them justice here. Therefore, we content ourselves with a sketch of the theory and point the reader to the references for details.

Suppose $f\colon X \to Y$ is a morphism (cf. section 0.3 for assumptions on X and Y). There are the following functors:

- $\mathbf{R}f_*\colon D^b_c(X, \overline{\mathbb{Q}}_l) \to D^b_c(Y, \overline{\mathbb{Q}}_l)$,
- $f^*\colon D^b_c(Y, \overline{\mathbb{Q}}_l) \to D^b_c(X, \overline{\mathbb{Q}}_l)$,
- $\mathbf{R}f_!\colon D^b_c(X, \overline{\mathbb{Q}}_l) \to D^b_c(Y, \overline{\mathbb{Q}}_l)$,
- $f^!\colon D^b_c(Y, \overline{\mathbb{Q}}_l) \to D^b_c(X, \overline{\mathbb{Q}}_l)$,
- $\mathbf{R}\mathscr{H}om(-,-)\colon D^b_c(X, \overline{\mathbb{Q}}_l)^{\mathrm{op}} \times D^b_c(X, \overline{\mathbb{Q}}_l) \to D^b_c(X, \overline{\mathbb{Q}}_l)$,
- $- \otimes^{\mathbf{L}} -\colon D^b_c(X, \overline{\mathbb{Q}}_l) \times D^b_c(X, \overline{\mathbb{Q}}_l) \to D^b_c(X, \overline{\mathbb{Q}}_l)$.

$\mathbf{R}f_*$ and f^* should be thought of as derived pushforward and pullback, respectively. They are adjoint, i.e., there is a natural bijection

$$\mathrm{Hom}_{D^b_c(X,\overline{\mathbb{Q}}_l)}(f^*\mathscr{K}^\bullet, \mathscr{L}^\bullet) \cong \mathrm{Hom}_{D^b_c(Y,\overline{\mathbb{Q}}_l)}(\mathscr{K}^\bullet, \mathbf{R}f_*\mathscr{L}^\bullet). \tag{2.5.1}$$

$\mathbf{R}f_!$ should be thought of as derived pushforward with proper supports. Recall the definition in the simpler setting of a torsion sheaf $\mathscr{F}$ (an actual sheaf): one chooses a *compactification* of $X \to Y$, i.e., a factorization $X \xrightarrow{j} \overline{X} \xrightarrow{g} Y$ of f into an open immersion j followed by a proper morphism g

which exists by Nagata's theorem (cf. [Con07]). One then extends $\mathscr{F}$ to $\overline{X}$ by zero outside of X to get a sheaf $j_!\mathscr{F}$ on $\overline{X}$. Finally, one sets $\mathbf{R}f_!\mathscr{F} := \mathbf{R}g_*(j_!\mathscr{F})$. Of course, one has to check that the definition is independent of the compactification chosen and this is done in [SGA$_{4\frac{1}{2}}$, Arcata, IV.5].

The functor $f^!$ can be interpreted formally as a right adjoint to $\mathbf{R}f_!$. That is, there is a natural bijection

$$\mathrm{Hom}_{D^b_c(Y,\overline{\mathbb{Q}}_l)}(\mathbf{R}f_!\mathscr{K}^\bullet,\mathscr{L}^\bullet) \cong \mathrm{Hom}_{D^b_c(X,\overline{\mathbb{Q}}_l)}(\mathscr{K}^\bullet, f^!\mathscr{L}^\bullet). \tag{2.5.2}$$

$\mathbf{R}\mathscr{H}om(-,-)$ and $-\otimes^{\mathbf{L}}-$ should be thought of as the derived sheaf-hom and derived tensor product. We state the following "adjunction" relating them, even though we don't need it later:

$$\mathbf{R}\mathscr{H}om(\mathscr{K}^\bullet \otimes^{\mathbf{L}} \mathscr{L}^\bullet, \mathscr{M}^\bullet) \cong \mathbf{R}\mathscr{H}om(\mathscr{K}^\bullet, \mathbf{R}\mathscr{H}om(\mathscr{L}^\bullet,\mathscr{M}^\bullet)). \tag{2.5.3}$$

As far as the actual constructions go, they follow the usual pattern that we have seen several times already: first do the construction for $D^b_{ctf}(X,\mathfrak{o}_i)$, make sure it behaves well with respect to $-\otimes^{\mathbf{L}}_{\mathfrak{o}_{i+1}}\mathfrak{o}_i$ and $-\otimes^{\mathbf{L}}_{\mathfrak{o}_i}\mathfrak{O}_{ie}$ so that you get a construction for $D^b_c(X,\mathfrak{o})$ compatibly with passing to a field extension, pass to $D^b_c(X,E)$, and finally put the constructions for each $D^b_c(X,E)$ together to get a desired construction for $D^b_c(X,\overline{\mathbb{Q}}_l)$. At the initial stage of $D^b_{ctf}(X,\mathfrak{o}_i)$ you of course start with the corresponding derived functor. The details, however, are intricate. For one thing, to get started with, say, $\mathbf{R}f_*$ one has to prove that the derived functor $\mathbf{R}f_*\colon D^+(X,\mathfrak{o}_i)\to D^+(Y,\mathfrak{o}_i)$ sends $D^b_{ctf}(X,\mathfrak{o}_i)$ to $D^b_{ctf}(Y,\mathfrak{o}_i)$. Part of what this is saying is that for a constructible sheaf $\mathscr{F}$ of $\mathfrak{o}_i$-modules all $\mathbf{R}^if_*\mathscr{F}$ are constructible which is a nontrivial theorem in étale cohomology proved by Deligne in [SGA$_{4\frac{1}{2}}$, Th. finitude].

References where the constructions are carried out are [SGA$_{4\frac{1}{2}}$, Th. finitude] where the "base cases" are done and [KW01, II.7-10, Theorem II.12.2, Appendices A and D]. See also [Eke90] for a different approach.

3. t-structures and perverse sheaves

3.1. **Triangulated structure on $D^b_c(X,\overline{\mathbb{Q}}_l)$.** We have seen in section 2.1 that the categories $D^b_{ctf}(X,\mathfrak{o}_i)$ are triangulated. We want to put their triangulated structures together to get a triangulated structure on $D^b_c(X,\mathfrak{o})$. First of all, the shift functor $(-)[1]$ will be defined by sending $\mathscr{K}^\bullet\in D^b_c(X,\mathfrak{o})$ to $\mathscr{K}^\bullet[1] := (\mathscr{K}^\bullet_i[1])_{i\geqslant 1}$. This is legitimate because the functors $-\otimes^{\mathbf{L}}_{\mathfrak{o}_{i+1}}\mathfrak{o}_i$ used to define structure morphisms of $\mathscr{K}^\bullet$ are triangulated (being derived functors of additive functors) and hence commute with shifts. Secondly, we declare the triangle $\mathscr{K}^\bullet\to\mathscr{L}^\bullet\to\mathscr{M}^\bullet\to\mathscr{K}^\bullet[1]$ to be distinguished if all its constituents $\mathscr{K}^\bullet_i\to\mathscr{L}^\bullet_i\to\mathscr{M}^\bullet_i\to\mathscr{K}^\bullet_i[1]$, $i\geqslant 1$ are distinguished.

Proposition 3.1.1. *With these definitions $D^b_c(X,\mathfrak{o})$ becomes a triangulated category.*

Proof. The only nontrivial axioms to verify are extension of morphisms, existence of mapping cones and the octahedral axiom. One has to use the finiteness condition discussed at the end of section 2.2. Using it one proves that all morphism sets $\mathrm{Hom}_{D^b_{ctf}(X,\mathfrak{o}_i)}(\mathscr{K}^\bullet_i,\mathscr{L}^\bullet_i)$ are finite (cf. [KW01, Theorem II.5.4]).

To see that every "morphism on two vertices"

$$\begin{tikzcd}
\mathscr{K}^\bullet \arrow[r] \arrow[d] & \mathscr{L}^\bullet \arrow[r] \arrow[d] & \mathscr{M}^\bullet \arrow[r] \arrow[d, dashed] & \mathscr{K}^\bullet[1] \arrow[d] \\
\widetilde{\mathscr{K}}^\bullet \arrow[r] & \widetilde{\mathscr{L}}^\bullet \arrow[r] & \widetilde{\mathscr{M}}^\bullet \arrow[r] & \widetilde{\mathscr{K}}^\bullet[1]
\end{tikzcd}$$

extends to a morphism of triangles as depicted, one knows that at every level i there is a finite nonempty set S_i of possible extensions, so that one can simply pick an element of $\varprojlim S_i \neq \varnothing$ to get a desired extension. The argument for the octahedral axiom is completely analogous.

The existence of a cone for a morphism $\mathscr{K}^\bullet \to \mathscr{L}^\bullet$ is seen by picking a cone $\mathscr{M}_i^\bullet$ at each level, applying $-\otimes^{\mathbf{L}}_{\mathfrak{o}_{i+1}} \mathfrak{o}_i$ to the triangle on the $(i+1)^{\text{st}}$ level to obtain a triangle that maps to the triangle on the i^{th} level at two vertices, using the extension of morphisms to get $\phi_i^{\mathscr{M}}$ on the third vertex and invoking the five lemma for triangulated categories to conclude that $\phi_i^{\mathscr{M}}$ is an isomorphism. ☐

The journey towards a triangulated structure on $D^b_c(X, \overline{\mathbb{Q}}_l)$ now follows the familiar pattern. One has a natural functor $D^b_c(X, \mathfrak{o}) \to D^b_c(X, E)$ which is identity on objects. Using it one declares a triangle in $D^b_c(X, E)$ to be distinguished if it is isomorphic to the image of a distinguished triangle in $D^b_c(X, \mathfrak{o})$. Verification of the axioms is effortless because if needed one can multiply appropriate morphisms by powers of π (multiplication by π is invertible by construction of $D^b_c(X, E)$) to reduce to appropriate axioms for $D^b_c(X, \mathfrak{o})$.

For a finite extension F/E the functor $T_{\mathfrak{O}/\mathfrak{o}} \colon D^b_c(X, \mathfrak{o}) \to D^b_c(X, \mathfrak{O})$ was constructed in section 2.3 using triangulated functors $- \otimes^{\mathbf{L}}_{\mathfrak{o}_i} \mathfrak{O}_{ie}$ and therefore is triangulated. Since $T_{\mathfrak{O}/\mathfrak{o}}$ is triangulated so is $T_{F/E}$.

Now we are ready to introduce the triangulated structure on $D^b_c(X, \overline{\mathbb{Q}}_l)$. The shift functor is defined taking a representing object in some $D^b_c(X, E)$ and applying the shift functor there; distinguished triangles are those which can be represented by a distinguished triangle in some $D^b_c(X, E)$ (and hence in every F/E). The axioms are immediately verified by reducing them to the axioms for some $D^b_c(X, F)$ because all transition functors $T_{F/E}$ are triangulated.

3.2. **The standard t-structure.** The standard t-structure on $D^b_c(X, \mathfrak{o})$ is defined by the pair of full subcategories

$$D^{\leqslant 0}(X, \mathfrak{o}) = \{\mathscr{K}^\bullet \in D^b_c(X, \mathfrak{o}) \mid H^n(\mathscr{K}^\bullet) = 0, n > 0\},$$
$$D^{\geqslant 0}(X, \mathfrak{o}) = \{\mathscr{K}^\bullet \in D^b_c(X, \mathfrak{o}) \mid H^n(\mathscr{K}^\bullet) = 0, n < 0\}.$$

Note that this is in complete analogy with how the standard t-structure on the derived category of an abelian category is defined in terms of vanishing of cohomology objects. By now the reader will have guessed the following result. Before stating it we recall that the *heart* of a t-structure $(D^{\leqslant 0}, D^{\geqslant 0})$ on a triangulated category D is $D^{\heartsuit} := D^{\leqslant 0} \cap D^{\geqslant 0}$ and is an abelian category.

Theorem 3.2.1. *The pair $(D^{\leqslant 0}(X, \mathfrak{o}), D^{\geqslant 0}(X, \mathfrak{o}))$ is a t-structure on $D^b_c(X, \mathfrak{o})$. By definition its heart is*

$$D^b_c(X, \mathfrak{o})^{\heartsuit} = \{\mathscr{K}^\bullet \in D^b_c(X, \mathfrak{o}) \mid H^n(\mathscr{K}^\bullet) = 0 \textit{ for } n \neq 0\}.$$

The functor $H^0(-)$ induces an equivalence of categories between the heart $D^b_c(X, \mathfrak{o})^{\heartsuit}$ and the abelian category $\mathrm{Sh}_{\text{A-R}}(X, \mathfrak{o})$ of A-R π-adic sheaves.

Proof. See [KW01, Theorem II.6.4]. ☐

The full subcategories $D^{\leqslant 0}(X, E), D^{\geqslant 0}(X, E) \subset D^b_c(X, E)$ are defined analogously and it follows formally from the theorem above that they define the *standard t-structure* on $D^b_c(X, E)$:

- $D^{\geqslant 1}(X, E) \subset D^{\geqslant 0}(X, E)$ and $D^{\leqslant -1}(X, E) \subset D^{\leqslant 0}(X, E)$ is a matter of unwinding definitions.

- $\mathrm{Hom}(D^{\leqslant 0}(X,E), D^{\geqslant 1}(X,E)) = 0$ follows from the corresponding axiom for $D^b_c(X,\mathfrak{o})$: indeed, pick a morphism f which you want to show to be zero. It suffices to show that some $\pi^n f = 0$. For big enough n this morphism will come from a morphism g in $D^b_c(X,\mathfrak{o})$. The source (resp., target) of g has a property that its cohomology sheaves in degrees > 0 (resp., < 1) vanish in $\mathrm{Sh}(X,E)$. Only finitely many of them are nonzero in $\mathrm{Sh}_{\text{A-R}}(X,\mathfrak{o})$ anyway (2.2.1), so the conclusion follows from the following proposition.

  Proposition 3.2.2. *A sheaf $\mathscr{F} \in \mathrm{Sh}_{\text{A-R}}(X,\mathfrak{o})$ is zero in $\mathrm{Sh}(X,E)$ if and only if it is* torsion *in the sense that $\pi^n\mathscr{F} \cong 0$ in $\mathrm{Sh}_{\text{A-R}}(X,\mathfrak{o})$ for some $n \geqslant 0$.*

  Proof. An object in a (pre)additive category is zero if and only if its identity morphism is zero. The identity morphism of a sheaf $\mathscr{F} \in \mathrm{Sh}_{\text{A-R}}(X,\mathfrak{o})$ becomes zero in $\mathrm{Sh}(X,E)$ if and only if for some $n \geqslant 0$ $\pi^n \,\mathrm{id}_{\mathscr{F}} = 0$, i.e., if and only if the image of the map $\mathscr{F} \xrightarrow{\pi^n} \mathscr{F}$ is zero. This image, however, is $\pi^n\mathscr{F}$. ☐

- The existence of truncation triangles follows from the corresponding axiom for $D^b_c(X,\mathfrak{o})$ because the natural functor $F\colon D^b_c(X,\mathfrak{o}) \to D^b_c(X,E)$ is triangulated (see section 3.1) and $F(D^{\leqslant 0}(X,\mathfrak{o})) \subset D^{\leqslant 0}(X,E)$, $F(D^{\geqslant 1}(X,\mathfrak{o})) \subset D^{\geqslant 1}(X,E)$ (on cohomology if an $\mathfrak{o}$-sheaf is zero then so is its corresponding E-sheaf).

3.2.1 also allows us to characterize the heart $D^b_c(X,E)^\heartsuit$:

Proposition 3.2.3. *The functor $H^0(-)$ induces an equivalence of categories between the heart $D^b_c(X,E)^\heartsuit$ of the standard t-structure on $D^b_c(X,E)$ and $\mathrm{Sh}(X,E)$.*

Proof. There is a commutative (up to natural isomorphism) diagram of categories and functors

$$\begin{tikzcd} D^b_c(X,\mathfrak{o})^\heartsuit \arrow[r, "H^0(-)"] \arrow[d] & \mathrm{Sh}(X,\mathfrak{o}) \arrow[d] \\ D^b_c(X,E)^\heartsuit \arrow[r, "H^0(-)"] & \mathrm{Sh}(X,E) \end{tikzcd}$$

The top arrow is an equivalence by 3.2.1. The effect of both vertical arrows is known: identity on objects, $-\otimes_{\mathfrak{o}} E$ on morphisms, moreover, the right one is surjective. Therefore, it suffices to show that the left arrow is essentially surjective. To do this pick an object downstairs, it comes from an object upstairs (possibly not from the heart), multiply that object by a sufficiently high power of π to get an object in the heart (cf. 3.2.2), at this point you have found a desired object upstairs lifting the one that you've picked downstairs (up to isomorphism). ☐

The functors $T_{F/E}$ preserve t-structures because of (2.4.1). Therefore, it makes sense to define full subcategories

$$D^{\leqslant 0}(X,\overline{\mathbb{Q}}_l) = \{\mathscr{K}^\bullet \in D^b_c(X,\overline{\mathbb{Q}}_l) \mid H^n(\mathscr{K}^\bullet) = 0, n > 0\} = \{\mathscr{L}^\bullet \otimes \overline{\mathbb{Q}}_l \mid \mathscr{L}^\bullet \otimes E \in D^{\leqslant 0}(X,E) \text{ for some } E\},$$
$$D^{\geqslant 0}(X,\overline{\mathbb{Q}}_l) = \{\mathscr{K}^\bullet \in D^b_c(X,\overline{\mathbb{Q}}_l) \mid H^n(\mathscr{K}^\bullet) = 0, n < 0\} = \{\mathscr{L}^\bullet \otimes \overline{\mathbb{Q}}_l \mid \mathscr{L}^\bullet \otimes E \in D^{\geqslant 0}(X,E) \text{ for some } E\}.$$

At this point the following theorem will come to the reader as no surprise.

Theorem 3.2.4. *$(D^{\leqslant 0}(X,\overline{\mathbb{Q}}_l), D^{\geqslant 0}(X,\overline{\mathbb{Q}}_l))$ defines a t-structure on $D^b_c(X,\overline{\mathbb{Q}}_l)$ called the* standard t-structure. *$H^0(-)\colon D^b_c(X,\overline{\mathbb{Q}}_l)^\heartsuit \to \mathrm{Sh}(X,\overline{\mathbb{Q}}_l)$ is an equivalence of categories.*

Proof. Only the second assertion requires proof. But it follows from 3.2.3 because in the commutative (up to natural isomorphisms) diagrams

$$\begin{tikzcd} D^b_c(X,E)^\heartsuit \arrow[r, "H^0(-)"] \arrow[d, "T_{F/E}"'] & \mathrm{Sh}(X,E) \arrow[d, "S_{F/E}"] \\ D^b_c(X,F)^\heartsuit \arrow[r, "H^0(-)"] & \mathrm{Sh}(X,F) \end{tikzcd}$$

the horizontal arrows are equivalences. □

A t-structure $(D^{\leqslant 0}, D^{\geqslant 0})$ on a triangulated category D defines truncation functors $\tau_{\leqslant n}$, $\tau_{\geqslant n}$ (right, resp., left adjoints to inclusions $D^{\leqslant 0}[-n] = D^{\leqslant n} \to D$, resp., $D^{\geqslant 0}[-n] = D^{\geqslant n} \to D$). In the case $D = D^b_c(X, \overline{\mathbb{Q}}_l)$ equipped with the standard t-structure those truncation functors will be denoted simply $\tau_{\leqslant n}$, $\tau_{\geqslant n}$. For $\mathscr{K}^\bullet \in D^b_c(X, \overline{\mathbb{Q}}_l)$ we identify $(\tau_{\leqslant n}\tau_{\geqslant n}\mathscr{K}^\bullet)[n] = \tau_{\leqslant 0}\tau_{\geqslant 0}(\mathscr{K}^\bullet[n])$ with $H^n(\mathscr{K}^\bullet)$ (a legitimate thing to do due to 3.2.4) and observe that for each distinguished triangle

$$\mathscr{K}^\bullet \to \mathscr{L}^\bullet \to \mathscr{M}^\bullet \to \mathscr{K}^\bullet[1]$$

in $D^b_c(X, \overline{\mathbb{Q}}_l)$ the general theory of t-structures gives the corresponding long exact cohomology sequence

$$\cdots \to H^{-1}(\mathscr{L}^\bullet) \to H^{-1}(\mathscr{M}^\bullet) \to H^0(\mathscr{K}^\bullet) \to H^0(\mathscr{L}^\bullet) \to H^0(\mathscr{M}^\bullet) \to H^1(\mathscr{K}^\bullet) \to H^1(\mathscr{L}^\bullet) \to \cdots \tag{3.2.5}$$

3.3. **Duality.** The adjunctions (2.5.1) and (2.5.2) admit "sheafified" versions analogous to (2.5.3). We explicate the one relating $\mathbf{R}f_!$ and $f^!$. The references for this section are the ones given at the end of section 2.5.

Theorem 3.3.1 (Relative Poincaré duality)**.** *For a morphism $f\colon X \to Y$ there is an isomorphism*

$$\mathbf{R}\mathscr{H}om(\mathbf{R}f_!\mathscr{K}^\bullet, \mathscr{L}^\bullet) \cong \mathbf{R}f_*\mathbf{R}\mathscr{H}om(\mathscr{K}^\bullet, f^!\mathscr{L}^\bullet) \tag{3.3.2}$$

in $D^b_c(Y, \overline{\mathbb{Q}}_l)$ which is natural in $\mathscr{K}^\bullet \in D^b_c(X, \overline{\mathbb{Q}}_l)$ and $\mathscr{L}^\bullet \in D^b_c(Y, \overline{\mathbb{Q}}_l)$.

Proof. The proof follows the usual pattern but we will not give it here. The "base case" is [SGA$_4$ III, Exposé XVIII, Proposition 3.1.10]. □

Let us illustrate with a computation why we are calling this relative Poincaré duality. Suppose that $Y = \operatorname{Spec} k$ where k is an algebraically closed field, for instance, k could be $\mathbb{F}$. Suppose also for simplicity that instead of $D^b_c(X, \overline{\mathbb{Q}}_l)$ we are dealing with $D^b_c(X, \mathfrak{o}_i)$ (and similarly for Y). Then let $\mathscr{K}^\bullet = \mathscr{F}$ be a constructible sheaf $\mathscr{F}$ of $\mathfrak{o}_i$-modules and set $\mathscr{L}^\bullet = \underline{\mathfrak{o}_i}_Y$ to be a constant sheaf on Y. Under these assumptions let us compute the i^{th} cohomology on both sides of (3.3.2). The left hand side becomes

$$H^i(\mathbf{R}\mathscr{H}om(\mathbf{R}f_!\mathscr{F}, \underline{\mathfrak{o}_i}_Y)) \cong H^i(\operatorname{Hom}_{\mathfrak{o}_i}(\mathbf{R}f_!\mathscr{F}, \mathfrak{o}_i)) \cong \operatorname{Hom}_{\mathfrak{o}_i}(H^{-i}(\mathbf{R}f_!\mathscr{F}), \mathfrak{o}_i) = (H^{-i}_c(X, \mathscr{F}))^{\vee},$$

where we have used that k is algebraically closed so that the category of sheaves of $\mathfrak{o}_i$-modules on $\operatorname{Spec} k$ is equivalent to the category of $\mathfrak{o}_i$-modules and $\mathfrak{o}_i$ is an injective $\mathfrak{o}_i$-module[4], i.e., $\operatorname{Hom}_{\mathfrak{o}_i}(-, \mathfrak{o}_i)$

[4]To show this use the ideal criterion: let $(\pi^k) \subset \mathfrak{o}_i$ be an ideal, we need to see that every $\mathfrak{o}_i$-homomorphism $h\colon (\pi^k) \to \mathfrak{o}_i$ extends to $\mathfrak{o}_i$. This will be the case once we know that $h(\pi^k)$ is a multiple of π^k. But it must be because $\pi^{i-k}h(\pi^k) = 0$.

is an exact functor. The cohomology with compact supports of $\mathscr{F}$ is by definition the cohomology of $\mathbf{R}f_!\mathscr{F}$. Letting $\mathbb{D}\mathscr{F}$ denote $\mathbf{R}\mathscr{H}om(\mathscr{F}, f^!\underline{\mathfrak{o}_i}_Y)$ the right hand side becomes

$$H^i(\mathbf{R}f_*(\mathbb{D}\mathscr{F})) = H^i(X, \mathbb{D}\mathscr{F}).$$

We conclude that (3.3.2) is giving us an isomorphism

$$(H_c^{-i}(X, \mathscr{F}))^{\vee} \cong H^i(X, \mathbb{D}\mathscr{F}),$$

which is of similar nature to Poincaré duality isomorphisms. In fact, if f is smooth and X is connected and one works out what $f^!\underline{\mathfrak{o}_i}_Y$ is one recovers the Poincaré duality isomorphism encountered in étale cohomology (after replacing i by $i - 2n$; here $n = \dim X$).

Motivated by the computation let $s\colon X \to \operatorname{Spec} k$ be the structure morphism ($k = \mathbb{F}_q$ or $k = \mathbb{F}$, cf. section 0.3) and for $\mathscr{K}^\bullet \in D_c^b(X, \overline{\mathbb{Q}}_l)$ let us define $\mathbb{D}_X\mathscr{K}^\bullet := \mathbf{R}\mathscr{H}om(\mathscr{K}^\bullet, s^!\overline{\mathbb{Q}}_l)$. Here $\overline{\mathbb{Q}}_l$ is a constant sheaf which could be thought of as represented by the projective system of flat constant sheaves concentrated in degree zero $\left\{\underline{\mathbb{Z}/l^i\mathbb{Z}}\right\}_{i\geqslant 1} \in D_c^b(\operatorname{Spec} k, \mathbb{Q}_l)$. We call $s^!\overline{\mathbb{Q}}_l \in D_c^b(X, \overline{\mathbb{Q}}_l)$ the *dualizing complex* and $\mathbb{D}_X\colon D_c^b(X, \overline{\mathbb{Q}}_l)^{\mathrm{op}} \to D_c^b(X, \overline{\mathbb{Q}}_l)$ the *dualizing functor* or the *duality functor*. The name is justified by the following:

Theorem 3.3.3. *The canonical natural transformation* $\mathrm{Id} \to \mathbb{D}_X \circ \mathbb{D}_X$ *is an isomorphism of functors. In particular,* $\mathbb{D}_X\colon D_c^b(X, \overline{\mathbb{Q}}_l)^{\mathrm{op}} \to D_c^b(X, \overline{\mathbb{Q}}_l)$ *is an equivalence of categories.*

Proof. See [KW01, II.10]. □

Suppose $f\colon X \to Y$ is a morphism of k-schemes. The following lemma shows that with this notion of duality the functors $\mathbf{R}f_*$ and $\mathbf{R}f_!$ as well as f^* and $f^!$ are dual to each other.

Lemma 3.3.4. *There are the following natural isomorphisms of functors*

1. $\mathbb{D}_Y \circ \mathbf{R}f_! \cong \mathbf{R}f_* \circ \mathbb{D}_X$, *or equivalently* $\mathbf{R}f_! \circ \mathbb{D}_X \cong \mathbb{D}_Y \circ \mathbf{R}f_*$;
2. $\mathbb{D}_X \circ f^! \cong f^* \circ \mathbb{D}_Y$, *or equivalently* $f^! \circ \mathbb{D}_Y \cong \mathbb{D}_X \circ f^*$.

Proof. The equivalence in both cases follows from 3.3.3. Let us first show **2.** assuming **1.** is known. By uniqueness of adjoints it suffices to show that $\mathbb{D}_X \circ f^! \circ \mathbb{D}_Y$ is left adjoint to $\mathbf{R}f_*$. This is demonstrated observing the following natural bijections:

$$\begin{aligned}
\operatorname{Hom}_{D_c^b(X,\overline{\mathbb{Q}}_l)}(\mathbb{D}_X(f^!(\mathbb{D}_Y(\mathscr{K}^\bullet))), \mathscr{L}^\bullet) &\cong \operatorname{Hom}_{D_c^b(X,\overline{\mathbb{Q}}_l)}(\mathbb{D}_X(f^!(\mathbb{D}_Y(\mathscr{K}^\bullet))), \mathbb{D}_X(\mathbb{D}_X(\mathscr{L}^\bullet))) \\
&\cong \operatorname{Hom}_{D_c^b(X,\overline{\mathbb{Q}}_l)}(\mathbb{D}_X(\mathscr{L}^\bullet), f^!(\mathbb{D}_Y(\mathscr{K}^\bullet))) \\
&\cong \operatorname{Hom}_{D_c^b(Y,\overline{\mathbb{Q}}_l)}(\mathbf{R}f_!(\mathbb{D}_X(\mathscr{L}^\bullet)), \mathbb{D}_Y(\mathscr{K}^\bullet)) \\
&\cong \operatorname{Hom}_{D_c^b(Y,\overline{\mathbb{Q}}_l)}(\mathbb{D}_Y(\mathbf{R}f_*(\mathscr{L}^\bullet)), \mathbb{D}_Y(\mathscr{K}^\bullet)) \\
&\cong \operatorname{Hom}_{D_c^b(Y,\overline{\mathbb{Q}}_l)}(\mathscr{K}^\bullet, \mathbf{R}f_*(\mathscr{L}^\bullet)).
\end{aligned}$$

To show **1.** we let $t\colon Y \to \operatorname{Spec} k$ be the structure morphism of Y (recall that s is the structure morphism of X) and note that $s^! \cong f^! \circ t^!$ because both are right adjoint to $\mathbf{R}s_! \cong \mathbf{R}t_! \circ \mathbf{R}f_!$ (this natural isomorphism is a by-product of the construction of $\mathbf{R}f_!$ which we haven't carried out and the corresponding identity in the "base case"). Put $\mathscr{L}^\bullet = t^!\overline{\mathbb{Q}}_l$ in (3.3.2) to get the natural isomorphisms

$$\begin{aligned}
\mathbb{D}_Y(\mathbf{R}f_!\mathscr{K}^\bullet) \cong \mathbf{R}\mathscr{H}om(\mathbf{R}f_!\mathscr{K}^\bullet, t^!\overline{\mathbb{Q}}_l) &\cong \mathbf{R}f_*\mathbf{R}\mathscr{H}om(\mathscr{K}^\bullet, f^!(t^!(\overline{\mathbb{Q}}_l))) \\
&\cong \mathbf{R}f_*\mathbf{R}\mathscr{H}om(\mathscr{K}^\bullet, s^!(\overline{\mathbb{Q}}_l)) \cong \mathbf{R}f_*(\mathbb{D}_X(\mathscr{K}^\bullet)).
\end{aligned}$$

This is the desired $\mathbb{D}_Y \circ \mathbf{R}f_! \cong \mathbf{R}f_* \circ \mathbb{D}_X$. □

3.4. **Perverse sheaves.** In section 3.2 we have defined the standard t-structure on $D^b_c(X, \overline{\mathbb{Q}}_l)$ which gave rise to cohomology functors $H^n(-): D^b_c(X, \overline{\mathbb{Q}}_l) \to \mathrm{Sh}(X, \overline{\mathbb{Q}}_l)$ (recall 3.2.4). There is another t-structure $({}^pD^{\leqslant 0}(X), {}^pD^{\geqslant 0}(X))$ on $D^b_c(X, \overline{\mathbb{Q}}_l)$ which gives rise to different cohomology functors and has an advantage of being self-dual in the sense that $\mathscr{K}^\bullet \in {}^pD^{\leqslant 0}(X)$ (resp., $\mathscr{K}^\bullet \in {}^pD^{\geqslant 0}(X)$) if and only if $\mathbb{D}_X(\mathscr{K}^\bullet) \in {}^pD^{\geqslant 0}(X)$ (resp., $\mathbb{D}_X(\mathscr{K}^\bullet) \in {}^pD^{\leqslant 0}(X)$). It is called the *perverse t-structure* and is described as follows:

$$
\begin{aligned}
{}^pD^{\leqslant 0}(X) &= \{\mathscr{K}^\bullet \in D^b_c(X, \overline{\mathbb{Q}}_l) \mid \dim \operatorname{supp} H^{-i}(\mathscr{K}^\bullet) \leqslant i, \text{ for all } i \in \mathbb{Z}\}, \\
{}^pD^{\geqslant 0}(X) &= \{\mathscr{K}^\bullet \in D^b_c(X, \overline{\mathbb{Q}}_l) \mid \dim \operatorname{supp} H^{-i}(\mathbb{D}_X(\mathscr{K}^\bullet)) \leqslant i, \text{ for all } i \in \mathbb{Z}\}.
\end{aligned}
$$

Here by $\operatorname{supp} \mathscr{F}$ we mean the support of the $\overline{\mathbb{Q}}_l$-sheaf $\mathscr{F}$ which is defined as follows: for each geometric point $j: \overline{x} \to X$ consider $j^*\mathscr{F} \in \mathrm{Sh}(\overline{x}, \overline{\mathbb{Q}}_l)$. This makes sense because j^* is an exact functor so it induces a functor from (A-R) π-adic sheaves on X to (A-R) π-adic sheaves on $\overline{x}$ and hence a functor $j^*: \mathrm{Sh}(X, \overline{\mathbb{Q}}_l) \to \mathrm{Sh}(\overline{x}, \overline{\mathbb{Q}}_l)$ because pull-back commutes with tensor products. Now $\operatorname{supp} \mathscr{F}$ is the closure of the set of all $j(\overline{x})$ for which $j^*\mathscr{F} \neq 0$.

Theorem 3.4.1. *With the definitions above $({}^pD^{\leqslant 0}(X), {}^pD^{\geqslant 0}(X))$ defines a t-structure on $D^b_c(X, \overline{\mathbb{Q}}_l)$.*

Proof. The proof is not very complicated but we won't give it here. The main ingredient is the glueing lemma which says that given an open embedding $j: U \to X$ with closed complement $i: Y \to X$ and t-structures on $D^b_c(U, \overline{\mathbb{Q}}_l)$ and $D^b_c(Y, \overline{\mathbb{Q}}_l)$ they can be "glued" to give a t-structure on $D^b_c(X, \overline{\mathbb{Q}}_l)$. More precisely, this t-structure is defined by

$$
\begin{aligned}
D^{\leqslant 0}(X) &:= \{\mathscr{K}^\bullet \in D^b_c(X, \overline{\mathbb{Q}}_l) \mid j^*\mathscr{K}^\bullet \in D^{\leqslant 0}(U), i^*\mathscr{K}^\bullet \in D^{\leqslant 0}(Y)\}, \\
D^{\geqslant 0}(X) &:= \{\mathscr{K}^\bullet \in D^b_c(X, \overline{\mathbb{Q}}_l) \mid j^*\mathscr{K}^\bullet \in D^{\geqslant 0}(U), i^!\mathscr{K}^\bullet \in D^{\geqslant 0}(Y)\}.
\end{aligned}
$$

Here $(D^{\leqslant 0}(U), D^{\geqslant 0}(U))$ and $(D^{\leqslant 0}(Y), D^{\geqslant 0}(Y))$ are the given t-structures on U and Y, respectively. The glueing lemma can be used to reduce to the case where $X \to \operatorname{Spec} k$ is smooth in which case the perverse t-structure is closely related to the standard t-structure. For the full argument see [KW01, III.2-3]. $\square$

The claimed self-duality of the perverse t-structure is immediate from the definition and 3.3.3. We will denote $D^b_c(X, \overline{\mathbb{Q}}_l)$ by ${}^pD^b_c(X, \overline{\mathbb{Q}}_l)$ when we think about it as being equipped with the perverse t-structure and we will denote by ${}^p\tau_{\leqslant n}$, ${}^p\tau_{\geqslant n}$ the corresponding truncation functors.

The heart of the perverse t-structure is the abelian category $\mathrm{Perv}(X) := {}^pD^b_c(X, \overline{\mathbb{Q}}_l)^\heartsuit = {}^pD^{\leqslant 0}(X) \cap {}^pD^{\geqslant 0}(X)$, the category of *perverse sheaves* on X. Note, however, that in general a perverse sheaf (an object of $\mathrm{Perv}(X)$) is not a sheaf but rather a complex of sheaves. The resulting cohomology functors

$$
{}^pH^n = {}^p\tau_{\leqslant n} {}^p\tau_{\geqslant n}((-)[n]): {}^pD^b_c(X, \overline{\mathbb{Q}}_l) \to \mathrm{Perv}(X)
$$

for each distinguished triangle

$$
\mathscr{K}^\bullet \to \mathscr{L}^\bullet \to \mathscr{M}^\bullet \to \mathscr{K}^\bullet[1]
$$

in $D^b_c(X, \overline{\mathbb{Q}}_l)$ give the corresponding long exact perverse cohomology sequence

$$
\cdots \to {}^pH^{-1}(\mathscr{L}^\bullet) \to {}^pH^{-1}(\mathscr{M}^\bullet) \to {}^pH^0(\mathscr{K}^\bullet) \to {}^pH^0(\mathscr{L}^\bullet) \to {}^pH^0(\mathscr{M}^\bullet) \to {}^pH^1(\mathscr{K}^\bullet) \to {}^pH^1(\mathscr{L}^\bullet) \to \cdots
$$

in $\mathrm{Perv}(X)$.

4. Weights

From now on we will let X_0 (or Y_0, etc.) denote a finite type separated scheme over $\mathbb{F}_q$.

4.1. **The Frobenius automorphism.** For a finite field $\mathbb{F}_q$ with an algebraic closure $\mathbb{F}$ the *geometric Frobenius* automorphism is $F \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ such that $F^{-1}(s) = s^q$ for $s \in \mathbb{F}$. As F is a topological generator for the profinite group $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ the fixed field of F is $\mathbb{F}_q$.

Let $|X_0|$ denote the set of closed points of X_0. For $x \in |X_0|$ the residue field $k(x)$ of x is a finite extension of $\mathbb{F}_q$ of order $N(x) := \#k(x)$ and we let $j\colon \operatorname{Spec}\mathbb{F} = \overline{x} \to X_0$ be a geometric point over x. We let $F_x \in \mathrm{Gal}(\mathbb{F}/k(x))$ be the geometric Frobenius of $k(x)$.

Suppose $\mathscr{F} \in \mathrm{Sh}(X_0, \overline{\mathbb{Q}}_l)$. When we discussed $\operatorname{supp}\mathscr{F}$ in section 3.4 we have observed that $j^*\mathscr{F} \in \mathrm{Sh}(\overline{x}, \overline{\mathbb{Q}}_l)$ makes sense. If we think of $j^*\mathscr{F}$ as being represented by a π-adic sheaf in $\mathrm{Sh}(\overline{x}, \mathfrak{o})$ then the *stalk* of $\mathscr{F}$ at $\overline{x}$ is the finite dimensional (because we are dealing with constructible sheaves) $\overline{\mathbb{Q}}_l$-vector space $\mathscr{F}_{\overline{x}} := j^*\mathscr{F} \otimes_{\mathfrak{o}} \overline{\mathbb{Q}}_l$. It is defind up to isomorphism and does not depend on $\mathfrak{o}$ that was used. The geometric Frobenius F_x induces an automorphism of $\overline{x}$ which fixes the morphism $\overline{x} \to \operatorname{Spec} k(x)$ induced by j. It therefore acts $\overline{\mathbb{Q}}_l$-linearly on the stalk $\mathscr{F}_{\overline{x}}$ and since we set $\mathbb{F}$ to be a *fixed* algebraic closure of $\mathbb{F}_q$ this action only depends on x (there is a unique choice for j). In particular, the eigenvalues of F_x are well-defined and they will play a major role in the sequel.

4.2. **Purity and mixedness.** Fix an isomorphism $\iota\colon \overline{\mathbb{Q}}_l \cong \mathbb{C}$ (algebraically closed fields are classified by characteristic and cardinality). When we talk of an absolute value of $y \in \overline{\mathbb{Q}}_l$ we always mean $|\iota y|$ where $|\cdot|$ is the usual absolute value on $\mathbb{C}$. A $\overline{\mathbb{Q}}_l$-sheaf $\mathscr{F}$ on X_0 is called *ι-pure* of weight β if for each $x \in |X_0|$ all eigenvalues α of F_x acting on the stalk $\mathscr{F}_{\overline{x}}$ have absolute value $|\iota\alpha| = N(x)^{\beta/2}$ (note that $\alpha \neq 0$ as F_x is invertible). A $\overline{\mathbb{Q}}_l$-sheaf $\mathscr{F}$ is called *ι-mixed* if it admits a finite filtration $0 = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_k = \mathscr{F}$ whose associated graded pieces $\mathscr{F}_j/\mathscr{F}_{j-1}$, $1 \leqslant j \leqslant k$ are ι-pure of weights β_j. The β_j corresponding to nonzero quotients $\mathscr{F}_j/\mathscr{F}_{j-1}$ are called the *(ι-)weights*[5] of $\mathscr{F}$. This definition is independent of the chosen filtration as the following proposition shows.

Proposition 4.2.1. *The weights of a ι-mixed sheaf $\mathscr{F}$ are*

$$\mathcal{W}_x := \{2\log_{N(x)} |\iota\alpha| : \alpha \text{ is an eigenvalue of } F_x\}$$

for any $x \in |X_0|$.

Proof. Let $j\colon \overline{x} \to X_0$ be a geometric point over x as in section 4.1. Then a filtration $0 = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_k = \mathscr{F}$ as above induces a filtration

$$0 = (\mathscr{F}_0)_{\overline{x}} \subset (\mathscr{F}_1)_{\overline{x}} \subset \cdots \subset (\mathscr{F}_k)_{\overline{x}} = \mathscr{F}_{\overline{x}}$$

of $\overline{\mathbb{Q}}_l$-vector spaces because $- \otimes_{\mathfrak{o}} \overline{\mathbb{Q}}_l$ is exact as well as j^* (the latter can be seen using the second part of 1.2.1 because the pullback of ordinary sheaves is exact and commutes with shifts $(-)[n]$). This filtration is stable under the action of F_x and therefore the set of eigenvalues of F_x acting on $\mathscr{F}_{\overline{x}}$ is the union of the eigenvalues of induced actions on $(\mathscr{F}_j)_{\overline{x}}/(\mathscr{F}_{j-1})_{\overline{x}} \cong (\mathscr{F}_j/\mathscr{F}_{j-1})_{\overline{x}}$. This gives the desired claim because the eigenvalues of F_x acting on $(\mathscr{F}_j/\mathscr{F}_{j-1})_{\overline{x}}$ are all of absolute value $N(x)^{\beta_j/2}$ because $\mathscr{F}_j/\mathscr{F}_{j-1}$ is ι-pure of weight β_j. □

Proposition 4.2.2. *ι-pure sheaves of weight β and ι-mixed sheaves on X_0 are stable under subobjects, quotients and extensions. In particular, both span abelian subcategories of* $\mathrm{Sh}(X_0, \overline{\mathbb{Q}}_l)$.

[5]We have resisted the temptation of dropping all the ι's and talking of pure or mixed (instead of ι-pure or ι-mixed) sheaves instead only because in the literature this means something else. Namely, one speaks of a pure (resp., mixed) sheaf when it is ι-pure (resp., ι-mixed) *for all* isomorphisms $\iota\colon \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Even though we never use this other notion (we might as well fix an isomorphism once and forget about it) we have decided to keep the ι's fearing to be confusing otherwise. However, when talking of ι-weights we usually drop the ι if it is clear that we are talking about ι-weights of a ι-pure or a ι-mixed sheaf.

Proof. Both subcategories contain the zero sheaf which is ι-pure of every weight and ι-mixed with an empty set of weights. Taking stalks is exact (see the proof of 4.2.1), so subobjects, quotients and extensions of ι-pure sheaves of weight β are ι-pure of weight β. This shows the claim for ι-pure sheaves of weight β.

Fixing a filtration of a ι-mixed sheaf and looking at the stalks, the stability of ι-mixedness under taking subobjects follows from the following triviality from linear algebra: suppose you have subspaces $V_1 \subset V_2$ of a vector space V and another subspace $W \subset V$ so that V_1, V_2, W are all stable under a linear operator $T\colon V \to V$, then the natural map $V_2 \cap W / V_1 \cap W \to V_2/V_1$ is injective and compatible with the endomorphisms induced by T. In a similar vein, stability of ι-mixedness under quotients can be reduced to the natural map $V_2/V_1 \to (V_2 + W)/(V_1 + W)$ being surjective and compatible with the endomorphisms induced by T (if $\mathscr{F}_i$ are the pieces of a filtration for $\mathscr{F}$ we take $(\mathscr{F}_i + \mathscr{G})/\mathscr{G}$ for the pieces of a filtration for $\mathscr{F}/\mathscr{G}$). Stability of ι-mixedness under extensions is trivial because one can simply combine the filtrations for the subobject and the quotient to get a desired filtration for the extension. □

A priori there is no reason to expect for the weights of a ι-pure (hence also of a ι-mixed) sheaf $\mathscr{F}$ to be integers. The following remarkable theorem of Deligne says (among other things) that in the case when they are integers and $\mathscr{F}$ is ι-mixed and smooth (see below) it admits a particularly nice filtration. Before stating it let us define an A-R π-adic sheaf to be *smooth* if it is isomorphic in the A-R category to a π-adic sheaf $\mathscr{F}$ for which all the $\mathscr{F}_i$ are locally constant.

Theorem 4.2.3 (Deligne)**.** *Let $\mathscr{F}$ be a ι-mixed sheaf on X_0.*

1. *There is a direct sum decomposition*
$$\mathscr{F} \cong \bigoplus_{b \in \mathbb{R}/\mathbb{Z}} \mathscr{F}(b)$$
in which each $\mathscr{F}(b)$ is ι-mixed and has all its weights in b. This decomposition is functorial and all but finitely many $\mathscr{F}(b)$ are zero.

2. *If moreover all the weights of $\mathscr{F}$ are integers and $\mathscr{F}$ is smooth then $\mathscr{F}$ has a unique finite ($\mathscr{F}_i = 0$ for i small enough, $\mathscr{F}_j = \mathscr{F}$ for j big enough) filtration*
$$\cdots \subset \mathscr{F}_{i-1} \subset \mathscr{F}_i \subset \mathscr{F}_{i+1} \subset \cdots \subset \mathscr{F}$$
where each $\mathscr{F}_i/\mathscr{F}_{i-1}$ is ι-pure of weight i and all $\mathscr{F}_i$ are smooth subsheaves of $\mathscr{F}$. This filtration is called the weight filtration *and is functorial in $\mathscr{F}$.*

Proof. See [Del80, Théorème 3.4.1]. □

4.3. **Weights and perverse sheaves.** An element $\mathscr{K}^\bullet$ of $D^b_c(X_0, \overline{\mathbb{Q}}_l)$ is called ι-mixed if all its cohomology sheaves $H^n(\mathscr{K}^\bullet)$ are ι-mixed. We consider the full subcategory
$$D_m(X_0) := \{\mathscr{K}^\bullet \in D^b_c(X_0, \overline{\mathbb{Q}}_l) \mid \mathscr{K}^\bullet \text{ is } \iota\text{-mixed}\}$$
spanned by the ι-mixed complexes. It is a triangulated subcategory of $D^b_c(X, \overline{\mathbb{Q}}_l)$: indeed, it is an additive subcategory closed under shifts so to see that the cone construction can be done within $D_m(X_0)$ it is sufficient to check that if two vertices of a triangle are ι-mixed then so is the third; this follows from the associated long exact cohomology sequence (3.2.5) and 4.2.2.

Proposition 4.3.1. *Suppose $f\colon X_0 \to Y_0$ is a morphism. Then the operations $\mathbf{R}f_!, \mathbf{R}f_*, f^*, f^!, \otimes^{\mathbf{L}}, \mathbb{D}$ preserve ι-mixedness.*

Proof. The claim for f^* is easy because pullback is exact, commutes with tensor products, and preserves flatness, so that one can show[6] that the claim for f^* boils down to showing that $f^*\mathscr{F}$ is ι-mixed if $\mathscr{F} \in \mathrm{Sh}(Y_0, \overline{\mathbb{Q}}_l)$ is ι-mixed. This is clear because the stalk of $f^*\mathscr{F}_j/f^*\mathscr{F}_{j-1}$ at a geometric point j is the stalk of $\mathscr{F}_j/\mathscr{F}_{j-1}$ at the geometric point $f \circ j$.

Given we know the claim for $\mathbb{D}$ Lemma 3.3.4 then yields the conclusion for $f^! \cong \mathbb{D}_{X_0} \circ f^* \circ \mathbb{D}_{Y_0}$.

The other parts are much harder. For one thing, the claim that $\mathbf{R}f_!$ preserves ι-mixedness is deduced from Deligne's generalization of the Riemann hypothesis part of the Weil conjectures in [Del80]. The full argument can be found in [KW01, Theorem II.12.2]. □

Proposition 4.3.2. *The perverse truncation operators* ${}^p\tau_{\leqslant n}$, ${}^p\tau_{\geqslant n}$ *preserve ι-mixedness, i.e.,* ${}^p\tau_{\leqslant n}\mathscr{K}^\bullet$, ${}^p\tau_{\geqslant n}\mathscr{K}^\bullet \in D_m(X_0)$ *if* $\mathscr{K}^\bullet \in D_m(X_0)$.

Proof. We will not give the proof here, see [KW01, Lemma III.3.2]. The main ingredient is the glueing lemma described in the proof of 3.4.1. □

In the view of the preceding proposition and 3.4.1 we see that the perverse t-structure on $D^b_c(X_0, \overline{\mathbb{Q}}_l)$ gives rise to a t-structure on $D_m(X_0)$ given by $({}^pD^{\leqslant 0}(X_0) \cap D_m(X_0), {}^pD^{\geqslant 0}(X_0) \cap D_m(X_0))$. Its heart ${}^pD_m(X_0)^\heartsuit = D_m(X_0) \cap \mathrm{Perv}(X_0)$ is the abelian category of *ι-mixed perverse sheaves.*

For a ι-mixed sheaf $\mathscr{F} \in \mathrm{Sh}(X_0, \overline{\mathbb{Q}}_l)$ let $w(\mathscr{F})$ denote its maximal weight, or ∞ if the set of weights of $\mathscr{F}$ is empty. For a ι-mixed complex $\mathscr{K}^\bullet \in D_m(X_0)$ we define the *weight* of $\mathscr{K}^\bullet$ to be $w(\mathscr{K}^\bullet) := \max_n(w(H^n(\mathscr{K}^\bullet)) - n)$ (recall that only finitely many $H^n(\mathscr{K}^\bullet)$ can be nonzero). For any real number β this allows to define two full subcategories of $D_m(X_0)$:

$$D_{\leqslant\beta}(X_0) = \{\mathscr{K}^\bullet \in D_m(X_0) \,|\, w(\mathscr{K}^\bullet) \leqslant \beta\},$$
$$D_{\geqslant\beta}(X_0) = \{\mathscr{K}^\bullet \in D_m(X_0) \,|\, w(\mathbb{D}_{X_0}(\mathscr{K}^\bullet)) \leqslant -\beta\}$$

The complexes in $D_{\leqslant\beta}(X_0) \cap D_{\geqslant\beta}(X_0)$ are called *ι-pure of weight* β.

We conclude these notes with the following theorem which in the spirit of 4.2.3 claims the existence of a weight filtration in the setting of perverse sheaves.

Theorem 4.3.3. *Let* $\mathscr{P} \in \mathrm{Perv}(X_0) \cap D_m(X_0)$ *be a ι-mixed perverse sheaf on* X_0*. There is a finite* weight filtration *of* $\mathscr{P}$

$$0 = \mathscr{P}_0 \subset \mathscr{P}_1 \subset \cdots \subset \mathscr{P}_k = \mathscr{P}$$

by perverse sheaves $\mathscr{P}_i \in \mathrm{Perv}(X_0)$ *whose graded pieces* $\mathscr{P}_i/\mathscr{P}_{i-1}$ *are ι-pure of weight* β_i*. Here the weights* β_i *satisfy* $\beta_i < \beta_j$ *for* $i < j$ *and the nonzero graded pieces* $\mathscr{P}_i/\mathscr{P}_{i-1}$ *are uniquely determined. The weight filtration is functorial in the sense that given any map* $f\colon \mathscr{P} \to \mathscr{R}$ *one can refine the weight filtrations of* $\mathscr{P}$ *and* $\mathscr{R}$ *by inserting finitely many degenerate pieces (so that nonzero graded pieces do not change) to make* f *respect the refined filtrations.*

Proof. See [BBD, Théorème 5.3.5] or [KW01, III.9 Lemma III]. □

Among other things the previous theorem is saying that a simple (having no proper subobjects) ι-mixed perverse sheaf on X_0 is ι-pure.

[6] We don't have means to show this, however, because we haven't constructed the truncation operators $\tau_{\leqslant n}$, $\tau_{\geqslant n}$ explicitly and therefore don't have a handle on how taking cohomology interacts with f^*.

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