PURITY FOR THE BRAUER GROUP

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Abstract. A purity conjecture due to Grothendieck and Auslander–Goldman predicts that the Brauer group of a regular scheme does not change after removing a closed subscheme of codimension $\geqslant 2$. The combination of several works of Gabber settles the conjecture except for some cases that concern p-torsion Brauer classes in mixed characteristic $(0, p)$. We establish the remaining cases by using the tilting equivalence for perfectoid rings. To reduce to perfectoids, we control the change of the Brauer group of the punctured spectrum of a local ring when passing to a finite flat cover.

1. The purity conjecture of Grothendieck and Auslander–Goldman

Grothendieck predicted in [Gro68b, §6] that the cohomological Brauer group of a regular scheme X is insensitive to removing a nowhere dense closed subscheme $Z \subset X$, with some exceptions for Z of codimension 1. Later examples suggested restricting this purity conjecture to Z of codimension $\geqslant 2$: by [DF84, Rem. 3], the Brauer group of $\mathbb{A}^2_{\mathbb{C}}$ does not agree with that of the complement of the coordinate axes. In turn, for Z of codimension $\geqslant 2$, such purity is known in many cases (as we discuss in detail below), for instance, for cohomology classes of order invertible on X. In this paper, we finish the remaining cases, that is, we complete the proof of the following theorem.

Theorem 1.1 (§5.5)**.** *For a scheme X and a closed subscheme $Z \subset X$ such that for every $z \in Z$ the local ring $\mathcal{O}_{X,z}$ of X at z is regular of dimension $\geqslant 2$, we have*

$$H^2_{\text{ét}}(X, \mathbb{G}_m) \xrightarrow{\sim} H^2_{\text{ét}}(X - Z, \mathbb{G}_m) \qquad \textit{and} \qquad H^3_{\text{ét}}(X, \mathbb{G}_m) \hookrightarrow H^3_{\text{ét}}(X - Z, \mathbb{G}_m).$$

The purity conjecture of Grothendieck builds on an earlier question of Auslander–Goldman pointed out in [AG60, 7.4]. Due to a result of Gabber [Gab81, II, Thm. 1], that is, due to the agreement of

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the Brauer group of an affine scheme with its cohomological counterpart, a positive answer to their question amounts to the affine case of the following consequence of Theorem 1.1.

Theorem 1.2 (§5.6)**.** *For a Noetherian, integral, regular scheme X with the function field K,*

$$H^2_{\text{ét}}(X, \mathbb{G}_m) = \textstyle\bigcap_{x \in X \text{ of height } 1} H^2_{\text{ét}}(\mathcal{O}_{X,x}, \mathbb{G}_m) \qquad \text{in} \qquad H^2_{\text{ét}}(K, \mathbb{G}_m).$$

The global Theorems 1.1 and 1.2 are known to readily reduce to the following key local purity result.

Theorem 1.3 (Theorem 5.3)**.** *For a strictly Henselian, regular, local ring R of dimension $\geqslant 2$,*

$$H^2_{\text{ét}}(U_R, \mathbb{G}_m) = 0, \qquad \text{where } U_R \text{ is the punctured spectrum of } R.$$

In turn, as we now summarize, many cases of Theorem 1.3 are already known.

(i) The case $\dim R = 2$ follows from the equivalence of categories between vector bundles on $\mathrm{Spec}(R)$ and on U_R, see [Gro68b, 6.1 b)].

(ii) The case $\dim R = 3$ was settled by Gabber in [Gab81, I, Thm. 2].

(iii) The vanishing of $H^2_{\text{ét}}(U_R, \mathbb{G}_m)[p^\infty]$ for the primes p that are invertible in R follows from the absolute purity conjecture whose proof, due to Gabber, is given in [Fuj02] or [ILO14, XVI] (special cases also follow from earlier [Gro68b, 6.1], [SGA 4_{III}, XVI, 3.7], [Tho84, 3.7]).

(iv) The vanishing of $H^2_{\text{ét}}(U_R, \mathbb{G}_m)[p^\infty]$ in the case when R is an $\mathbb{F}_p$-algebra is given by [Gab93, 2.5] (and, under further assumptions, also by the earlier [Hoo80, Cor. 2]).

(v) The case when R is formally smooth over a discrete valuation ring is given by [Gab93, 2.10].

(vi) Gabber announced further cases in an Oberwolfach abstract [Gab04, Thm. 5 and Thm. 6] whose proofs have not been published: the case $\dim R \geqslant 5$ and the case when R is of dimension 4, of mixed characteristic $(0, p)$, and contains a primitive p-th root of unity.

An example of an R that is not covered by these published or announced results is

$$R = (W(\overline{\mathbb{F}}_p)[\![x_1, x_2, x_3, x_4]\!])/(x_1x_2x_3x_4 - p).$$

For proving Theorem 1.3, we will use its known cases (i)–(iii) but not (iv)–(vi).

Our proof has two main steps. The first is to show that the validity of Theorem 1.3 for R of dimension $\geqslant 4$ is insensitive to replacing R by a regular R' that is finite flat over R. Such a reduction has also been announced in [Gab04, Thm. 4], but our argument seems simpler and gives a more broadly applicable result. More precisely, we argue in §2 that passage to R' is controlled by the U_R-points of a certain homogeneous R-space X, show that X is affine, and then conclude by deducing that $X(U_R) = X(R)$; the restriction $\dim R \geqslant 4$ comes from using the vanishing of the Picard group of the punctured spectrum of the local complete intersection $R' \otimes_R R'$ that intervenes in reducing to X. In comparison, the argument sketched for *loc. cit.* uses deformation theory and a local Lefschetz theorem from [SGA 2_{new}, X] to eventually obtain passage to R' from the known cases of Theorem 1.3. Since the p-primary Brauer group of a perfect $\mathbb{F}_p$-algebra vanishes, the first step suffices in characteristic p.

The second step is to use the tilting equivalence of Scholze introduced in [Sch12] (which, in turn, is a version of the almost purity theorem of Faltings [Fal02]) to show that for a p-torsion free perfectoid ring A, the p-primary cohomological Brauer group $H^2_{\text{ét}}(A[\frac{1}{p}], \mathbb{G}_m)[p^\infty]$ vanishes (see Theorem 4.10). This vanishing ultimately comes from the fact that the étale p-cohomological dimension of an affine, Noetherian scheme of characteristic p is $\leqslant 1$. The intervening comparisons between the étale cohomology of (non-Noetherian) affinoid adic spaces and of their underlying coordinate rings add to the technical details required for the second step but not much to the length of the overall argument because, modulo limit arguments, the comparisons we need were proved by Huber in [Hub96].

The flexibility of the first step leads to a finer result that seems new even for the $\ell \neq p$ parts:

Theorem 1.4 (§5.4)**.** *For a Henselian, regular, local ring R of dimension $\geqslant 2$ whose residue field is of dimension $\leqslant 1$ (in the sense recalled in Definition A.1) and an R-torus T, we have*

$$H^1(U_R, T) = H^2(U_R, T) = 0.$$

The vanishing of $H^1(U_R, T)$, included here for completeness, follows already from [CTS79, 6.9].

1.5. Notation and conventions. For a semilocal ring R, we let

$$U_R \subset \operatorname{Spec}(R)$$

be the open complement of the closed points. For most schemes S that we consider, we have $H^2(S, \mathbb{G}_m) = H^2(S, \mathbb{G}_m)_{\mathrm{tors}}$ (see Lemma 3.2), so we phrase our results about the (cohomological) Brauer group in terms of étale cohomology. Other than in the proof of Lemma 3.1, we do not use the relationship with Azumaya algebras. For a scheme morphism $S' \to S$, we let $(-)_{S'}$ denote base change to S' and let $\operatorname{Res}_{S'/S}(-)$ denote the restriction of scalars. For a field k, we let $\overline{k}$ denote a fixed choice of its algebraic closure. We let $W(-)$ denote the p-typical Witt vectors. When no confusion seems likely, we let $\mathcal{O}$ abbreviate the structure sheaf $\mathcal{O}_S$ of a scheme S.

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2. Passage to a finite flat cover

The perfectoid approach to the purity conjecture hinges on the ability to pass to an infinitely ramified cover of a regular local ring R without killing Brauer classes of its punctured spectrum. The results of the present section facilitate this. To highlight the inputs to their proofs, we chose an axiomatic approach when presenting the key Propositions 2.2 and 2.3. Concrete situations in which these propositions apply are described in Corollaries 2.4 and 2.5 and Remarks 2.6 and 2.7.

Lemma 2.1. *For a finite, locally free scheme morphism $\pi\colon S' \to S$ and an S-affine S-group scheme G, the homogeneous space $X := (\operatorname{Res}_{S'/S}(G_{S'}))/G$ is representable by an S-affine scheme that is smooth if so is G. In addition, if π has a section, then*

$$\operatorname{Res}_{S'/S}(G_{S'}) \cong G \times_S X \tag{2.1.1}$$

as S-schemes and, in the case when G is commutative, even as S-group schemes.

Proof. Both the representability by an S-affine scheme and the smoothness are properties that are fppf local on S, so, by base change along π, we assume that π has a section:

$$\begin{tikzcd} S \arrow[r, "s"] \arrow[dr, equal] & S' \arrow[d, "\pi"] \\ & S. \end{tikzcd}$$

Then the adjunction map $i\colon G \hookrightarrow \operatorname{Res}_{S'/S}(G_{S'})$ has a section $j\colon \operatorname{Res}_{S'/S}(G_{S'}) \twoheadrightarrow \operatorname{Res}_{S/S}(G) \cong G$, which is a group morphism. It follows that $X \cong \operatorname{Ker} j$ over S, compatibly with group structures if G is commutative. Since $\operatorname{Res}_{S'/S}(G_{S'})$ is an S-affine S-group scheme (see [BLR90, 7.6/4 and its proof]), the representability of X by an S-affine scheme and the decomposition (2.1.1) follow. If G is smooth, then so is $\mathrm{Res}_{S'/S}(G_{S'})$, and hence X is, too (see [BLR90, 7.6/5] and [SGA $3_{\text{I new}}$, $\mathrm{VI_B}$, 9.2 (xii)]). $\square$

Proposition 2.2. *For a finite, flat map $R \to R'$ of local rings, an open subscheme $V \subset \operatorname{Spec} R$, and an affine, smooth R-group scheme G, if*

(1) $\Gamma(\operatorname{Spec} R, \mathcal{O}) \cong \Gamma(V, \mathcal{O})$ *via pullback*; *and*

(2) *every G-torsor is trivial over R*;

then the following pullback is injective:

$$H^1_{\text{ét}}(V, G) \hookrightarrow H^1_{\text{ét}}(V_{R'}, G). \tag{2.2.1}$$

Proof. By Lemma 2.1, the homogeneous space $X := (\mathrm{Res}_{R'/R}(G_{R'}))/G$ is an affine R-scheme. Thus, due to (1), we have $X(R) \cong X(V)$ via pullback. However, due to (2), every element of $X(R)$ lifts to $(\mathrm{Res}_{R'/R}(G_{R'}))(R)$. Consequently, every element of $X(V)$ lifts to $(\mathrm{Res}_{R'/R}(G_{R'}))(V)$, so, by [Gir71, III.3.2.2], the map

$$H^1_{\text{ét}}(V, G) \to H^1_{\text{ét}}(V, \mathrm{Res}_{R'/R}(G_{R'})) \tag{2.2.2}$$

is injective. However, the projection $\pi \colon V_{R'} \to V$ is finite, so, as may be checked on strict Henselizations at points of V, the étale sheaf $R^1\pi_*(G_{V_{R'}})$ vanishes. By [Gir71, V.3.1.3], this implies that $H^1_{\text{ét}}(V, \mathrm{Res}_{R'/R}(G_{R'})) \cong H^1_{\text{ét}}(V_{R'}, G)$, so the injectivity of (2.2.1) follows from that of (2.2.2). $\square$

Proposition 2.3. *For a finite, flat map $R \to R'$ of local rings, an open subscheme $V \subset \operatorname{Spec} R$, and an R-torus T that splits over R', if*

(1) $\Gamma(\operatorname{Spec} R, \mathcal{O}) \cong \Gamma(V, \mathcal{O})$ *via pullback;*

(2) *every $((\mathrm{Res}_{R'/R}(T_{R'}))/T)$-torsor is trivial over R; and*

(3) $\mathrm{Pic}(V_{R'\otimes_R R'}) = 0$*;*

then the following pullback is injective:

$$H^2_{\text{ét}}(V, T) \hookrightarrow H^2_{\text{ét}}(V_{R'}, T); \tag{2.3.1}$$

if instead of (3) we have

(3′) $\mathrm{Pic}(V_{R'\otimes_R R'})$ *is torsion free and* $\mathrm{Pic}(V_{R'})$ *is torsion*;

(*in addition to (1) and (2)*), *then the pullback is injective on the torsion subgroups*:

$$H^2_{\text{ét}}(V, T)_{\text{tors}} \hookrightarrow H^2_{\text{ét}}(V_{R'}, T)_{\text{tors}}. \tag{2.3.2}$$

Proof. By Lemma 2.1, the quotient $G := (\mathrm{Res}_{R'/R}(T_{R'}))/T$ is representable by an affine, smooth R-group scheme and $G_{R'}$ is a direct factor R'-group scheme of $(\mathrm{Res}_{(R'\otimes_R R')/R'}(\mathbb{G}_m))^{\mathrm{rk}\, T}$. In particular, (1)–(2) ensure that Proposition 2.2 applies to G, and we conclude the injectivity of the maps

$$H^1_{\text{ét}}(V, G) \hookrightarrow H^1_{\text{ét}}(V_{R'}, G) \hookrightarrow \textstyle\bigoplus_{i=1}^{\mathrm{rk}\, T} H^1_{\text{ét}}(V_{R'}, \mathrm{Res}_{(R'\otimes_R R')/R'}(\mathbb{G}_m)) \cong (\mathrm{Pic}(V_{R'\otimes_R R'}))^{\mathrm{rk}\, T},$$

where the identification follows from the exactness in the étale topology of the pushforward along a finite morphism (see [SGA 4_{II}, VIII, 5.5]). The assumption (3) then gives $H^1_{\text{ét}}(V, G) = 0$ and hence, due to the cohomology sequence

$$\ldots \to H^1_{\text{ét}}(V_{R'}, T) \to H^1_{\text{ét}}(V, G) \to H^2_{\text{ét}}(V, T) \to H^2_{\text{ét}}(V_{R'}, T) \to \ldots, \tag{2.3.3}$$

implies the claimed (2.3.1). In addition, since T splits over R', we have $H^1_{\text{ét}}(V_{R'}, T) \cong (\mathrm{Pic}(V_{R'}))^{\mathrm{rk}\, T}$. Thus, if (3′) holds instead, then $H^1_{\text{ét}}(V, G)$ is torsion free and injects into $H^2_{\text{ét}}(V, T)$, to the effect that then (2.3.3) implies (2.3.2). $\square$

Corollary 2.4. *For a Henselian, regular, local ring R of dimension $\geqslant 2$ whose residue field k is of dimension $\leqslant 1$, a finite étale R-algebra R', and an R-torus T, the following pullbacks are injective*:

$$H^1_{\text{ét}}(U_R, T) \hookrightarrow H^1_{\text{ét}}(U_{R'}, T) \qquad \textit{and} \qquad H^2_{\text{ét}}(U_R, T) \hookrightarrow H^2_{\text{ét}}(U_{R'}, T).$$

Proof. We set $V := U_R$, so that, due to the R-finiteness of R', we have $V_{R'} = U_{R'}$. We lose no generality by enlarging R', so we assume that R' is local and T splits over R'. Thus, the claim follows from Propositions 2.2 and 2.3 once we explain why their assumptions (1)–(2) and (1)–(3) hold.

The assumption (1) holds because R is Noetherian of depth $\geqslant 2$ (see [EGA IV_2, 5.10.5]). Since R'/R is finite étale, $(\mathrm{Res}_{R'/R}(T_{R'}))/T$ and T are both tori, so, by Lemma A.2, their torsors are trivial over k. Thus, since R is Henselian, the same is true over R (see [EGA IV_4, 18.5.17]), so (2) holds. Finally, (3) holds because $R' \otimes_R R'$ is a product of regular local rings of dimension $\geqslant 2$. ☐

Corollary 2.5. *For a finite, flat map $f\colon R \to R'$ of strictly Henselian, regular, local rings of dimension $\geqslant 4$, the following pullback is injective*:

$$H^2_{\text{ét}}(U_R, \mathbb{G}_m) \hookrightarrow H^2_{\text{ét}}(U_{R'}, \mathbb{G}_m).$$

Proof. We apply Proposition 2.3 with $V = U_R$. Its assumption (1) holds because R is of depth $\geqslant 2$. Since R is strictly Henselian and $(\mathrm{Res}_{R'/R}((\mathbb{G}_m)_{R'}))/\mathbb{G}_m$ is R-smooth and affine (see Lemma 2.1), the assumption (2) holds, too. Since R and R' are regular, f is necessarily a local complete intersection morphism (see [SP, 0E9K]), so the local ring $R' \otimes_R R'$ is a local complete intersection of dimension $\geqslant 4$ (see [SP, 069I, 07D3, 09Q7]). However, by [SGA 2_{new}, XI, 3.13 (ii)], the Picard group of the punctured spectrum of a local complete intersection of dimension $\geqslant 4$ vanishes, so (3) holds. ☐

Remarks.

2.6. As is clear from its proof, one may strengthen Corollary 2.5 by assuming instead that f is a finite, flat, local complete intersection morphism of strictly Henselian, Noetherian, local rings that are local complete intersections of dimension $\geqslant 4$.

2.7. A conjecture of Gabber [Gab04, Conj. 3] predicts that $\mathrm{Pic}(U_A)$ is torsion free for any Noetherian local ring A that is a local complete intersection of dimension 3. Thanks to (3′) and the proof above (as well as Lemma 3.2 below), this conjecture implies that the dimension requirement in Corollary 2.5 may be weakened to $\geqslant 3$.

3. Passage to the completion

We will need the flexibility of replacing a Henselian ring by its completion without killing Brauer classes. The following standard results achieve this. Their general theme goes back at least to [Elk73] and they rely on the work of Gabber [Gab81] and Gabber–Ramero [GR03].

Lemma 3.1. *For a ring R that is Henselian along a principal ideal $(f) \subset R$ generated by a nonzerodivisor $f \in R$, the following pullback, where $\widehat{R}$ denotes the f-adic completion of R, is injective*:

$$H^2_{\text{ét}}(R[\tfrac{1}{f}], \mathbb{G}_m)_{\text{tors}} \hookrightarrow H^2_{\text{ét}}(\widehat{R}[\tfrac{1}{f}], \mathbb{G}_m)_{\text{tors}}. \tag{3.1.1}$$

Proof. By [GR03, 5.4.41], for every $n \geqslant 0$, the following pullback is bijective:

$$H^1_{\text{ét}}(R[\tfrac{1}{f}], \mathrm{GL}_n) \xrightarrow{\sim} H^1_{\text{ét}}(\widehat{R}[\tfrac{1}{f}], \mathrm{GL}_n).$$

In addition, for any two $(\mathrm{PGL}_n)_{R[\frac{1}{f}]}$-torsors X and X', their isomorphism functor $\mathrm{Isom}_{\mathrm{PGL}_n}(X, X')$ is representable by an affine, smooth $R[\frac{1}{f}]$-scheme (which étale locally on $R[\frac{1}{f}]$ is isomorphic to

$(\mathrm{PGL}_n)_{R[\frac{1}{f}]})$. Consequently, by [GR03, 5.4.21], if X and X' are not isomorphic, they cannot become isomorphic over $\widehat{R}[\frac{1}{f}]$, to the effect that the following pullback map is injective:

$$H^1_{\text{ét}}(R[\tfrac{1}{f}], \mathrm{PGL}_n) \hookrightarrow H^1_{\text{ét}}(\widehat{R}[\tfrac{1}{f}], \mathrm{PGL}_n).$$

Thus, the nonabelian cohomology exact sequences of [Gir71, IV.4.2.10] that result from the central extension $1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$ fit into the commutative diagram

$$\begin{tikzcd}
H^1_{\text{ét}}(R[\frac{1}{f}], \mathrm{GL}_n) \arrow[r] \arrow[d, "\wr"] & H^1_{\text{ét}}(R[\frac{1}{f}], \mathrm{PGL}_n) \arrow[r] \arrow[d, hook] & H^2_{\text{ét}}(R[\frac{1}{f}], \mathbb{G}_m) \arrow[d] \\
H^1_{\text{ét}}(\widehat{R}[\frac{1}{f}], \mathrm{GL}_n) \arrow[r] & H^1_{\text{ét}}(\widehat{R}[\frac{1}{f}], \mathrm{PGL}_n) \arrow[r] & H^2_{\text{ét}}(\widehat{R}[\frac{1}{f}], \mathbb{G}_m).
\end{tikzcd}$$

This diagram shows that no nonzero element of the image of $H^1_{\text{ét}}(R[\frac{1}{f}], \mathrm{PGL}_n)$ in $H^2_{\text{ét}}(R[\frac{1}{f}], \mathbb{G}_m)$ maps to zero in $H^2_{\text{ét}}(\widehat{R}[\frac{1}{f}], \mathbb{G}_m)$. By [Gab81, II, Thm. 1], as n varies, these images sweep out $H^2_{\text{ét}}(R[\frac{1}{f}], \mathbb{G}_m)_{\text{tors}}$, so the desired injectivity (3.1.1) follows. $\square$

To deduce Proposition 3.3 from Lemma 3.1, we will use the following widely-known result.

Lemma 3.2 ([Gro68a, 1.8])**.** *For a Noetherian, integral, regular scheme X and its function field K, the pullback*

$$H^2_{\text{ét}}(X, \mathbb{G}_m) \hookrightarrow H^2_{\text{ét}}(K, \mathbb{G}_m)$$

is injective; in particular, $H^2_{\text{ét}}(X, \mathbb{G}_m)$ is torsion. $\square$

Proposition 3.3. *For a Henselian, regular, local ring $(R, \mathfrak{m})$, the following pullback, where $\widehat{R}$ denotes the $\mathfrak{m}$-adic completion of R, is injective:*

$$H^2_{\text{ét}}(U_R, \mathbb{G}_m) \hookrightarrow H^2(U_{\widehat{R}}, \mathbb{G}_m). \tag{3.3.1}$$

Proof. Let $f_1, \dots, f_{\dim(R)} \in \mathfrak{m}$ be a regular sequence that generates $\mathfrak{m}$, set $R_0 := R$, and, for each $1 \leqslant i \leqslant \dim(R)$, let R_i be the f_i-adic completion of R_{i-1}. Explicitly, each R_i is the $(f_1, \dots, f_i)$-adic completion of R: indeed, by induction on i, this follows by forming $\varprojlim_n$ of the short exact sequences

$$0 \to R/(f_1^n, \dots, f_{i-1}^n) \xrightarrow{f_i^m} R/(f_1^n, \dots, f_{i-1}^n) \to R/(f_1^n, \dots, f_{i-1}^n, f_i^m) \to 0 \quad \text{for} \quad i > 1, \quad m \geqslant 1.$$

In particular, $R_{\dim(R)} \cong \widehat{R}$ and each R_i is local, regular, and Henselian (see [SP, 0AGX, 07NY, 0DYD]). Consequently, for $1 \leqslant i \leqslant \dim(R)$, Lemmas 3.1 and 3.2 give the commutative diagram

$$\begin{tikzcd}
H^2_{\text{ét}}(U_{R_{i-1}}, \mathbb{G}_m) \arrow[r] \arrow[d, hook] & H^2_{\text{ét}}(U_{R_i}, \mathbb{G}_m) \arrow[d, hook] \\
H^2_{\text{ét}}(R_{i-1}[\frac{1}{f_i}], \mathbb{G}_m) \arrow[r, hook] & H^2_{\text{ét}}(R_i[\frac{1}{f_i}], \mathbb{G}_m),
\end{tikzcd}$$

which shows the injectivity of its top horizontal map. Induction on i then gives (3.3.1). $\square$

4. The p-primary Brauer group in the perfectoid case

While §§2–3 facilitate passage to perfect or perfectoid rings, the present one investigates the p-primary part of the Brauer group of such a ring. We begin with the simpler positive characteristic case.

Proposition 4.1. *For a prime p and a perfect $\mathbb{F}_p$-scheme X, we have*

$$H^i_{\text{ét}}(X, \mathbb{G}_m)[p^\infty] = 0 \quad \textit{for every} \quad i \in \mathbb{Z}.$$

Proof. Every étale X-scheme inherits perfectness from X (see [SGA 5, XV, §1, Prop. 2 c) 2)]). Therefore, on the étale site of X, the p-power map is an automorphism of the sheaf $\mathbb{G}_m$. $\square$

A mixed characteristic analogue of Proposition 4.1 is Theorem 4.10 below, which concerns perfectoid rings. The latter were introduced by Scholze in [Sch12] in the context of rigid geometry, with variants in other contexts appearing afterwards. Their axiomatics that suit our purposes are captured by the following definition and discussion, which are related to [BMS16, §3.2].

Definition 4.2. For a prime p, a p-torsion free ring R is *perfectoid* if R is p-adically complete and the divisor $(p) \subset R$ has a p-th root in the sense that there is a $\varpi \in R$ with $(\varpi^p) = (p)$ and

$$R/\varpi \xrightarrow[\sim]{x \mapsto x^p} R/p. \tag{4.2.1}$$

(Since $(\varpi) \subset R$ is the preimage of the kernel of the Frobenius of R/p, it is uniquely determined.)

Remarks.

4.3. The p-torsion freeness, the p-adic completeness, and (4.2.1) imply that R is reduced.

4.4. The p-adic completeness of R implies that the reduction modulo p map

$$\varprojlim_{x \mapsto x^p} R \xrightarrow{\sim} \varprojlim_{x \mapsto x^p} (R/p)$$

is an isomorphism of multiplicative monoids (see the proof of [Sch12, 3.4 (i)]). Thus, due to the surjectivity of (4.2.1), there is a p-power compatible sequence $(\dots, \varpi_2, \varpi_1)$ of elements of R with $\varpi_1 \equiv \varpi \bmod p$. Since $(p) = (\varpi^p)$, this gives

$$(\varpi) = (\varpi_1) \subset (\varpi_2) \subset \dots \qquad \text{and} \qquad (\varpi_n^{p^n}) = (p) \quad \text{for every} \quad n > 0.$$

In particular, each $(\varpi_n) \subset R$ is uniquely determined: by induction on n, it is the preimage of the kernel of the p^n-power Frobenius of R/p, so that

$$R/\varpi_n \xrightarrow[\sim]{x \mapsto x^{p^n}} R/p. \tag{4.4.2}$$

4.5. By (4.2.1), modulo p^2 every element of R is of the form $x^p + py^p$ or, equivalently, $x^p + \varpi^p y'^p$. In particular, modulo $p\varpi$ every element of R is a p-th power (a special case of [BMS16, 3.9]).

4.6. By [BMS16, 3.10 (ii)], an R as in Definition 4.2 is perfectoid in the sense of the definition [BMS16, 3.5]. Conversely, a p-torsion free ring that is perfectoid in the sense of *loc. cit.* is perfectoid in the sense of Definition 4.2 due to [BMS16, 3.9 and 3.10 (i)].

The following simple lemma often helps to recognize perfectoid rings in nature.

Lemma 4.7. *For a prime p and a p-torsion free ring R such that $(\varpi^p) = (p)$ for some $\varpi \in R$, if R is integrally closed in $R[\frac{1}{p}]$, then the map $R/\varpi \xrightarrow{x \mapsto x^p} R/p$ is injective; if, in addition, every element of R/p is a p-th power, then the p-adic completion $\widehat{R}$ of R is perfectoid.*

Proof. If $r \in R$ represents a class in the kernel of $R/\varpi \xrightarrow{x \mapsto x^p} R/p$, then $r^p = \varpi^p s$ for some $s \in R$. Since $R[\frac{1}{\varpi}] = R[\frac{1}{p}]$ and R is integrally closed in $R[\frac{1}{p}]$, this implies that $\frac{r}{\varpi} \in R$. Thus, $r \in (\varpi)$, and the injectivity follows. Since $\widehat{R}/\varpi \cong R/\varpi$ and $\widehat{R}/p \cong R/p$, the second assertion follows as well. $\square$

To study the p-primary Brauer group of $R[\frac{1}{p}]$, we will use the tilting equivalence of Scholze [Sch12, 7.12]. More precisely, since we do not wish to restrict to $R[\frac{1}{p}]$ that are algebras over some perfectoid field, we will use the version of this equivalence presented by Kedlaya and Liu in [KL15]. We will review its precise statement in §4.9, after discussing the following auxiliary reduction.

4.8. A reduction to the case $R = (R[\frac{1}{p}])^\circ$. We endow a p-torsion free perfectoid ring R with its p-adic topology and $R[\frac{1}{p}]$ with the unique ring topology for which $R \subset R[\frac{1}{p}]$ is open, so that $R[\frac{1}{p}]$ is a Tate ring in the sense of Huber (see [Hub93]). Due to (4.4.2), if $x \in R$ is such that $x^{p^n} \in p^{p^n} R$, then $x \in pR$; in particular, R contains the topologically nilpotent elements $(R[\frac{1}{p}])^{\circ\circ}$. Thus, since the subring $(R[\frac{1}{p}])^\circ \subset R[\frac{1}{p}]$ of powerbounded elements is the union of the open, bounded subrings of $R[\frac{1}{p}]$ that contain R (see [Hub93, 1.2–1.3]), we conclude that, in the notation of Remark 4.4,

$$\text{each} \quad \varpi_n \quad \text{kills the cokernel of the inclusion} \quad R \subset (R[\tfrac{1}{p}])^\circ \tag{4.8.1}$$

(a special case of [BMS16, 3.21]). In particular, the subring $(R[\frac{1}{p}])^\circ \subset R[\frac{1}{p}]$ is bounded (that is, $R[\frac{1}{p}]$ is *uniform*), so $(R[\frac{1}{p}])^\circ$ is p-adically complete. In fact, $(R[\frac{1}{p}])^\circ$ is even perfectoid: indeed, the map

$$(R[\tfrac{1}{p}])^\circ/\varpi \xrightarrow{x \mapsto x^p} (R[\tfrac{1}{p}])^\circ/p$$

is injective because $x^p = \varpi^p y$ in $(R[\frac{1}{p}])^\circ$ implies $\frac{x}{\varpi} \in (R[\frac{1}{p}])^\circ$; it is also surjective because, by (4.8.1) and Remark 4.5, for every $x \in (R[\frac{1}{p}])^\circ$ we have $\varpi_1 x = y^p + p\varpi_1 z$ with $y, z \in R$, so that $\frac{y}{\varpi_2} \in (R[\frac{1}{p}])^\circ$.

In conclusion, by replacing R by $(R[\frac{1}{p}])^\circ$, we reduce the study of $R[\frac{1}{p}]$ to the case when $R = (R[\frac{1}{p}])^\circ$. Then R is a ring of integral elements, so that $(R[\frac{1}{p}], R)$ is an affinoid Tate ring (see [Hub93, §3]).

4.9. The tilting equivalence. Let R be a p-torsion free perfectoid ring with $R = (R[\frac{1}{p}])^\circ$. The norm function

$$x \mapsto \inf(\{2^n \mid n \in \mathbb{Z} \text{ with } p^n x \in R\}) \tag{4.9.1}$$

makes $R[\frac{1}{p}]$ a Banach $\mathbb{Q}_p$-algebra (in the sense of [KL15, 2.2.1]) whose unit ball is R. In particular, the pair $(R[\frac{1}{p}], R)$ becomes a *perfectoid Banach $\mathbb{Q}_p$-algebra* in the sense of [KL15, 3.6.1] (see [KL15, 3.6.2 (e)]). Its *tilt* is

$$(R^\flat[\tfrac{1}{\varpi^\flat}], R^\flat), \qquad \text{where} \qquad R^\flat := \varprojlim_{x \mapsto x^p} (R/p) \quad \text{and} \quad \varpi^\flat \overset{4.4}{:=} (\ldots, \varpi_2 \bmod p, \varpi_1 \bmod p) \in R^\flat,$$

so that $\varpi^\flat \in R^\flat$ is a nonzerodivisor and $R^\flat$ is a $\varpi^\flat$-adically complete, perfect $\mathbb{F}_p$-algebra. We endow $R^\flat$ with its $\varpi^\flat$-adic topology and $R^\flat[\frac{1}{\varpi^\flat}]$ with the unique ring topology for which $R^\flat \subset R^\flat[\frac{1}{\varpi^\flat}]$ is open. Due to the compatible multiplicative monoid isomorphisms

$$R^\flat \overset{4.4}{\cong} \varprojlim_{x \mapsto x^p} R \qquad \text{and} \qquad R^\flat[\tfrac{1}{\varpi^\flat}] \cong \varprojlim_{x \mapsto x^p} (R[\tfrac{1}{p}]),$$

$R^\flat = (R^\flat[\frac{1}{\varpi^\flat}])^\circ$. The norm (4.9.1) with $\varpi^\flat$ in place of p makes $(R^\flat[\frac{1}{\varpi^\flat}], R^\flat)$ a Banach $\mathbb{F}_p$-algebra.

By [KL15, 3.1.13, 3.6.15], the structure presheaves of the spaces

$$\operatorname{Spa}(R[\tfrac{1}{p}], R) \qquad \text{and} \qquad \operatorname{Spa}(R^\flat[\tfrac{1}{\varpi^\flat}], R^\flat) \tag{4.9.2}$$

are sheaves, that is, these spaces are adic. Moreover, by [KL15, 3.6.14], the two spaces in (4.9.2) are naturally (and functorially in R) homeomorphic in such a way that rational subsets correspond to rational subsets. In addition, by the almost purity theorem in this context [KL15, 3.6.23], this homeomorphism extends to an equivalence of étale sites[1]

$$\mathrm{Spa}(R[\tfrac{1}{p}], R)_{\text{ét}} \cong \mathrm{Spa}(R^\flat[\tfrac{1}{\varpi^\flat}], R^\flat)_{\text{ét}} \tag{4.9.3}$$

that identifies finite étale $(R[\frac{1}{p}])$-algebras and finite étale $(R^\flat[\frac{1}{\varpi^\flat}])$-algebras.

Theorem 4.10. *For a p-torsion free perfectoid ring R and a commutative, finite, étale $R[\frac{1}{p}]$-group scheme G of p-power order, we have*

$$H^i_{\text{ét}}(R[\tfrac{1}{p}], G) = 0 \quad \textit{for} \quad i \geqslant 2, \qquad \textit{so also} \qquad H^i_{\text{ét}}(R[\tfrac{1}{p}], \mathbb{G}_m)[p^\infty] = 0 \quad \textit{for} \quad i \geqslant 2. \tag{4.10.1}$$

Proof. By §4.8, we may assume that $R = (R[\frac{1}{p}])^\circ$. Then [Hub96, 3.2.9] (granted that we explain why it applies, as we do below; we choose $U = \mathrm{Spa}\, A$ in *loc. cit.*) gives the identification

$$H^i_{\text{ét}}(R[\tfrac{1}{p}], G) \cong H^i_{\text{ét}}(\mathrm{Spa}(R[\tfrac{1}{p}], R), G). \tag{4.10.2}$$

Since the equivalence (4.9.3) identifies finite étale $(R[\frac{1}{p}])$-algebras and finite étale $(R^\flat[\frac{1}{\varpi^\flat}])$-algebras, G determines a commutative, finite, étale $(R^\flat[\frac{1}{\varpi^\flat}])$-group scheme $G^\flat$ of p-power order such that

$$H^i_{\text{ét}}(\mathrm{Spa}(R[\tfrac{1}{p}], R), G) \cong H^i_{\text{ét}}(\mathrm{Spa}(R^\flat[\tfrac{1}{\varpi^\flat}], R^\flat), G^\flat). \tag{4.10.3}$$

By [Hub96, 3.2.9] again (with the same caveat),

$$H^i_{\text{ét}}(\mathrm{Spa}(R^\flat[\tfrac{1}{\varpi^\flat}], R^\flat), G^\flat) \cong H^i_{\text{ét}}(R^\flat[\tfrac{1}{\varpi^\flat}], G^\flat). \tag{4.10.4}$$

However, by [SGA 4_{III}, X, 5.1], the étale cohomological p-dimension of an affine Noetherian $\mathbb{F}_p$-scheme is at most 1, so, by a limit argument, $H^i_{\text{ét}}(R^\flat[\frac{1}{\varpi^\flat}], G^\flat) = 0$ for $i \geqslant 2$. Due to (4.10.2)–(4.10.4), this gives the desired $H^i_{\text{ét}}(R[\frac{1}{p}], G) = 0$ for $i \geqslant 2$. The second part of (4.10.1) follows by choosing $G = \mu_p$.

In order to ensure that the structure presheaves of adic spectra are sheaves, the book [Hub96] is written under a blanket Noetherianness assumption [Hub96, 1.1.1]. Thus, the deduction of (4.10.2) and (4.10.4) above from [Hub96, 3.2.9] implicitly involves the following limit argument.

The ring R is a filtered direct limit of p-torsion free $\mathbb{Z}_p$-subalgebras R_j of finite type that are integrally closed in $R_j[\frac{1}{p}]$ (to ensure the latter, we use the reducedness of R and [EGA IV_2, 7.8.6 (ii), 7.8.3 (ii)–(iii)]). We may assume that G descends to each $R_j[\frac{1}{p}]$, so, by [SGA 4_{II}, VII, 5.9],

$$H^i_{\text{ét}}(R[\tfrac{1}{p}], G) \cong \varinjlim_j H^i_{\text{ét}}(R_j[\tfrac{1}{p}], G) \qquad \text{for every } i. \tag{4.10.5}$$

We equip each R_j with the p-adic topology and $R_j[\frac{1}{p}]$ with the unique ring topology for which $R_j \subset R_j[\frac{1}{p}]$ is open. Then a valuation of $R_j[\frac{1}{p}]$ whose values on R_j are $\leqslant 1$ is continuous if and only if the values of $\{p^n\}_{n>0}$ are not bounded below, and likewise for $R[\frac{1}{p}]$. In particular, the map

$$\mathrm{Spa}(R[\tfrac{1}{p}], R) \to \varprojlim_j (\mathrm{Spa}(R_j[\tfrac{1}{p}], R_j)) \tag{4.10.6}$$

is a homeomorphism that respects rational subsets. In fact, by following the arguments of [Sch17, proof of 6.4 (ii)], we see that (4.10.6) extends to an equivalence between the étale site $\mathrm{Spa}(R[\frac{1}{p}], R)_{\text{ét}}$ and the 2-limit of the étale sites $\mathrm{Spa}(R_j[\frac{1}{p}], R_j)_{\text{ét}}$ granted that we restrict to quasi-compact and

[1]The étale sites are defined as in [Sch12, 7.1, 7.11]: e.g., a morphism to $\mathrm{Spa}(R[\frac{1}{p}], R)$ is étale if and only if on an open cover of the source it is an open immersion followed by a finite étale map to a rational subspace of $\mathrm{Spa}(R[\frac{1}{p}], R)$; for stability of étale morphisms under compositions and fiber products, see [KL15, 8.2.17 (c)].

quasi-separated adic spaces in these sites (this does not change the associated topoi). As in the proof of [SGA 4_{II}, VII, 5.7], generalities on projective limits of fibered topoi then imply[2] that

$$H^i_{\text{ét}}(\mathrm{Spa}(R[\tfrac{1}{p}], R), G) \cong \varinjlim_j H^i_{\text{ét}}(\mathrm{Spa}(R_j[\tfrac{1}{p}], R_j), G) \qquad \text{for every } i. \tag{4.10.7}$$

Since [Hub96, 3.2.9] applies to each $(R_j[\frac{1}{p}], R_j)$ and the definition of $\mathrm{Spa}(R_j[\frac{1}{p}], R_j)_{\text{ét}}$ that we are using agrees with the one in *op. cit.* (see footnote 1 and [Hub96, 2.2.8]), the combination of (4.10.5) and (4.10.7) gives (4.10.2). The proof for (4.10.4) is analogous: after expressing $R^\flat$ as a filtered direct limit of finite type $\mathbb{F}_p$-subalgebras R'_j that contain $\varpi^\flat$ and are integrally closed in $R'_j[\frac{1}{\varpi^\flat}]$, one repeats the same arguments. $\square$

5. Passage to perfect or perfectoid towers

We are ready to combine the results of the previous sections into a proof of the remaining cases of the purity conjecture for the Brauer group. We begin with auxiliary lemmas that build suitable towers.

Lemma 5.1. *For a complete, regular, local ring $(R, \mathfrak{m})$, there is a filtered direct system of finite, flat R-algebras R_i such that each $(R_i, \mathfrak{m}R_i)$ is a regular local ring and $(\varinjlim_i R_i, \mathfrak{m}(\varinjlim_i R_i))$ is a regular local ring with an algebraically closed residue field.*

Proof. We set $k := R/\mathfrak{m}$ and $p := \mathrm{char}\, k$ and begin with the case when R is of equicharacteristic. By the Cohen structure theorem [Mat89, 29.7], then $R \simeq k[\![x_1, \ldots, x_n]\!]$. We let k_i range over the finite subextensions of $\overline{k}/k$ and set $R_i := k_i[\![x_1, \ldots x_n]\!]$. The $\mathfrak{m}$-adic completion of $\varinjlim_i R_i$ is $\overline{k}[\![x_1, \ldots x_n]\!]$, so $\varinjlim_i R_i$ is Noetherian (see [SP, 033E, 05UU, 00MK]), and hence also a regular local ring.

Now we turn to the case when R is of mixed characteristic and $p \in \mathfrak{m}\backslash\mathfrak{m}^2$. Then, by [Mat89, 29.7] again, $R \simeq W[\![x_1, \ldots, x_n]\!]$ for some complete discrete valuation ring W with p as a uniformizer. By [Mat89, proof of 29.1], there is an integral extension W'/W of discrete valuation rings such that W' has p as a uniformizer and $\overline{k}$ as the residue field. Letting W_i/W range over the finite discrete valuation ring subextensions of W'/W, we argue as in the equicharacteristic case that the local ring $(\varinjlim_i (W_i[\![x_1, \ldots, x_n]\!]), \mathfrak{m}(\varinjlim_i (W_i[\![x_1, \ldots, x_n]\!])))$ has $\widehat{W'}[\![x_1, \ldots, x_n]\!]$ as its completion and is regular.

In the remaining case when R is of mixed characteristic and $p \in \mathfrak{m}^2$, by [Mat89, 29.3 and the proof of 29.8 (ii)], there is a W as above such that $R \simeq W[\![x_1, \ldots, x_n]\!]/(p - f)$ with $f \in (p, x_1, \ldots, x_n)^2$. Then, with the same W' and W_i as before, each $R_i := W_i[\![x_1, \ldots, x_n]\!]/(p - f)$ is a finite, flat R-algebra that is a regular local ring. In addition, by the previous case, $\varinjlim_i (W_i[\![x_1, \ldots, x_n]\!])$ is a regular local ring with the maximal ideal generated by the system of parameters $(p, x_1, \ldots, x_n)$, so $\varinjlim_i R_i$ is a regular local ring with the maximal ideal generated by the system of parameters $(x_1, \ldots, x_n)$. $\square$

The following variant of [And16, 3.4.5 (3)] supplies the perfectoid covers we will need.

Lemma 5.2. *For a mixed characteristic $(0, p)$, complete, regular, local ring $(R, \mathfrak{m})$ with a perfect residue field k, there is a tower $\{R_m\}_{m \in \mathbb{Z}_{\geq 0}}$ of finite, flat R-algebras R_m each of which is a regular, local ring such that the p-adic completion $\widehat{R}_\infty$ of $R_\infty := \varinjlim_m R_m$ is perfectoid (see Definition 4.2).*

Proof. We begin with the case when $p \in \mathfrak{m}\backslash\mathfrak{m}^2$, in which, by the proof of Lemma 5.1 and the perfectness of k, we have $R \simeq W[\![x_1, \ldots, x_n]\!]$ with $W \cong W(k)$. We set $R_m := (W[p^{1/p^m}])[\![x_1^{1/p^m}, \ldots, x_n^{1/p^m}]\!]$ and use Lemma 4.7 to confirm that the resulting $\widehat{R}_\infty$ is perfectoid.

[2] Alternatively and more concretely, one may use hypercoverings and [SP, 01H0] to deduce (4.10.7).

In the remaining case when $p \in \mathfrak{m}^2$, we have $R \simeq W[\![x_1, \dots, x_n]\!]/(p-f)$ with $W \cong W(k)$ and some $f \in (p, x_1, \dots, x_n)^2$, and we set $R_m := W[\![x_1^{1/p^m}, \dots, x_n^{1/p^m}]\!]/(p-f)$. Due to Lemma 4.7, to show that the p-adic completion of the local ring $(R_\infty, \mathfrak{m}_\infty)$ is perfectoid, we only need to argue that for some $u \in R_\infty^\times$ the element up is a p-th power in R_∞. For this, we follow the argument of [Shi16, 4.9]: every element of R_∞/p is a p-th power and $p \in \mathfrak{m}_\infty^2$, so, by writing $p = \sum_i ((s_i^p + pt_i)(s_i'^p + pt_i'))$ with $s_i, s_i' \in \mathfrak{m}_\infty$ and $t_i, t_i' \in R_\infty$, we find that $p = s^p + pt$ with $s, t \in \mathfrak{m}_\infty$ and may set $u := 1 - t$. □

The key purity conclusion for the Brauer group is the following result.

Theorem 5.3. *For a strictly Henselian, regular, local ring $(R, \mathfrak{m})$ of dimension $\geqslant 2$, we have*

$$H^2_{\text{ét}}(U_R, \mathbb{G}_m) = 0.$$

Proof. We set $k := R/\mathfrak{m}$ and $p := \operatorname{char} k$ and use Proposition 3.3 to assume that R is complete. The case $\dim R = 2$ follows from [Gro68b, 6.1 b)] and the case $\dim R = 3$ then follows from [Gab81, I, Thm. 2], so we assume further that $\dim R \geqslant 4$. We then combine Corollary 2.5 with a limit argument and Lemma 5.1 to reduce to the case when $k = \overline{k}$ (to preserve completeness, we again use Proposition 3.3). The absolute purity conjecture proved by Gabber, more precisely, [Fuj02, 2.1.1], implies that for every prime $\ell \neq p$ we have $H^2_{\text{ét}}(U_R, \mu_\ell) = 0$, so also $H^2_{\text{ét}}(U_R, \mathbb{G}_m)[\ell] = 0$. Therefore, since $H^2_{\text{ét}}(U_R, \mathbb{G}_m)$ is torsion (see Lemma 3.2), we will focus on the vanishing of $H^2_{\text{ét}}(U_R, \mathbb{G}_m)[p]$.

We may assume that $p > 0$ and begin with the case when R is an $\mathbb{F}_p$-algebra, so that, as in the proof of Lemma 5.1, we have $R \simeq k[\![x_1, \dots, x_n]\!]$. Since $k = \overline{k}$, the p^m-Frobenius of R is finite and flat for every $m > 0$, so, by combining Corollary 2.5 with a limit argument, we reduce to proving that the perfection V of U_R satisfies $H^2_{\text{ét}}(V, \mathbb{G}_m)[p] = 0$. This, in turn, is a special case of Proposition 4.1.

In the remaining case when R is of mixed characteristic $(0, p)$, let $\{R_m\}$ be a tower supplied by Lemma 5.2. By Corollary 2.5 and a limit argument, it suffices to show that $H^2_{\text{ét}}(U_{R_\infty}, \mathbb{G}_m)[p] = 0$. Each R_m is regular, so, by Lemma 3.2,

$$H^2_{\text{ét}}(U_{R_\infty}, \mathbb{G}_m) \hookrightarrow H^2_{\text{ét}}(R_\infty[\tfrac{1}{p}], \mathbb{G}_m), \quad \text{and, by Lemma 3.1,} \quad H^2_{\text{ét}}(R_\infty[\tfrac{1}{p}], \mathbb{G}_m) \hookrightarrow H^2_{\text{ét}}(\widehat{R}_\infty[\tfrac{1}{p}], \mathbb{G}_m).$$

However, by Theorem 4.10, the group $H^2_{\text{ét}}(\widehat{R}_\infty[\tfrac{1}{p}], \mathbb{G}_m)$ has no nonzero p-torsion. □

5.4. Proof of Theorem 1.4. We have a Henselian, regular, local ring R of dimension $\geqslant 2$ whose residue field is of dimension $\leqslant 1$ and an R-torus T, and we need to show that

$$H^1_{\text{ét}}(U_R, T) = H^2_{\text{ét}}(U_R, T) = 0.$$

By passing to the limit over all the finite, étale, local R-algebras R' and using Corollary 2.4, we reduce to the case when R is strictly Henselian. In this case, T is split, $H^1_{\text{ét}}(U_R, \mathbb{G}_m) = 0$ because every line bundle on U_R extends to R, and $H^2_{\text{ét}}(U_R, \mathbb{G}_m) = 0$ by Theorem 5.3. □

The following standard arguments deduce Theorems 1.1 and 1.2 from Theorem 5.3.

5.5. Proof of Theorem 1.1. We have a scheme X and a closed subscheme $Z \subset X$ such that each local ring $\mathcal{O}_{X,z}$ with $z \in Z$ is regular of dimension $\geqslant 2$, and we need to show that

$$H^2_{\text{ét}}(X, \mathbb{G}_m) \xrightarrow{\sim} H^2_{\text{ét}}(X - Z, \mathbb{G}_m) \qquad \text{and} \qquad H^3_{\text{ét}}(X, \mathbb{G}_m) \hookrightarrow H^3_{\text{ét}}(X - Z, \mathbb{G}_m).$$

By [SGA 4_{II}, V, 6.5], for each X-étale X' and the preimage $Z' \subset X'$ of Z, we have the exact sequence

$$\dots \to H^2_{Z'}(X', \mathbb{G}_m) \to H^2_{\text{ét}}(X', \mathbb{G}_m) \to H^2_{\text{ét}}(X' - Z', \mathbb{G}_m) \to H^3_{Z'}(X', \mathbb{G}_m) \to \dots, \tag{5.5.1}$$

so it suffices to show that $H^2_Z(X, \mathbb{G}_m) = H^3_Z(X, \mathbb{G}_m) = 0$. Thus, letting $\mathcal{H}^q_Z(-, \mathbb{G}_m)$ denote the étale sheafification of the presheaf $X' \mapsto H^q_{Z'}(X', \mathbb{G}_m)$, the local-to-global spectral sequence

$$H^p_{\text{ét}}(X, \mathcal{H}^q_Z(X, \mathbb{G}_m)) \Rightarrow H^{p+q}_Z(X, \mathbb{G}_m)$$

of [SGA 4_{II}, V, 6.4] reduces us to showing that $\mathcal{H}^q_Z(-, \mathbb{G}_m) = 0$ for $q \leqslant 3$. This, in turn, may be checked on stalks: the sheaves $\mathcal{H}^q_Z(-, \mathbb{G}_m)$ are supported on Z and, for each geometric point $\overline{z}$ of Z, the direct limit of the exact sequences (5.5.1) gives the exact sequence

$$\ldots \to \mathcal{H}^2_Z(X, \mathbb{G}_m)_{\overline{z}} \to H^2_{\text{ét}}(\mathcal{O}^{\text{sh}}_{X,\overline{z}}, \mathbb{G}_m) \to H^2_{\text{ét}}(U_{\mathcal{O}^{\text{sh}}_{X,\overline{z}}}, \mathbb{G}_m) \to \mathcal{H}^3_Z(X, \mathbb{G}_m)_{\overline{z}} \to \ldots,$$

so, since $\mathcal{O}^{\text{sh}}_{X,\overline{z}}$ is regular of dimension $\geqslant 2$, the bijectivity of $H^0(\mathcal{O}^{\text{sh}}_{X,\overline{z}}, \mathbb{G}_m) \xrightarrow{\sim} H^0_{\text{ét}}(U_{\mathcal{O}^{\text{sh}}_{X,\overline{z}}}, \mathbb{G}_m)$, the vanishing of $H^i_{\text{ét}}(\mathcal{O}^{\text{sh}}_{X,\overline{z}}, \mathbb{G}_m)$ for $i > 0$, and the vanishing of $H^i_{\text{ét}}(U_{\mathcal{O}^{\text{sh}}_{X,\overline{z}}}, \mathbb{G}_m)$ for $i = 1, 2$ supplied by the extendability of line bundles and Theorem 5.3 give the desired $\mathcal{H}^q_Z(X, \mathbb{G}_m)_{\overline{z}} = 0$ for $q \leqslant 3$. $\square$

5.6. Proof of Theorem 1.2. We have a Noetherian, integral, regular scheme X with the function field K and, bearing Lemma 3.2 in mind, we need to show that

$$H^2_{\text{ét}}(X, \mathbb{G}_m) \xrightarrow{\sim} \textstyle\bigcap_{x \in X \text{ of height } 1} H^2_{\text{ét}}(\mathcal{O}_{X,x}, \mathbb{G}_m) \qquad \text{in} \qquad H^2_{\text{ét}}(K, \mathbb{G}_m).$$

Let α be an element of the intersection in the target. If $U, V \subset X$ are open subschemes such that α extends to an element of both $H^2_{\text{ét}}(U, \mathbb{G}_m)$ and $H^2_{\text{ét}}(V, \mathbb{G}_m)$, then α extends to an element of $H^2_{\text{ét}}(U \cup V, \mathbb{G}_m)$: indeed, this follows from the Mayer–Vietoris sequence

$$\ldots \to H^2_{\text{ét}}(U \cup V, \mathbb{G}_m) \to H^2_{\text{ét}}(U, \mathbb{G}_m) \oplus H^2_{\text{ét}}(V, \mathbb{G}_m) \to H^2_{\text{ét}}(U \cap V, \mathbb{G}_m) \to H^3_{\text{ét}}(U \cup V, \mathbb{G}_m) \to \ldots$$

that results from the Čech-to-derived spectral sequence $\check{H}^p(\{U, V\}, H^q_{\text{ét}}(-, \mathbb{G}_m)) \Rightarrow H^{p+q}_{\text{ét}}(U \cup V, \mathbb{G}_m)$ for the cover $\{U, V\}$ of $U \cup V$. Thus, α extends to an element of $H^2_{\text{ét}}(U, \mathbb{G}_m)$ for some open $U \subset X$ that covers all the height 1 points of X. Then, by Theorem 1.1, it also extends to $H^2_{\text{ét}}(X, \mathbb{G}_m)$. $\square$

Remark 5.7. For a discrete valuation ring $\mathcal{O}$ with the fraction field K and the residue field k, by [Gro68b, 2.1], there is the residue sequence

$$0 \to H^2_{\text{ét}}(\mathcal{O}, \mathbb{G}_m) \to H^2_{\text{ét}}(K, \mathbb{G}_m) \to H^1_{\text{ét}}(k, \mathbb{Q}/\mathbb{Z})$$

that is exact granted that one excludes the $(\operatorname{char} k)$-primary parts in the case when k is imperfect (this exclusion is necessary, see [Poo17, 6.8.2]). Therefore, Theorem 1.2 implies that, as predicted in [Poo17, 6.8.4], for a Noetherian, integral, regular scheme X with the function field K, the sequence

$$0 \to H^2_{\text{ét}}(X, \mathbb{G}_m) \to H^2_{\text{ét}}(K, \mathbb{G}_m) \to \textstyle\bigoplus_{x \in X \text{ of height } 1} H^1_{\text{ét}}(k(x), \mathbb{Q}/\mathbb{Z})$$

is exact granted that one excludes the p-primary parts for every prime p for which some point $x \in X$ of height 1 has an imperfect residue field $k(x)$ of characteristic p.

Appendix A. Fields of dimension $\leqslant 1$

The formulation of Theorem 1.4 above involves the following well-known class of fields.

Definition A.1 ([Ser02, II.§3.1, Prop. 5 and I.§3.1, Prop. 11]). A field k is *of dimension* $\leqslant 1$ if

$$H^i_{\text{ét}}(k, G) = 0 \quad \text{for} \quad i \geqslant 2 \quad \text{and every commutative, finite, étale } k\text{-group scheme } G$$

and if also, when $\operatorname{char} k$ is positive, $H^2_{\text{ét}}(K, \mathbb{G}_m) = 0$ for every finite, separable extension K/k.

In this short appendix, we record Lemma A.2 in the form convenient for its use in the proof of Corollary 2.4 and give an equivalent definition of a field of dimension $\leqslant 1$ in Theorem A.3, which, we believe, deserves to be known more widely.

Lemma A.2. *For a field k of dimension $\leqslant 1$ and a k-torus T, we have*

$$H^i_{\text{ét}}(k, T) = 0 \quad \textit{for every} \quad i \geqslant 1. \tag{A.2.1}$$

Proof. The strict cohomological dimension of k is $\leqslant 2$ (see [Ser02, I.§3.2, Prop. 13]), so the $i \geqslant 3$ case follows. Thus, so does the case $T = \mathbb{G}_m$. Consequently, by choosing a finite Galois extension K/k that splits T and considering the norm map $\mathrm{Res}_{K/k}(T_K) \to T$, at the cost of changing T, we may replace $H^i_{\text{ét}}$ by $H^{i+1}_{\text{ét}}$ in (A.2.1), and then likewise by $H^{i+2}_{\text{ét}}$. Thus, the settled $i \geqslant 3$ case suffices. □

Theorem A.3. *A field k is of dimension $\leqslant 1$ if and only if*

$$H^i_{\text{fppf}}(k, G) = 0 \quad \textit{for} \quad i \geqslant 2 \quad \textit{and every commutative, finite } k\textit{-group scheme } G.$$

Proof. We may assume that $p := \mathrm{char}\, k$ is positive. The displayed condition implies that k is of dimension $\leqslant 1$: indeed, for K/k finite, separable, each $H^2_{\text{ét}}(K, \mathbb{G}_m) \cong H^2_{\text{ét}}(k, \mathrm{Res}_{K/k}(\mathbb{G}_m))$ is torsion, and hence vanishes as soon as $H^2_{\text{fppf}}(k, (\mathrm{Res}_{K/k}(\mathbb{G}_m))[\ell])$ vanishes for every prime ℓ (including $\ell = p$).

For the converse, we assume that k is of dimension $\leqslant 1$ and, by decomposing and filtering G, that G is killed by p, connected, and with $G^\vee$ that is either connected or étale. If $G^\vee$ is also connected, then, by [SGA 3_{II}, XVII, 4.2.1 ii) $\Leftrightarrow$ iv)], the group G is a successive extension of the copies of the Frobenius kernel α_p of $\mathbb{G}_a$. The vanishing of the coherent cohomology $H^i(k, \mathbb{G}_a) = 0$ for $i \geqslant 1$ then gives the claim. If $G^\vee$ is étale, then G is the kernel of a map of k-tori and Lemma A.2 suffices. □

Corollary A.4. *A field k is of dimension $\leqslant 1$ if and only if*

$$H^i_{\text{fppf}}(k, G) = 0 \quad \textit{for} \quad i \geqslant 2 \quad \textit{and every commutative, finite type } k\textit{-group scheme } G.$$

Proof. We may focus on the 'only if.' In addition, by [SGA $3_{\mathrm{I\ new}}$, $\mathrm{VII_A}$, 8.3] and Theorem A.3, we may assume that G is smooth, and then also connected. Then $H^i_{\text{fppf}}(k, G) \cong H^i_{\text{ét}}(k, G)$ is torsion for $i \geqslant 1$, so consideration of the ℓ-torsion $G[\ell]$ settles the case when $\mathrm{char}\, k = 0$ or G is semiabelian. Thus, the "anti-Chevalley theorem" [CGP15, A.3.9] reduces further to affine G. Grothendieck's theorem on maximal tori [SGA 3_{II}, XIV, 1.1] then allows us to assume that G is unipotent (see [SGA 3_{II}, XVII, 4.1.1]). For unipotent G, the filtration of [SGA 3_{II}, XVII, 3.5 ii)] suffices. □

Remark A.5. For vanishing results for $H^1(k, G)$ with k of dimension $\leqslant 1$, see [Ser02, III.§2.3].

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