THE A_{inf}-COHOMOLOGY IN THE SEMISTABLE CASE

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Abstract. For a proper, smooth scheme X over a p-adic field K, we show that any proper, flat, semistable $\mathcal{O}_K$-model $\mathcal{X}$ of X whose logarithmic de Rham cohomology is torsion free determines the same $\mathcal{O}_K$-lattice inside $H^i_{\mathrm{dR}}(X/K)$ and, moreover, that this lattice is functorial in X. For this, we extend the results of Bhatt–Morrow–Scholze on the construction and the analysis of an A_{inf}-valued cohomology theory of p-adic formal, proper, smooth $\mathcal{O}_{\overline{K}}$-schemes $\mathfrak{X}$ to the semistable case. The relation of the A_{inf}-cohomology to the p-adic étale and the logarithmic crystalline cohomologies allows us to reprove the semistable conjecture of Fontaine–Jannsen.

1. Introduction

1.1. Integral relations between p-adic cohomology theories. For a proper smooth scheme X over a complete discretely valued extension K of $\mathbb{Q}_p$ with a perfect residue field k, comparison isomorphisms of p-adic Hodge theory relate the p-adic étale, de Rham, and, in the case of semistable reduction, also crystalline cohomologies of X. For instance, they show that for $i \in \mathbb{Z}$, the $\mathrm{Gal}(\overline{K}/K)$-representation $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ functorially determines the filtered K-vector space $H^i_{\mathrm{dR}}(X/K)$. Even though the "integral" analogues of these isomorphisms are known to fail in general, one may still consider their hypothetical consequences, for instance, one may ask the following.

- For proper, flat, semistable $\mathcal{O}_K$-models $\mathcal{X}$ and $\mathcal{X}'$ of X endowed with their "standard" log structures, do the images of $H^i_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ and $H^i_{\log \mathrm{dR}}(\mathcal{X}'/\mathcal{O}_K)$ in $H^i_{\mathrm{dR}}(X/K)$ agree?

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One of the main goals of the present paper is to show that the answer is positive if the logarithmic de Rham cohomology of the models $\mathcal{X}$ and $\mathcal{X}'$ is torsion free (see (8.6.3) and Theorem 8.7). More precisely, in this case we show that both $H^i_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ and $H^i_{\log \mathrm{dR}}(\mathcal{X}'/\mathcal{O}_K)$ agree with the $\mathcal{O}_K$-lattice in $H^i_{\mathrm{dR}}(X/K)$ that is functorially determined by $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)$. The good reduction case of this result may be derived from the work of Bhatt–Morrow–Scholze [BMS16] on integral p-adic Hodge theory, and our approach, as well as the bulk of this paper, is concerned with extending the framework of *op. cit.* to the semistable case.

1.2. The A_{inf}-cohomology in the semistable case. To approach the question above, we set $C := \widehat{\overline{K}}$, let $A_{\mathrm{inf}} := W(\mathcal{O}_C^\flat)$ be the basic period ring of Fontaine, and, for a semistable $\mathcal{O}_K$-model $\mathcal{X}$ of X, similarly to the smooth case treated in [BMS16], construct the A_{inf}-cohomology

$$R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X}) \in D^{\geq 0}(A_{\mathrm{inf}}).$$

We show that various base changes of $R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X})$ recover other cohomology theories:

$$\begin{aligned} R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} W(C^\flat) &\cong R\Gamma_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} W(C^\flat); \\ R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta} \mathcal{O}_C &\cong R\Gamma_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}_K) \otimes^{\mathbb{L}}_{\mathcal{O}_K} \mathcal{O}_C; \\ R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} W(\overline{k}) &\cong R\Gamma_{\log \mathrm{cris}}(\mathcal{X}_k/W(k)) \otimes^{\mathbb{L}}_{W(k)} W(\overline{k}), \end{aligned} \tag{1.2.1}$$

where $R\Gamma_{\log \mathrm{cris}}$ denotes the logarithmic crystalline (that is, Hyodo–Kato) cohomology, $W(k)$ is endowed with the log structure associated to $\mathbb{N}_{\geq 0} \xrightarrow{0} W(k)$, and $\mathcal{X}_k$ is endowed with the base change of the "standard" log structure $\mathcal{O}_{\mathcal{X},\text{ét}} \cap (\mathcal{O}_{\mathcal{X},\text{ét}}[\frac{1}{p}])^\times$ of $\mathcal{X}$.

If the cohomology of $R\Gamma_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ is torsion free, then that of $R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X})$ is A_{inf}-free and the base changes (1.2.1) hold in each individual cohomological degree (see §7.6). In this case, the Fargues equivalence and the formalism of Breuil–Kisin–Fargues $\mathrm{Gal}(\overline{K}/K)$-modules allow us to prove that

$$\text{the } \mathrm{Gal}(\overline{K}/K)\text{-representation} \quad H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \quad \text{determines} \quad H^i_{A_{\mathrm{inf}}}(\mathcal{X}).$$

It follows that then $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)$ also determines $H^i_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ (together with $H^i_{\log \mathrm{cris}}(\mathcal{X}_k/W(k))$). Since the same reasoning applies to another model $\mathcal{X}'$, this leads to the result claimed in §1.1.

The base changes (1.2.1) also allow us to extend the cohomology specialization results obtained in the good reduction case in [BMS16]. Qualitatively, in Proposition 7.7 we show that $H^i_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ is torsion free if and only if so is $H^i_{\log \mathrm{cris}}(\mathcal{X}_k/W(k))$, in which case $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)$ is torsion free. Quantitatively, in Theorems 7.10 and 7.13 we show that for every $n \geq 0$,

$$\begin{aligned} \mathrm{length}_{\mathbb{Z}_p}((H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)_{\mathrm{tors}})/p^n) &\leq \mathrm{length}_{W(k)}((H^i_{\log \mathrm{cris}}(\mathcal{X}_k/W(k))_{\mathrm{tors}})/p^n), \\ \mathrm{length}_{\mathbb{Z}_p}((H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)_{\mathrm{tors}})/p^n) &\leq \frac{1}{\mathrm{length}_{\mathcal{O}_K}(\mathcal{O}_K/p)} \cdot \mathrm{length}_{\mathcal{O}_K}((H^i_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)_{\mathrm{tors}})/p^n). \end{aligned}$$

1.3. The semistable comparison isomorphism. The analysis of $R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X})$, specifically, its relation to the p-adic étale and the logarithmic crystalline cohomologies, permits us to reprove in Theorem 9.5 the "semistable conjecture" of Fontaine–Jansen [Kat94a, Conj. 1.1]:

$$R\Gamma_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} B_{\mathrm{st}} \cong R\Gamma_{\log \mathrm{cris}}(\mathcal{X}_k/W(k)) \otimes^{\mathbb{L}}_{W(k)} B_{\mathrm{st}}. \tag{1.3.1}$$

Other proofs of this conjecture have been given in [Tsu99], [Fal02], [Niz08], [Bha12], [Bei13a], and [CN17], whereas [BMS16] used $R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X})$ to reprove the "crystalline conjecture." Similarly to [CN17], we establish (1.3.1) for a suitable class of proper, flat, "semistable" formal $\mathcal{O}_K$-schemes $\mathcal{X}$.

The key result that leads to (1.3.1) is the so-called absolute crystalline comparison isomorphism

$$R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}} \cong R\Gamma_{\log \mathrm{cris}}(\mathcal{X}_{\mathcal{O}_{\overline{K}}/p}/A_{\mathrm{cris}}) \tag{1.3.2}$$

of Corollary 5.43, whose construction in §5 forms the technical core of this paper. This construction is based on the "all possible coordinates" technique that is a variant of its analogue used to establish (1.3.2) in the smooth case in [BMS16, §12]. The presence of singularities and log structures creates additional complications that do not appear in the smooth case and are overviewed in §5.

Using the absolute crystalline comparison, in Corollary 6.7 we compare the A_{inf}-cohomology of $\mathcal{X}$ with the B_{dR}^{+}-cohomology of X defined by Bhatt–Morrow–Scholze in [BMS16, §13]:

$$R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} B_{\mathrm{dR}}^{+} \cong R\Gamma_{\mathrm{cris}}(X_C^{\mathrm{ad}}/B_{\mathrm{dR}}^{+}). \tag{1.3.3}$$

The identification (1.3.3) is important for ensuring that our semistable comparison (1.3.1) is compatible with the de Rham comparison proved in [Sch13], and hence that it respects filtrations.

As for the question posed in §1.1, even though it only involves the étale and the de Rham cohomologies, the resolution of its "torsion free case" outlined in §1.2 uses both (1.3.2) and (1.3.3) (so also the bulk of the material of this paper). This is because we need to ensure that the determination of $H^i_{\mathrm{dR}}(X/K)$ by $H^i_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbb{Q}_p)$ via the de Rham comparison of p-adic Hodge theory is compatible with the determination of $H^i_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ and $H^i_{\log \mathrm{dR}}(\mathcal{X}'/\mathcal{O}_K)$ by $H^i_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbb{Z}_p)$ via A_{inf}-cohomology and Breuil–Kisin–Fargues modules. In fact, even for showing that the cohomology modules of $R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X})$ are Breuil–Kisin–Fargues, we already use the absolute crystalline comparison (1.3.2).

1.4. The object $A\Omega_{\mathfrak{X}}$ and its base changes. Even though above we have focused on schemes, the construction and the analysis of $R\Gamma_{A_{\mathrm{inf}}}(-)$ works for any p-adic formal $\mathcal{O}_C$-scheme $\mathfrak{X}$ that is semistable in the sense described in §1.5 below (see (1.5.1)) and that, whenever needed, is assumed to be proper. Specifically, for such an $\mathfrak{X}$, in §2.2 we use the (variant for the étale topology of the) definition of Bhatt–Morrow–Scholze from [BMS16] to build an object

$$A\Omega_{\mathfrak{X}} \in D^{\geq 0}(\mathfrak{X}_{\mathrm{\acute{e}t}}, A_{\mathrm{inf}}), \qquad \text{and to set} \qquad R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) := R\Gamma(\mathfrak{X}_{\mathrm{\acute{e}t}}, A\Omega_{\mathfrak{X}}).$$

As in the smooth case of [BMS16], the relation of $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ to the p-adic étale cohomology of the adic generic fiber $\mathfrak{X}_C^{\mathrm{ad}}$ of $\mathfrak{X}$ follows from the results of [Sch13] (see §2). In turn, the relations to the logarithmic de Rham and crystalline cohomologies are the subjects of §4 and §5, respectively, and rest on the following identifications established in Theorems 4.16 and 5.4:

$$A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\, \theta} \mathcal{O}_C \cong \Omega^{\bullet}_{\mathfrak{X}/\mathcal{O}_C,\, \log} \qquad \text{and} \qquad A\Omega_{\mathfrak{X}} \widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}} \cong Ru_*(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}), \tag{1.4.1}$$

where $u\colon (\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}})_{\log \mathrm{cris}} \to \mathfrak{X}_{\mathrm{\acute{e}t}}$ is the forgetful map of topoi. The arguments for (1.4.1) are built on the same general skeleton as in [BMS16] but differ, among other aspects, in handling the interaction of the Deligne–Berthelot–Ogus décalage functor $L\eta$ used in the definition of $A\Omega_{\mathfrak{X}}$ with the intervening base changes and with the almost isomorphisms supplied by the almost purity theorem. Namely, for this, the nonflatness over the singular points of $\mathfrak{X}$ of the explicit perfectoid proétale covers that we construct makes it difficult to directly adapt the arguments from *op. cit.* Instead, we take advantage of several general results about $L\eta$ from [Bha16]. Verifying their assumptions in our case amounts to the analysis in §3 of a number of continuous group cohomology modules built using the aforementioned perfectoid cover. The typical conclusion of this analysis is that these modules have no nonzero "almost torsion" and that the element $\mu \in A_{\mathrm{inf}}$ kills their "nonintegral parts."

Further and more specific overviews of our arguments are given in the beginning parts of the sections that follow. In the rest of this introduction, we fix the precise notational setup for the rest of the paper (see §1.5), discuss the logarithmic structure on $\mathfrak{X}$ that we later use without notational explication (see §1.6), and review the relevant general notational conventions (see §1.7).

1.5. The setup. In what follows, we fix the following notational setup.

- We fix an algebraically closed field k of characteristic $p > 0$, let C be the completed algebraic closure of $W(k)[\frac{1}{p}]$, and let $\mathfrak{m} \subset \mathcal{O}_C$ be the maximal ideal in the valuation ring of C.
- For convenience, we fix an embedding $p^{\mathbb{Q}} \subset C$, that is, for every prime ℓ, we fix a system of compatible ℓ^n-power roots $p^{1/\ell^\infty} := (p^{1/\ell^n})_{n>0}$ of p in $\mathcal{O}_C$.
- We fix a p-adic formal scheme $\mathfrak{X}$ over $\mathcal{O}_C$ that in the étale topology may be covered by open affines $\mathfrak{U}$ which admit an étale $\mathcal{O}_C$-morphism
$$\mathfrak{U} = \mathrm{Spf}(R) \to \mathrm{Spf}(R^\square) \quad \text{with} \quad R^\square := \mathcal{O}_C\{t_0, \dots, t_r, t_{r+1}^{\pm 1}, \dots, t_d^{\pm 1}\}/(t_0 \cdots t_r - p^q) \quad (1.5.1)$$
for some $d \geq 0$, some $0 \leq r \leq d$, and some $q \in \mathbb{Q}_{>0}$ (where d, r, and q may depend on $\mathfrak{U}$).

For example, C could be the completed algebraic closure of any discretely valued field K of mixed characteristic $(0, p)$ with a perfect residue field. In addition, no generality is gained by replacing p^q in (1.5.1) by *any* nonunit $\pi \in \mathcal{O}_C \setminus \{0\}$. The role of the embedding $p^{\mathbb{Q}} \subset C$ is to simplify arguments with explicit charts for the log structure on $\mathfrak{X}$ (defined in §1.6); this is particularly useful in §5, especially in §§5.25–5.26. Our C is less general than in [BMS16], where any complete algebraically closed nonarchimedean extension of $\mathbb{Q}_p$ is typically allowed. One of the main reasons for this is that we want to be able to apply, especially in §5, certain auxiliary results from [Bei13b] (besides, relations $t_0 \cdots t_r - \pi$ in which π has, say, a transcendental valuation go beyond what is typically understood by "semistable reduction").

The existence of the étale local *semistable coordinates* (1.5.1) implies that each $\mathfrak{X}_{\mathcal{O}_C/p^n}$ is locally of finite type and flat over $\mathcal{O}_C/p^n$ and $\mathfrak{X}^{\mathrm{sm}}_{\mathcal{O}_C/p^n}$ is dense in $\mathfrak{X}_{\mathcal{O}_C/p^n}$. By [SP, 04D1] and limit arguments, the map (1.5.1) is the formal p-adic completion of the $\overline{W(k)}$-base change of an étale $\mathcal{O}$-morphism
$$U \to \mathrm{Spec}\left(\mathcal{O}[t_0, \dots, t_r, t_{r+1}^{\pm 1}, \dots, t_d^{\pm 1}]/(t_0 \cdots t_r - p^q)\right) \quad (1.5.2)$$
for some discrete valuation subring $\mathcal{O} \subset \overline{W(k)}$ that contains p^q. *Loc. cit.* and [GR03, 7.1.6 (i)] also imply that R is $R^\square$-flat. In addition, if R/p is not $\mathcal{O}_C/p$-smooth, then R determines q.[1]

Any smooth p-adic formal $\mathcal{O}_C$-scheme $\mathfrak{X}$ meets the requirements above: indeed, then the cover $\{\mathfrak{U}\}$ exists already for the Zariski topology with $r = 0$ and $q = 1$ for all $\mathfrak{U}$, see [FK17, I.5.3.18]. Another key example is
$$\mathfrak{X} = \widehat{\mathcal{X}_{\mathcal{O}_C}} \quad (1.5.3)$$
for some discrete valuation subring $\mathcal{O} \subset \mathcal{O}_C$ with a perfect residue field and a uniformizer $\pi \in \mathcal{O}$ and a locally of finite type, flat $\mathcal{O}$-scheme $\mathcal{X}$ that is *semistable* in the sense that $\mathcal{X}_{\mathcal{O}/\pi}$ is a normal crossings divisor in $\mathcal{X}$ (as defined in [SP, 0BSF]), so that, in particular, $\mathcal{X}$ is regular at every point of $\mathcal{X}_{\mathcal{O}/\pi}$.[2] Moreover, if $\mathcal{X}$ is even *strictly semistable* in the sense that $\mathcal{X}_{\mathcal{O}/\pi}$ is even a strict normal

[1] The following argument justifies this. Choose an $n \in \mathbb{Z}_{>q}$ and let A be the local ring of $\mathrm{Spec}(R/p^n)$ at some singular point. Without loss of generality, all the t_i with $0 \leq i \leq r$ are noninvertible in A, so, in particular, $r \geq 1$. The d-th Fitting ideal $\mathrm{Fitt}_d(\Omega^1_{(R^\square/p^n)/(\mathcal{O}_C/p^n)}) \subset R^\square/p^n$ is generated by the r-fold partial products $t_0 \cdots \widehat{t_i} \cdots t_r$ with $0 \leq i \leq r$, so the same holds for $\mathrm{Fitt}_d(\Omega^1_{A/(\mathcal{O}_C/p^n)}) \subset A$ (see [SGA 7_{I}, VI, 5.1 (a)]). Consequently, the quotient $(R^\square/p^n)/(\mathrm{Fitt}_d(\Omega^1_{(R^\square/p^n)/(\mathcal{O}_C/p^n)}))$ is faithfully flat over $\mathcal{O}_C/(p^q)$, and hence so is $A/(\mathrm{Fitt}_d(\Omega^1_{A/(\mathcal{O}_C/p^n)}))$. It follows that $(p^q) \subset \mathcal{O}_C$ is the preimage of $\mathrm{Fitt}_d(\Omega^1_{A/(\mathcal{O}_C/p^n)}) \subset A$, to the effect that R determines q.

[2] To justify that any $\mathfrak{X}$ as in (1.5.3) meets the requirements, we first note that étale locally on $\mathcal{X}$ there exists a regular sequence such that the product its $r+1$ first terms cuts out $\mathcal{X}_{\mathcal{O}/\pi}$. Thus, since any finite extension of $\mathcal{O}/\pi$ is separable, the miracle flatness theorem [EGA IV_2, 6.1.5] ensures that every $x \in \mathcal{X}_{\mathcal{O}/\pi}$ has an étale neighborhood $U \to \mathcal{X}$ that admits an étale $\mathcal{O}$-morphism $U \to \mathrm{Spec}(\mathcal{O}[t_0, \dots, t_d]/(t_0 \cdots t_r - \pi))$ or, equivalently, an étale morphism
$$U \to \mathrm{Spec}(\mathcal{O}[t_0, \dots, t_r, t_{r+1}^{\pm 1}, \dots, t_d^{\pm 1}]/(t_0 \cdots t_r - \pi))). \quad (1.5.4)$$

crossings divisor in $\mathcal{X}$ (as defined in [SP, 0BI9]), then the étale maps (1.5.4) exist even Zariski locally on $\mathcal{X}$, and so also the cover $\{\mathfrak{U}\}$ exists already for the Zariski topology of $\mathfrak{X}$.

- We let $\mathfrak{X}_C^{\mathrm{ad}}$ denote the adic generic fiber of $\mathfrak{X}$. By (1.5.1) and [Hub96, 3.5.1], the adic space $\mathfrak{X}_C^{\mathrm{ad}}$ is smooth over C; by [Hub96, 1.3.18 ii)], if $\mathfrak{X}$ is $\mathcal{O}_C$-proper, then $\mathfrak{X}_C^{\mathrm{ad}}$ is C-proper.
- We let $(\mathfrak{X}_C^{\mathrm{ad}})_{\mathrm{pro\acute{e}t}}$ denote the proétale site of $\mathfrak{X}_C^{\mathrm{ad}}$ (reviewed in [BMS16, §5.1] and defined in [Sch13, 3.9] and [Sch13e, (1)]) and let

$$\nu\colon (\mathfrak{X}_C^{\mathrm{ad}})_{\mathrm{pro\acute{e}t}} \to \mathfrak{X}_{\mathrm{\acute{e}t}} \tag{1.5.5}$$

be the morphism to the étale site of $\mathfrak{X}$ that sends any étale $\mathfrak{U} \to \mathfrak{X}$ to the constant pro-system associated to its adic generic fiber. By [SP, 00X6], this functor indeed defines a morphism of sites: by [Hub96, 3.5.1], it preserves coverings, commutes with fiber products, and respects final objects. Thus, ν induces a morphism of topoi (ν^{-1}, ν_*) (see [SP, 00XC]).

1.6. The logarithmic structure on $\mathfrak{X}$. Unless noted otherwise, we always equip

(1) the ring $\mathcal{O}_C$ (resp., $\mathcal{O}_C/p^n$ or k) with the log structure $\mathcal{O}_C \setminus \{0\} \hookrightarrow \mathcal{O}_C$ (resp., its pullback);

(2) the formal scheme $\mathfrak{X}$ (resp., $\mathfrak{X}_{\mathcal{O}_C/p^n}$ or $\mathfrak{X}_k$) with the log structure given by the subsheaf associated to the subpresheaf[3] $\mathcal{O}_{\mathfrak{X},\,\mathrm{\acute{e}t}} \cap (\mathcal{O}_{\mathfrak{X},\,\mathrm{\acute{e}t}}[\frac{1}{p}])^\times \hookrightarrow \mathcal{O}_{\mathfrak{X},\,\mathrm{\acute{e}t}}$ (resp., its pullback log structure).

Both (1) and (2) determine the same log structure on $\mathrm{Spf}(\mathcal{O}_C)$, so the map $\mathfrak{X} \to \mathrm{Spf}(\mathcal{O}_C)$ is that of log formal schemes. Moreover, étale locally on $\mathfrak{X}$, the log structure may be made explicit: in the presence of a coordinate morphism (1.5.1), Claims 1.6.1 and 1.6.3 below give an explicit chart for the log structure of $\mathfrak{U}$, namely, the chart (1.6.2) in which we replace $\mathcal{O}$ by $\mathcal{O}_C$, replace U by $\mathfrak{U}$, and set $\pi := p^q$. This chart shows, in particular, that $\mathfrak{U}$ and $\mathcal{O}_C$ may be endowed with fine log structures whose base changes along a "change of log structure" self-map of $\mathcal{O}_C$ recover the log structures described in (1)–(2). In practice this means that we may deal with the log structures in (1)–(2) as if they were fine and, in particular, we may cite [Kat89] for certain purposes.

By the preceding discussion, all the log structures above are quasi-coherent and integral. Moreover, by [Kat89, 3.7 (2)], each $\mathfrak{X}_{\mathcal{O}_C/p^n}$ is log smooth over $\mathcal{O}_C/p^n$, so that, by [Kat89, 3.10], the $\mathcal{O}_{\mathfrak{X}}$-module $\Omega^1_{\mathfrak{X}/\mathcal{O}_C,\,\log}$ of logarithmic differentials is finite locally free. We set

$$\Omega^i_{\mathfrak{X}/\mathcal{O}_C,\,\log} := \textstyle\bigwedge^i \Omega^1_{\mathfrak{X}/\mathcal{O}_C,\,\log},$$

let $\Omega^\bullet_{\mathfrak{X}/\mathcal{O}_C,\,\log}$ denote the logarithmic de Rham complex, and set

$$R\Gamma_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C) := R\Gamma(\mathfrak{X}_{\mathrm{\acute{e}t}}, \Omega^\bullet_{\mathfrak{X}/\mathcal{O}_C,\,\log}).$$

Claim 1.6.1. *For a valuation subring $\mathcal{O} \subset \overline{W(k)}$ and an $\mathcal{O}$-scheme U that has an étale morphism*

$$U \to \mathrm{Spec}\left(\mathcal{O}[t_0, \dots, t_r, t_{r+1}^{\pm 1}, \dots, t_d^{\pm 1}]/(t_0 \cdots t_r - \pi)\right) \qquad \text{for some nonunit} \qquad \pi \in \mathcal{O} \setminus \{0\},$$

the log structure on U associated to $\mathcal{O}_{U,\,\mathrm{\acute{e}t}} \cap (\mathcal{O}_{U,\,\mathrm{\acute{e}t}}[\frac{1}{p}])^\times$ has the chart

$$\mathbb{N}_{\geq 0}^{r+1} \sqcup_{\mathbb{N}_{\geq 0}} (\mathcal{O} \setminus \{0\}) \to \Gamma(U, \mathcal{O}_U) \tag{1.6.2}$$

given by $(a_i)_{0 \leq i \leq r} \mapsto \prod_{0 \leq i \leq r} t_i^{a_i}$ on $\mathbb{N}_{\geq 0}^{r+1}$, the diagonal $\mathbb{N}_{\geq 0} \to \mathbb{N}_{\geq 0}^{r+1}$ and $\mathbb{N}_{\geq 0} \xrightarrow{a \mapsto \pi^a} (\mathcal{O} \setminus \{0\})$ on $\mathbb{N}_{\geq 0}$, and the structure map $(\mathcal{O} \setminus \{0\}) \to \Gamma(U, \mathcal{O}_U)$ on $\mathcal{O} \setminus \{0\}$.

[3] The subpresheaf and its associated subsheaf necessarily agree on every quasi-compact object $\mathfrak{U}$ of $\mathfrak{X}_{\mathrm{\acute{e}t}}$.

Proof. Without loss of generality, U is affine, so, by a limit argument, we may assume that $\mathcal{O}$ is discretely valued. Then U, endowed with the log structure associated to (1.6.2), is logarithmically regular in the sense of [Kat94b, 2.1] (compare with [Bei12, §4.1, proof of Lemma]). Therefore, since the locus of triviality of this log structure is $U[\frac{1}{p}]$, the claim follows from [Kat94b, 11.6]. □

Claim 1.6.3. *For $\mathcal{O}$ as in Claim 1.6.1, a flat $\mathcal{O}$-scheme U (resp., and its formal p-adic completion $\mathfrak{U}$) endowed with the log structure associated to $\mathcal{O}_{U,\,\text{ét}} \cap (\mathcal{O}_{U,\,\text{ét}}[\frac{1}{p}])^\times$ (resp., $\mathcal{O}_{\mathfrak{U},\,\text{ét}} \cap (\mathcal{O}_{\mathfrak{U},\,\text{ét}}[\frac{1}{p}])^\times$),*

$$\text{the formal } p\text{-adic completion morphism} \quad j\colon \mathfrak{U} \to U \quad \text{of log ringed étale sites is strict.} \tag{1.6.4}$$

Proof. For a geometric point $\overline{u}$ of $\mathfrak{U}$, due to [SP, 04D1], the stalk map $\mathcal{O}_{U,\overline{u}} \cong j^{-1}(\mathcal{O}_{U,\overline{u}}) \to \mathcal{O}_{\mathfrak{U},\overline{u}}$ induces an isomorphism $\mathcal{O}_{U,\overline{u}}/p^n \cong \mathcal{O}_{\mathfrak{U},\overline{u}}/p^n$ for every $n > 0$. We consider the stalk map

$$\mathcal{O}_{U,\overline{u}} \cap (\mathcal{O}_{U,\overline{u}}[\tfrac{1}{p}])^\times \cong j^{-1}(\mathcal{O}_{U,\overline{u}} \cap (\mathcal{O}_{U,\overline{u}}[\tfrac{1}{p}])^\times) \to \mathcal{O}_{\mathfrak{U},\overline{u}} \cap (\mathcal{O}_{\mathfrak{U},\overline{u}}[\tfrac{1}{p}])^\times. \tag{1.6.5}$$

Every element x of the target of (1.6.5) satisfies the equation $xy = p^n$ for some $n > 0$. We choose an $\widetilde{x} \in \mathcal{O}_{U,\overline{u}}$ congruent to x modulo p^{n+1}, so that $\widetilde{x}\widetilde{y} = p^n + p^{n+1}\widetilde{z}$ for some $\widetilde{y}, \widetilde{z} \in \mathcal{O}_{U,\overline{u}}$. Since $1 + p\widetilde{z} \in \mathcal{O}_{U,\overline{u}}^\times$, we adjust $\widetilde{y}$ to get $\widetilde{x}\widetilde{y} = p^n$, which shows that $\widetilde{x} \in \mathcal{O}_{U,\overline{u}} \cap (\mathcal{O}_{U,\overline{u}}[\frac{1}{p}])^\times$ and $(p^n) \subset (\widetilde{x})$. Thus, the image of $\widetilde{x}$ in $\mathcal{O}_{\mathfrak{U},\overline{u}}$ and x generate the same ideal, and hence are unit multiples of each other. Conversely, if $\widetilde{x}_1, \widetilde{x}_2 \in \mathcal{O}_{U,\overline{u}} \cap (\mathcal{O}_{U,\overline{u}}[\frac{1}{p}])^\times$ are unit multiples of each other in $\mathcal{O}_{\mathfrak{U},\overline{u}}$, then, by reducing modulo p^n for a large enough n, we see that they generate the same ideal in $\mathcal{O}_{U,\overline{u}}$, so are unit multiples of each other already in $\mathcal{O}_{U,\overline{u}}$. In conclusion, the map (1.6.5) induces an isomorphism

$$(\mathcal{O}_{U,\overline{u}} \cap (\mathcal{O}_{U,\overline{u}}[\tfrac{1}{p}])^\times)/\mathcal{O}_{U,\overline{u}}^\times \xrightarrow{\sim} (\mathcal{O}_{\mathfrak{U},\overline{u}} \cap (\mathcal{O}_{\mathfrak{U},\overline{u}}[\tfrac{1}{p}])^\times)/\mathcal{O}_{\mathfrak{U},\overline{u}}^\times,$$

to the effect that the map (1.6.4) is indeed strict, as claimed. □

1.7. Conventions and additional notation. For a field K, we let $\overline{K}$ be its algebraic closure (taken inside C if K is given as a subfield of C). If K has a valuation, we let $\mathcal{O}_K$ be its valuation subring and write $\overline{\mathcal{O}}_K$ for the integral closure of $\mathcal{O}_K$ in $\overline{K}$. In mixed characteristic, we normalize the valuations by requiring that $v(p) = 1$. We let $(-)^{\text{sm}}$ denote the smooth locus of a (formal) scheme over an implicitly understood base. For power series rings, we use $\{-\}$ to indicate decaying coefficients. For a topological ring R, we let R° denote the subset of powerbounded elements.

We let $W(-)$ (resp. $W_n(-)$) denote p-typical Witt vectors (resp., their length n truncation), and let $[-]$ denote Teichmüller representatives. We let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at p, let μ_{p^n} be the group scheme of p^n-th roots of unity, and let ζ_{p^n} denote a primitive p^n-th root of unity. For brevity, we set $\mathbb{Z}_p(1) := \varprojlim (\mu_{p^n}(C))$. We let $\widehat{M}$ denote the (by default, p-adic) completion of a module M and, similarly, let $\widehat{\bigoplus}$ denote the completion of a direct sum. Unless specified otherwise, we endow a p-adically complete module with the inverse limit of the discrete topologies.

We use the definition of a perfectoid ring given in [BMS16, 3.5] (the compatibility with prior definitions is discussed in [BMS16, 3.20]). Explicitly, by [BMS16, 3.9 and 3.10], a p-torsion free ring S is *perfectoid* if and only if S is p-adically complete and the divisor $(p) \subset S$ has a p-power root in the sense that there is a $\varpi \in S$ with $(\varpi^p) = (p)$ and $S/\varpi S \xrightarrow[x \mapsto x^p]{\sim} S/pS$. In particular, for such an S, any p-adically formally étale S-algebra S' that is p-adically complete is also perfectoid.

For a ring object R of a topos $\mathscr{T}$, we write $D(\mathscr{T}, R)$, or simply $D(R)$, for the derived category of R-modules. For an object M of a derived category, we denote its derived p-adic completion by

$$\widehat{M} := R\lim_n (M \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}), \qquad \text{and also set} \qquad * \mathbin{\widehat{\otimes}^{\mathbb{L}}_{\cdot}} - := R\lim_n ((* \otimes^{\mathbb{L}}_{\cdot} -) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}) \tag{1.7.1}$$

(see [SP, 0940] for the definition of $R\lim$). For a morphism f of ringed topoi, we use the commutation of the functor Rf_* with derived limits and derived completions, see [SP, 0A07 and 0944].

For a profinite group H and a continuous H-module M, we write $R\Gamma_{\mathrm{cont}}(H, M)$ for the continuous cochain complex. Whenever convenient, we also view $R\Gamma_{\mathrm{cont}}(H, -)$ as the derived global sections functor of the site of profinite H-sets (see [Sch13, 3.7 (iii)] and [Sch13e, (1)]).

For commuting endomorphisms $f_1, \dots, f_n$ of an abelian group A, we recall the *Koszul complex*:

$$K_A(f_1, \dots, f_n) := A \otimes_{\mathbb{Z}[x_1, \dots, x_n]} \bigotimes_{i=1}^{n} \left(\mathbb{Z}[x_1, \dots, x_n] \xrightarrow{x_i} \mathbb{Z}[x_1, \dots, x_d] \right), \qquad (1.7.2)$$

where A is regarded as a $\mathbb{Z}[x_1, \dots, x_n]$-module by letting x_j act as f_j, the tensor products are over $\mathbb{Z}[x_1, \dots, x_n]$, and the factor complexes are concentrated in degrees 0 and 1.

For an ideal I of a ring R and an R-module complex $(M^\bullet, d^\bullet)$ with $M^j \cong 0$ for $j < 0$, the subcomplex

$$\eta_I(M^\bullet) \subset M^\bullet \qquad \text{is defined by} \qquad (\eta_I(M^\bullet))^j := \{m \in I^j M^j \mid d^j(m) \in I^{j+1} M^{j+1}\}. \qquad (1.7.3)$$

We will mostly (but possibly not always, see Proposition 5.34) use $\eta_I(M^\bullet)$ in the same context as in [BMS16, 6.2]: when I is generated by a nonzerodivisor and the M^j have no nonzero I-torsion.

A *logarithmic divided power thickening* (or, for brevity, a *log PD thickening*) is an exact closed immersion of logarithmic (often abbreviated to log) schemes equipped with a divided power structure on the quasi-coherent sheaf of ideals that defines the underlying closed immersion of schemes.

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2. The object $A\Omega_{\mathfrak{X}}$ and the p-adic étale cohomology of $\mathfrak{X}$

As in the case when $\mathfrak{X}$ is smooth treated in [BMS16], the eventual construction of the A_{inf}-cohomology modules of $\mathfrak{X}$ rests on the object $A\Omega_{\mathfrak{X}}$ that lives in a derived category of A_{inf}-module sheaves on $\mathfrak{X}$. In this short section, we review the definition of $A\Omega_{\mathfrak{X}}$ in §2.2 and then, in the case when $\mathfrak{X}$ is proper, review the connection between $A\Omega_{\mathfrak{X}}$ and the integral p-adic étale cohomology of $\mathfrak{X}_C^{\mathrm{ad}}$ in Theorem 2.3. We begin by fixing the basic notation that concerns the ring A_{inf} of integral p-adic Hodge theory. The setup of §§2.1–2.2 will be used freely in the rest of the paper.

2.1. The ring A_{inf}. We denote the tilt of $\mathcal{O}_C$ by

$$\mathcal{O}_C^\flat := \varprojlim_{y \mapsto y^p} (\mathcal{O}_C/p), \quad \text{so that, by reduction mod } p, \quad \varprojlim_{y \mapsto y^p} \mathcal{O}_C \xrightarrow{\sim} \varprojlim_{y \mapsto y^p} (\mathcal{O}_C/p) = \mathcal{O}_C^\flat$$

as multiplicative monoids (see [Sch12, 3.4 (i)]). We regard p^{1/p^∞} fixed in §1.5 as an element of $\mathcal{O}_C^\flat$. Due to the fixed embedding $p^{\mathbb{Q}_{\geq 0}} \subset \mathcal{O}_C$, this element comes equipped with well-defined powers $(p^{1/p^\infty})^q \in \mathcal{O}_C^\flat$ for $q \in \mathbb{Q}_{\geq 0}$. For each $x \in \mathcal{O}_C^\flat$, we let $(\dots, x^{(1)}, x^{(0)})$ denote its preimage in $\varprojlim_{y \mapsto y^p} \mathcal{O}_C$. The map $x \mapsto \mathrm{val}_{\mathcal{O}_C}(x^{(0)})$ makes $\mathcal{O}_C^\flat$ a complete valuation ring of height 1 whose fraction field $C^\flat := \mathrm{Frac}(\mathcal{O}_C^\flat)$ is algebraically closed (see [Sch12, 3.4 (iii), 3.7 (ii)]). We let $\mathfrak{m}^\flat$ denote the maximal ideal of $\mathcal{O}_C^\flat$.

The basic period ring A_{inf} of Fontaine is defined by

$$A_{\mathrm{inf}} := W(\mathcal{O}_C^\flat) \qquad \text{and comes equipped with the Witt vector Frobenius} \qquad \varphi\colon A_{\mathrm{inf}} \xrightarrow{\sim} A_{\mathrm{inf}}.$$

We equip the local domain A_{inf} with the product of the valuation topologies via the Witt coordinate bijection $W(\mathcal{O}_C^\flat) \cong \prod_{n=1}^{\infty} \mathcal{O}_C^\flat$. Then A_{inf} is complete and its topology agrees with the $(p,[x])$-adic topology for any nonzero nonunit $x \in \mathcal{O}_C^\flat$. We fix (once and for all) a compatible system $\epsilon = (\dots, \zeta_{p^2}, \zeta_p, 1)$ of p-power roots of unity in $\mathcal{O}_C$, so that $\epsilon \in \mathcal{O}_C^\flat$, and set

$$\mu := [\epsilon] - 1 \in A_{\mathrm{inf}}. \tag{2.1.1}$$

Since $(p, \mu) = (p, [\epsilon - 1])$, the topology of A_{inf} is (p,μ)-adic. By forming the limit of the sequences

$$0 \to W_n(\mathcal{O}_C^\flat) \xrightarrow{\mu} W_n(\mathcal{O}_C^\flat) \to W_n(\mathcal{O}_C^\flat)/\mu \to 0, \tag{2.1.2}$$

we see that A_{inf}/μ is p-adically complete and that the ideal $(\mu) \subset A_{\mathrm{inf}}$ does not depend on the choice of ϵ (use the fact that the valuation of $\zeta_p - 1$ does not depend on ζ_p).

The assignment $[x] \mapsto x^{(0)}$ extends uniquely to a ring homomorphism

$$\theta\colon A_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_C, \qquad \text{the so-called } \textit{de Rham specialization} \text{ map of } A_{\mathrm{inf}}, \tag{2.1.3}$$

which is surjective, as indicated, and intertwines the Frobenius φ of A_{inf} with the absolute Frobenius of $\mathcal{O}_C/p$. Its kernel $\mathrm{Ker}(\theta) \subset A_{\mathrm{inf}}$ is principal and generated by the element

$$\xi := \textstyle\sum_{i=0}^{p-1} [\epsilon^{i/p}] \tag{2.1.4}$$

(see [BMS16, 3.16]). Analogues of the sequences (2.1.2) show that each A_{inf}/ξ^n is p-adically complete. In fact, the map θ identifies A_{inf}/ξ^n with the initial p-adically complete infinitesimal thickening of $\mathcal{O}_C$ of order $n-1$, see [SZ17, 3.13]. The composition

$$\theta \circ \varphi^{-1}\colon A_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_C \qquad \text{is the so-called } \textit{Hodge–Tate specialization} \text{ map of } A_{\mathrm{inf}},$$

and its kernel is generated by the element $\varphi(\xi) = \sum_{i=0}^{p-1} [\epsilon^i]$.

Due to the nature of our C (see §1.5), the ring $\mathcal{O}_C/p$ is a k-algebra, so A_{inf} is a $W(k)$-algebra.

2.2. The object $A\Omega_{\mathfrak{X}}$. The operations that define $\mathcal{O}_C^\flat$ and A_{inf} make sense on the proétale site $(\mathfrak{X}_C^{\mathrm{ad}})_{\text{proét}}$: namely, as in [Sch13, 4.1, 5.10, and 6.1], we have the integral completed structure sheaf

$$\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}} := \varprojlim_n (\mathcal{O}^+_{\mathfrak{X}_C^{\mathrm{ad}},\,\text{proét}}/p^n), \qquad \text{its tilt} \qquad \widehat{\mathcal{O}}^{+,\flat}_{\mathfrak{X}_C^{\mathrm{ad}}} := \varprojlim_{y \mapsto y^p} (\mathcal{O}^+_{\mathfrak{X}_C^{\mathrm{ad}},\,\text{proét}}/p), \tag{2.2.1}$$

and the basic period sheaf

$$\mathbb{A}_{\mathrm{inf},\,\mathfrak{X}_C^{\mathrm{ad}}} := W(\widehat{\mathcal{O}}^{+,\flat}_{\mathfrak{X}_C^{\mathrm{ad}}}). \tag{2.2.2}$$

For brevity, we often denote these sheaves simply by $\widehat{\mathcal{O}}^+$, $\widehat{\mathcal{O}}^{+,\flat}$, and $\mathbb{A}_{\mathrm{inf}}$. Affinoid perfectoids form a basis for $(\mathfrak{X}_C^{\mathrm{ad}})_{\text{proét}}$ (see [Sch13, 4.7]) and the construction of the map θ of (2.1.3) makes sense for any perfectoid $\mathcal{O}_C$-algebra (see [BMS16, §3]). In particular, $\mathbb{A}_{\mathrm{inf},\,\mathfrak{X}_C^{\mathrm{ad}}}$ comes equipped with the map

$$\theta_{\mathfrak{X}_C^{\mathrm{ad}}}\colon \mathbb{A}_{\mathrm{inf},\,\mathfrak{X}_C^{\mathrm{ad}}} \to \widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}}, \tag{2.2.3}$$

which, by construction, is compatible with the map $\theta\colon A_{\mathrm{inf}} \to \mathcal{O}_C$, intertwines the Witt vector Frobenius φ of $\mathbb{A}_{\mathrm{inf},\,\mathfrak{X}_C^{\mathrm{ad}}}$ with the absolute Frobenius of $\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}}/p$, and, by [Sch13, 6.3 and 6.5], is surjective with $\mathrm{Ker}(\theta_{\mathfrak{X}_C^{\mathrm{ad}}}) = \xi \cdot \mathbb{A}_{\mathrm{inf},\,\mathfrak{X}_C^{\mathrm{ad}}}$.

The key object that we are going to study in this paper is

$$A\Omega_{\mathfrak{X}} := L\eta_{(\mu)}(R\nu_*(\mathbb{A}_{\mathrm{inf},\,\mathfrak{X}_C^{\mathrm{ad}}})) \in D^{\geq 0}(\mathfrak{X}_{\text{ét}}, A_{\mathrm{inf}}), \tag{2.2.4}$$

where the décalage functor $L\eta$ of [BMS16, §6] is formed with respect to the ideal (μ) of the constant sheaf A_{inf} of $\mathfrak{X}_{\text{ét}}$ (the definition of $L\eta_{(\mu)}$ builds on the formula (1.7.3) for $\eta_{(\mu)}$). The formula (2.2.4) may also be executed with the Zariski site $\mathfrak{X}_{\mathrm{Zar}}$ as the target of ν, and it then defines the object

$$A\Omega_{\mathfrak{X}_{\mathrm{Zar}}} \in D^{\geq 0}(\mathfrak{X}_{\mathrm{Zar}}, A_{\mathrm{inf}}), \tag{2.2.5}$$

which is the $A\Omega_{\mathfrak{X}}$ that was used in [BMS16]. We will only use $A\Omega_{\mathfrak{X}_{\mathrm{Zar}}}$ in Corollary 4.20 (and in some results that lead to it) for comparison to $A\Omega_{\mathfrak{X}}$.

Since $\varphi(\mu) = \varphi(\xi)\mu$, the Frobenius automorphism of $\mathbb{A}_{\mathrm{inf},\, \mathfrak{X}_C^{\mathrm{ad}}}$ gives the "Frobenius" morphism

$$A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\, \varphi} A_{\mathrm{inf}} \cong L\eta_{(\varphi(\xi))}(A\Omega_{\mathfrak{X}}) \xrightarrow{\text{[BMS16, 6.11, 6.10, and 3.17 (ii)]}} A\Omega_{\mathfrak{X}} \quad \text{in} \quad D^{\geq 0}(\mathfrak{X}_{\text{ét}}, A_{\mathrm{inf}}), \tag{2.2.6}$$

which, by [BMS16, 6.14], induces an isomorphism

$$(A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\, \varphi} A_{\mathrm{inf}})[\tfrac{1}{\varphi(\xi)}] \xrightarrow{\sim} (A\Omega_{\mathfrak{X}})[\tfrac{1}{\varphi(\xi)}]. \tag{2.2.7}$$

In addition, by *loc. cit.*, we also have

$$A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}[\tfrac{1}{\mu}] \cong (R\nu_*(\mathbb{A}_{\mathrm{inf},\, \mathfrak{X}_C^{\mathrm{ad}}})) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}[\tfrac{1}{\mu}], \tag{2.2.8}$$

so a result of Scholze [BMS16, 5.6] supplies the following relation to integral p-adic étale cohomology:

Theorem 2.3. *If $\mathfrak{X}$ is $\mathcal{O}_C$-proper, then there is an identification*

$$R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}[\tfrac{1}{\mu}] \cong R\Gamma_{\text{ét}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} A_{\mathrm{inf}}[\tfrac{1}{\mu}]. \tag{2.3.1}$$

In broad strokes, the proof of Theorem 2.3 given in *loc. cit.* goes as follows: one considers the map

$$R\Gamma_{\text{ét}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} A_{\mathrm{inf}} \cong R\Gamma_{\text{proét}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} A_{\mathrm{inf}} \to R\Gamma_{\text{proét}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{A}_{\mathrm{inf},\, \mathfrak{X}_C^{\mathrm{ad}}}) \tag{2.3.2}$$

induced by the inclusion $A_{\mathrm{inf}} \hookrightarrow \mathbb{A}_{\mathrm{inf},\, \mathfrak{X}_C^{\mathrm{ad}}}$ and deduces from the almost purity theorem with, for instance, Lemma 3.17 below that the ideal

$$W(\mathfrak{m}^\flat) := \mathrm{Ker}(W(\mathcal{O}_C^\flat) \twoheadrightarrow W(k)) \quad \text{of} \quad A_{\mathrm{inf}} \tag{2.3.3}$$

kills the cohomology of its cone. Since μ lies in $W(\mathfrak{m}^\flat)$ and we have the identification (2.2.8), it follows that the map (2.3.2) induces the identification (2.3.1).

Remark 2.4. In practice, $\mathfrak{X}$ often arises as the formal p-adic completion of a proper, finitely presented $\mathcal{O}_C$-scheme $\mathcal{X}$. In this situation, $\mathfrak{X}_C^{\mathrm{ad}}$ agrees with the adic space associated to $\mathcal{X}_C$ (see [Con99, 5.3.1 4.], [Hub94, 4.6 (i)], and [Hub96, 1.9.2 ii)]) and, by [Hub96, 3.7.2], we have

$$R\Gamma_{\text{ét}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{Z}_p) \cong R\Gamma_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}_p).$$

3. The local analysis of $A\Omega_{\mathfrak{X}}$

Even though the definition of the object $A\Omega_{\mathfrak{X}}$ given in (2.2.4) is global, the key computations that will eventually relate it to the logarithmic de Rham and crystalline cohomologies are local and are presented in this section. Under the assumption that $\mathfrak{X}$ has a coordinate morphism as in (1.5.1) (or (3.1.1) below), their basic goal is to express the cohomology of the proétale sheaf $\mathbb{A}_{\mathrm{inf},\, \mathfrak{X}_C^{\mathrm{ad}}}$, at least after applying $L\eta_{(\mu)}$, in terms of continuous group cohomology formed using an explicit perfectoid proétale cover $\mathfrak{X}_{C,\infty}^{\mathrm{ad}}$ of $\mathfrak{X}_C^{\mathrm{ad}}$ (see Theorem 3.20). The basic relation of this sort is supplied by the almost purity theorem, so the key point is to explicate the appearing group cohomology modules well enough in order to eliminate the "almost" ambiguities inherent in this theorem with the help of Lemma 3.18 below that comes from [Bha16]. We first carry out this program for the simpler sheaf $\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}}$, and then build on this case to address $\mathbb{A}_{\mathrm{inf},\, \mathfrak{X}_C^{\mathrm{ad}}}$.

In comparison to the local analysis carried out in the smooth case in [BMS16], one complication is that the perfectoid cover of $\mathfrak{X}$ that gives rise to $\mathfrak{X}^{\mathrm{ad}}_{C,\infty}$ is not flat over the singular points of $\mathfrak{X}_k$. This makes it difficult to transfer various arguments with "q-de Rham complexes" across the coordinate morphism (3.1.1). In fact, we avoid q-de Rham complexes altogether and instead phrase the intermediate steps of the local analysis purely in terms of continuous group cohomology modules.

3.1. The local setup. We assume throughout §3 that $\mathfrak{X} = \operatorname{Spf} R$ and for some $d \geq 0$, some $0 \leq r \leq d$, and some $q \in \mathbb{Q}_{>0}$, there is an étale $\operatorname{Spf}(\mathcal{O}_C)$-morphism as in (1.5.1):

$$\mathfrak{X} = \operatorname{Spf}(R) \to \operatorname{Spf}(R^{\square}) =: \mathfrak{X}^{\square} \quad \text{with} \quad R^{\square} := \mathcal{O}_C\{t_0, \dots, t_r, t_{r+1}^{\pm 1}, \dots, t_d^{\pm 1}\}/(t_0 \cdots t_r - p^q). \tag{3.1.1}$$

Due to our assumptions from §1.5, a general $\mathfrak{X}$ is of this form on a basis for its étale topology.

3.2. The perfectoid cover $\mathfrak{X}^{\mathrm{ad}}_{C,\infty}$. For each $m \geq 0$, we consider the $R^{\square}$-algebra

$$R^{\square}_m := \mathcal{O}_C\{t_0^{1/p^m}, \dots, t_r^{1/p^m}, t_{r+1}^{\pm 1/p^m}, \dots, t_d^{\pm 1/p^m}\}/(t_0^{1/p^m} \cdots t_r^{1/p^m} - p^{q/p^m}), \quad \text{and} \quad R^{\square}_\infty := (\varinjlim R^{\square}_m)^\wedge.$$

Explicitly, we have the p-adically completed direct sum decomposition

$$R^{\square}_\infty \cong \widehat{\bigoplus}_{\substack{(a_0,\dots,a_d) \in (\mathbb{Z}[\frac{1}{p}]_{\geq 0})^{\oplus(r+1)} \oplus (\mathbb{Z}[\frac{1}{p}])^{\oplus(d-r)},\\ a_j = 0 \text{ for some } 0 \leq j \leq r}} \mathcal{O}_C \cdot t_0^{a_0} \cdots t_d^{a_d}, \tag{3.2.1}$$

which shows that $R^{\square}_\infty$ is perfectoid (see §1.7) and that, for each $m \geq 0$, the map $R^{\square}_m \to R^{\square}_\infty$ is an inclusion of an $R^{\square}_m$-module direct summand comprised of those summands $\mathcal{O}_C \cdot t_0^{a_0} \cdots t_d^{a_d}$ of (3.2.1) for which $p^m a_j \in \mathbb{Z}$ for all j.

The corresponding R-algebras are

$$R_m := R \otimes_{R^{\square}} R^{\square}_m \qquad \text{and} \qquad R_\infty := (\varinjlim R_m)^\wedge \cong (R \otimes_{R^{\square}} R^{\square}_\infty)^\wedge.$$

Each R_m (resp., R_∞) is a p-torsion free p-adically formally étale $R^{\square}_m$-algebra (resp., $R^{\square}_\infty$-algebra). In particular, R_∞ is perfectoid (see §1.7). By [GR03, 7.1.6 (ii)], each R_m is p-adically complete.

The summands in (3.2.1) with $a_j \notin \mathbb{Z}$ for some $0 \leq j \leq d$ comprise an $R^{\square}$-submodule $M^{\square}_\infty$ of $R^{\square}_\infty$, and we set $M_\infty := R \widehat{\otimes}_{R^{\square}} M^{\square}_\infty$. Thus, we have the $R^{\square}$-module (resp., R-module) decomposition

$$R^{\square}_\infty \cong R^{\square} \oplus M^{\square}_\infty \qquad (\text{resp.,} \qquad R_\infty \cong R \oplus M_\infty). \tag{3.2.2}$$

The profinite group

$$\Delta := \left\{ (\epsilon_0, \dots, \epsilon_d) \in \left(\varprojlim_{m \geq 0} \mu_{p^m}(\mathcal{O}_C) \right)^{\oplus(d+1)} \,\middle|\, \epsilon_0 \cdots \epsilon_r = 1 \right\} \simeq \mathbb{Z}_p^{\oplus d}$$

acts $R^{\square}$-linearly on $R^{\square}_m$ by scaling each t_j^{1/p^m} by the μ_{p^m}-component of ϵ_j. The induced actions of Δ on $R^{\square}_\infty$ and R_∞ are continuous, compatible, and preserve the decompositions (3.2.1) and (3.2.2). In terms of the element ϵ fixed in §2.1, Δ is topologically freely generated by the following d elements:

$$\delta_i := (\epsilon^{-1}, 1, \dots, 1, \epsilon, 1, \dots, 1) \quad \text{for } i = 1, \dots, r, \quad \text{where the 0-th and } i\text{-th entries are nonidentity;}$$
$$\delta_i := (1, \dots, 1, \epsilon, 1, \dots, 1) \quad \text{for } i = r+1, \dots, d, \quad \text{where the } i\text{-th entry is nonidentity.}$$

After inverting p, for each $m \geq 0$, we have

$$R^{\square}_m[\tfrac{1}{p}] \cong \bigoplus_{a_1, \dots, a_d \in \{0, \frac{1}{p^m}, \dots, \frac{p^m-1}{p^m}\}} R^{\square}[\tfrac{1}{p}] \cdot t_1^{a_1} \cdots t_d^{a_d},$$

so $R^{\square}_m[\frac{1}{p}]$ is the $R^{\square}[\frac{1}{p}]$-algebra obtained by adjoining the $(p^m)^{\mathrm{th}}$ roots of $t_1, \dots, t_d \in (R^{\square}[\frac{1}{p}])^{\times}$, and hence is finite étale over $R^{\square}[\frac{1}{p}]$. Therefore, $\varinjlim_m \left(R^{\square}_m[\frac{1}{p}]\right)$ is a pro-(finite étale) Δ-cover of $R^{\square}[\frac{1}{p}]$. The explicit description (3.2.1) implies that $R^{\square}_m = (R^{\square}_m[\frac{1}{p}])^{\circ}$, so the pro-object

$$(\mathfrak{X}^{\square})^{\mathrm{ad}}_{C,\infty} := \text{“}\varprojlim \operatorname{Spa}(R^{\square}_m[\tfrac{1}{p}], R^{\square}_m)\text{”}$$

is an affinoid perfectoid pro-(finite étale) Δ-cover of the adic generic fiber $(\mathfrak{X}^{\square})^{\mathrm{ad}}_C$ of $\operatorname{Spf}(R^{\square})$. Consequently, the $\mathfrak{X}^{\mathrm{ad}}_C$-base change of $(\mathfrak{X}^{\square})^{\mathrm{ad}}_{C,\infty}$, namely, the tower

$$\mathfrak{X}^{\mathrm{ad}}_{C,\infty} := \text{“}\varprojlim \operatorname{Spa}(R_m[\tfrac{1}{p}], R_m),\text{”}$$

is an affinoid perfectoid pro-(finite étale) Δ-cover of $\mathfrak{X}^{\mathrm{ad}}_C$.

By [Sch13, 4.10 (iii)], the value on $\mathfrak{X}^{\mathrm{ad}}_{C,\infty}$ of the sheaf $\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{ad}}_C}$ reviewed in (2.2.1) is the ring R_∞.

3.3. The cohomology of $\widehat{\mathcal{O}}^+$ and continuous group cohomology. By [Sch13, 3.5, 3.7 (iii) and its proof, 6.6], the Čech complex of the sheaf $\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{ad}}_C}$ with respect to the pro-(finite étale) affinoid perfectoid cover

$$\mathfrak{X}^{\mathrm{ad}}_{C,\infty} \twoheadrightarrow \mathfrak{X}^{\mathrm{ad}}_C$$

is identified with the continuous cochain complex $R\Gamma_{\mathrm{cont}}(\Delta, R_\infty)$. In particular, by using [SP, 01GY], we obtain the edge map to the proétale cohomology of $\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{ad}}_C}$:

$$e\colon R\Gamma_{\mathrm{cont}}(\Delta, R_\infty) \to R\Gamma_{\mathrm{pro\acute{e}t}}(\mathfrak{X}^{\mathrm{ad}}_C, \widehat{\mathcal{O}}^+), \qquad (3.3.1)$$

which on the level of cohomology is described by the Cartan–Leray spectral sequence (see *loc. cit.* or [SGA 4_{II}, V.3.3]). By the almost purity theorem [Sch13, 4.10 (v)], the maximal ideal $\mathfrak{m} \subset \mathcal{O}_C$ kills the cohomology groups of $\operatorname{Cone}(e)$.

We will show in Theorem 3.9 that $L\eta_{(\zeta_p-1)}(e)$ is an isomorphism, so that $L\eta_{(\zeta_p-1)}(R\Gamma_{\mathrm{pro\acute{e}t}}(\mathfrak{X}^{\mathrm{ad}}_C, \widehat{\mathcal{O}}^+))$ is computed in terms of continuous group cohomology. For this, we will use the following lemma.

Lemma 3.4 ([BMS16, 8.11 (i)]). *An $\mathcal{O}_C$-module map $f\colon M \to M'$ with $M[\mathfrak{m}] = \left(\frac{M}{(\zeta_p-1)M}\right)[\mathfrak{m}] = 0$ and both* $\operatorname{Ker} f$ *and* $\operatorname{Coker} f$ *killed by $\mathfrak{m}$ induces an isomorphism $\frac{M}{M[\zeta_p-1]} \xrightarrow{\sim} \frac{M'}{M'[\zeta_p-1]}$.* ☐

In order to apply Lemma 3.4, we will check in Proposition 3.8 that the cohomology modules $H^i_{\mathrm{cont}}(\Delta, R_\infty)$ have no nonzero $\mathfrak{m}$-torsion. This will use the following general lemmas.

Lemma 3.5. *For an inclusion $\mathfrak{o} \subset \mathfrak{O}$ of a discrete valuation ring into a nondiscrete valuation ring of rank 1, if N is an $\mathfrak{o}$-module and $\mathfrak{M} \subset \mathfrak{O}$ denotes the maximal ideal, then $(N \otimes_{\mathfrak{o}} \mathfrak{O})[\mathfrak{M}] = 0$.*

Proof. The $\mathfrak{o}$-flatness of $\mathfrak{O}$ reduces us to the case when N is finitely generated, so it suffices to observe that $(\mathfrak{O}/(a))[\mathfrak{M}] = 0$ whenever $a \in \mathfrak{O}$. ☐

Lemma 3.6. *Fix an $i \in \mathbb{Z}_{\geq 0}$, let H be a profinite group, let $\{M_j\}_{j\in J}$ be p-adically complete, p-torsion free, continuous H-modules, and suppose that either*

(i) *the group $H^i_{\mathrm{cont}}(H, M_j)$ is p-torsion free for every j, or*

(ii) *some p^n kills $H^i_{\mathrm{cont}}(H, M_j)$ for every j.*

Then the following map is injective:

$$H^i_{\mathrm{cont}}(H, \widehat{\bigoplus}_{j\in J} M_j) \hookrightarrow \prod_{j\in J} H^i_{\mathrm{cont}}(H, M_j), \qquad \textit{where the completion is } p\textit{-adic.}$$

In particular, in the case (i) (*resp.,* (ii)), $H^i_{\mathrm{cont}}(H, \widehat{\bigoplus}_{j\in J} M_j)$ *is* p*-torsion free* (*resp., killed by* p^n).

Proof. Let c be a continuous $\left(\widehat{\bigoplus}_{j\in J} M_j\right)$-valued i-cocycle that represents an element of the kernel. For each j, let c_j be the "j-th coordinate" of c. We discard the j with $c_j = 0$ and, for each remaining j, we choose the maximal $n_j \in \mathbb{Z}_{\ge 0}$ such that c_j is $(p^{n_j} M_j)$-valued, so that the function $j \mapsto n_j$ is finite-to-one. Since each M_j is p-torsion free, each $p^{-n_j} c_j$ is an M_j-valued continuous i-cocycle.

In the case (i), the class of $p^{-n_j} c_j$ in $H^i_{\mathrm{cont}}(H, M_j)$ vanishes, so each c_j is the coboundary of a $(p^{n_j} M_j)$-valued continuous $(i-1)$-cochain b_j. In the case (ii), p^n kills $H^i_{\mathrm{cont}}(H, M_j)$, so c_j is the coboundary of a $(p^{n_j - n} M_j)$-valued continuous $(i-1)$-cochain b_j whenever $n_j \ge n$.

In both cases, the b_j exhibit c as a continuous coboundary. □

Lemma 3.7 ([BMS16, 7.3 (ii)])**.** *Let H be a profinite group isomorphic to $\mathbb{Z}_p^{\oplus d}$ for some $d > 0$, and let $M \cong \varprojlim_{n\ge 1} M_n$ be a continuous H-module with each M_n a discrete, p^n-torsion, continuous H-module. For any $\gamma_1, \dots, \gamma_d \in H$ that topologically freely generate H, there is a natural identification*

$$R\Gamma_{\mathrm{cont}}(H, M) \cong K_M(\gamma_1 - 1, \dots, \gamma_d - 1), \quad \textit{so also} \quad H^j_{\mathrm{cont}}(H, M) \cong H^j(K_M(\gamma_1 - 1, \dots, \gamma_d - 1)),$$

in the derived category (*see* §1.7 *for the notation*). □

Proposition 3.8. *The element $\zeta_p - 1$ kills the $\mathcal{O}_C$-modules $H^i_{\mathrm{cont}}(\Delta, M_\infty)$. Moreover, for each $b \in \mathcal{O}_C$, the $\mathcal{O}_C$-modules R_∞/b and $H^i_{\mathrm{cont}}(\Delta, R_\infty/b)$ have no nonzero $\mathfrak{m}$-torsion.*

Proof. Let $S := \mathcal{O}_C \cdot t_0^{a_0} \cdots t_d^{a_d}$ be a summand of (3.2.1). By Lemma 3.7, the $\mathcal{O}_C$-module $H^i_{\mathrm{cont}}(\Delta, S)$ is the i-th cohomology of the $\mathcal{O}_C$-tensor product of d complexes of the form $\mathcal{O}_C \xrightarrow{\zeta - 1} \mathcal{O}_C$ for suitable p-power roots of unity ζ. Moreover, since the d complexes may be defined over some discrete valuation subring of $\mathcal{O}_C$, Lemma 3.5 ensures that

$$H^i_{\mathrm{cont}}(\Delta, S) \qquad \text{has no nonzero } \mathfrak{m}\text{-torsion.} \tag{3.8.1}$$

If S contributes to M_∞, that is, if $a_j \notin \mathbb{Z}$ for some j, then some ζ is not 1, and the corresponding factor complex is quasi-isomorphic to $\mathcal{O}_C/(\zeta - 1)$. Thus, in this case,

$$\zeta - 1, \quad \text{and hence also} \quad \zeta_p - 1, \quad \text{kills} \quad H^i_{\mathrm{cont}}(\Delta, S). \tag{3.8.2}$$

For $m > 0$, let $M_m^\square$ denote the p-adically completed direct sum of those summands $\mathcal{O}_C \cdot t_0^{a_0} \cdots t_d^{a_d}$ of (3.2.1) for which m is the smallest *nonnegative* integer with $p^m \cdot (a_0, \dots, a_d) \in \mathbb{Z}^{\oplus(d+1)}$. Lemma 3.6 and (3.8.1)–(3.8.2) imply that the $\mathcal{O}_C$-module

$$H^i_{\mathrm{cont}}(\Delta, M_m^\square) \qquad \text{has no nonzero } \mathfrak{m}\text{-torsion and is killed by } \zeta_p - 1. \tag{3.8.3}$$

Since R is $R^\square$-flat and $R \otimes_{R^\square} M_m^\square$ is p-adically complete (see §1.5 and §3.2), Lemma 3.7 gives

$$H^i_{\mathrm{cont}}(\Delta, R \otimes_{R^\square} M_m^\square) \cong R \otimes_{R^\square} H^i_{\mathrm{cont}}(\Delta, M_m^\square). \tag{3.8.4}$$

Since $M_\infty \cong \widehat{\bigoplus}_m (R \otimes_{R^\square} M_m^\square)$, (3.8.3)–(3.8.4) and Lemma 3.6 imply that $\zeta_p - 1$ kills $H^i_{\mathrm{cont}}(\Delta, M_\infty)$.

Since R_∞/b is p-adically complete and each of the summands of the decomposition

$$R_\infty/(b, p^n) \cong R/(b, p^n) \oplus \bigoplus_{m>0} (R \otimes_{R^\square} M_m^\square)/(b, p^n) \qquad \text{for } n > 0$$

may be defined over a suitably large discrete valuation subring of $\mathcal{O}_C$, Lemma 3.5 ensures that R_∞/b has no nonzero $\mathfrak{m}$-torsion. In addition, the Δ-action on each summand may be defined over a possibly larger such subring, so, by Lemmas 3.5 and 3.7, in the case $b \neq 0$ each

$$H^i_{\mathrm{cont}}(\Delta, (R \otimes_{R^\square} M_m^\square)/b), \quad \text{so also} \quad H^i_{\mathrm{cont}}(\Delta, M_\infty/b), \quad \text{has no nonzero } \mathfrak{m}\text{-torsion.}$$

This conclusion extends to the case $b = 0$ because the $(\zeta_p - 1)$-annihilation of $H^i_{\mathrm{cont}}(\Delta, M_\infty)$ supplies the injection $H^i_{\mathrm{cont}}(\Delta, M_\infty) \hookrightarrow H^i_{\mathrm{cont}}(\Delta, M_\infty/(\zeta_p - 1))$. It remains to observe that the $\mathcal{O}_C$-module $H^i_{\mathrm{cont}}(\Delta, R/b)$ also has no nonzero $\mathfrak{m}$-torsion: Δ acts trivially on R/b, so Lemma 3.7 ensures that $H^i_{\mathrm{cont}}(\Delta, R/b)$ is a direct sum of copies of R/b. $\square$

Theorem 3.9. *The edge map e defined in (3.3.1) induces the isomorphism*

$$L\eta_{(\zeta_p-1)}(e)\colon L\eta_{(\zeta_p-1)}(R\Gamma_{\mathrm{cont}}(\Delta, R_\infty)) \xrightarrow{\sim} L\eta_{(\zeta_p-1)}(R\Gamma_{\mathrm{pro\acute{e}t}}(\mathfrak{X}^{\mathrm{ad}}_C, \widehat{\mathcal{O}}^+)). \tag{3.9.1}$$

Proof. Proposition 3.8 ensures that the $\mathcal{O}_C$-modules $H^i_{\mathrm{cont}}(\Delta, R_\infty)$ have no nonzero $\mathfrak{m}$-torsion and that $\frac{H^i_{\mathrm{cont}}(\Delta, R_\infty)}{H^i_{\mathrm{cont}}(\Delta, R_\infty)[\zeta_p - 1]} \cong \frac{H^i_{\mathrm{cont}}(\Delta, R)}{H^i_{\mathrm{cont}}(\Delta, R)[\zeta_p - 1]}$. Since Δ acts trivially on R, this last quotient is a finite direct sum of copies of R (see Lemma 3.7), so, by Proposition 3.8, it has no nonzero $\mathfrak{m}$-torsion. Consequently, since $\mathfrak{m}$ kills the kernel and the cokernel of each map

$$H^i(e)\colon H^i_{\mathrm{cont}}(\Delta, R_\infty) \to H^i(\mathfrak{X}^{\mathrm{ad}}_C, \widehat{\mathcal{O}}^+)$$

(see §3.3), Lemma 3.4 applies to these maps and gives the desired conclusion. $\square$

Remark 3.10. Theorem 3.9 extends as follows: for a pro-(finite étale) affinoid perfectoid Δ'-cover

$$\mathrm{Spa}(R'_\infty[\tfrac{1}{p}], R'_\infty) \to \mathrm{Spa}(R[\tfrac{1}{p}], R) \cong \mathfrak{X}^{\mathrm{ad}}_C \qquad \text{that refines} \qquad \mathfrak{X}^{\mathrm{ad}}_{C,\infty} \to \mathfrak{X}^{\mathrm{ad}}_C \qquad \text{of §3.2}, \tag{3.10.1}$$

the edge map e' defined analogously to (3.3.1) induces the isomorphism

$$L\eta_{(\zeta_p-1)}(e')\colon L\eta_{(\zeta_p-1)}(R\Gamma_{\mathrm{cont}}(\Delta', R'_\infty)) \xrightarrow{\sim} L\eta_{(\zeta_p-1)}(R\Gamma_{\mathrm{pro\acute{e}t}}(\mathfrak{X}^{\mathrm{ad}}_C, \widehat{\mathcal{O}}^+)).$$

Indeed, by the almost purity theorem [Sch13, 4.10 (v)], the ideal $\mathfrak{m}$ kills the cohomology of $\mathrm{Cone}(e')$ (in addition to that of $\mathrm{Cone}(e)$), so the octahedral axiom (see [BBD82, 1.1.7.1]) ensures that it also kills the cohomology of the cone of the map $R\Gamma_{\mathrm{cont}}(\Delta, R_\infty) \to R\Gamma_{\mathrm{cont}}(\Delta', R'_\infty)$; Lemma 3.4 then applies to this map and combines with Theorem 3.9 to give the claim.

The main goal of this section is an analogue of Theorem 3.9 for the sheaf $\mathbb{A}_{\mathrm{inf},\, \mathfrak{X}^{\mathrm{ad}}_C}$ (see Theorem 3.20). To prepare for it, in §3.11 and §3.14 we describe the values of the sheaves $\widehat{\mathcal{O}}^{+,\flat}_{\mathfrak{X}^{\mathrm{ad}}_C}$ and $\mathbb{A}_{\mathrm{inf},\, \mathfrak{X}^{\mathrm{ad}}_C}$ on $\mathfrak{X}^{\mathrm{ad}}_{C,\infty}$.

3.11. The tilt $R^\flat_\infty$. Thanks to the explicit description (3.2.1) of the perfectoid ring $R^\square_\infty$, its tilt $(R^\square_\infty)^\flat := \varprojlim_{y \mapsto y^p}(R^\square_\infty/p)$ is described explicitly by the identification

$$(R^\square_\infty)^\flat \cong \Big(\varinjlim_m \Big(\mathcal{O}^\flat_C[x_0^{1/p^m}, \ldots, x_r^{1/p^m}, x_{r+1}^{\pm 1/p^m}, \ldots, x_d^{\pm 1/p^m}]/(x_0^{1/p^m} \cdots x_r^{1/p^m} - (p^{1/p^\infty})^{q/p^m})\Big)\Big)^{\widehat{}}$$
$$\cong \widehat{\bigoplus}_{\substack{(a_0, \ldots, a_d) \in (\mathbb{Z}[\frac{1}{p}]_{\geq 0})^{\oplus(r+1)} \oplus (\mathbb{Z}[\frac{1}{p}])^{\oplus(d-r)}, \\ a_j = 0 \text{ for some } 0 \leq j \leq r}} \mathcal{O}^\flat_C \cdot x_0^{a_0} \cdots x_d^{a_d},$$

where x_i^{1/p^m} corresponds to the p-power compatible sequence $(\ldots, t_i^{1/p^{m+1}}, t_i^{1/p^m})$ of elements of $R^\square_\infty$, the completions are p^{1/p^∞}-adic, and the decomposition is as $\mathcal{O}^\flat_C$-modules. Thus,

$$\text{the tilt} \quad R^\flat_\infty := \varprojlim_{y \mapsto y^p}(R_\infty/p) \quad \text{of the perfectoid ring} \quad R_\infty$$

is identified with the p^{1/p^∞}-adic completion of any lift of the étale $R^\square_\infty/p$-algebra R_∞/p to an étale $(R^\square_\infty)^\flat$-algebra (such a lift exists, see [SP, 04D1]). By [Sch13, 5.11 (i)], the value on $\mathfrak{X}^{\mathrm{ad}}_{C,\infty}$ of the sheaf $\widehat{\mathcal{O}}^{+,\flat}_{\mathfrak{X}^{\mathrm{ad}}_C}$ reviewed in (2.2.1) is the ring $R^\flat_\infty$.

By functoriality, the group Δ acts continuously and $\mathcal{O}^\flat_C$-linearly on $(R^\square_\infty)^\flat$ and $R^\flat_\infty$. Explicitly, Δ respects the completed direct sum decomposition and an $(\epsilon_0, \dots, \epsilon_d) \in \Delta$ scales $x_j^{a_j}$ by $\epsilon_j^{a_j} \in \mathcal{O}^\flat_C$.

Our analysis in §3.14 of the value on $\mathfrak{X}^{\mathrm{ad}}_{C,\infty}$ of the sheaf $\mathbb{A}_{\mathrm{inf},\mathfrak{X}^{\mathrm{ad}}_C}$ will hinge on the following lemmas.

Lemma 3.12. *Both $R^\flat_\infty/b$ and $H^i_{\mathrm{cont}}(\Delta, R^\flat_\infty/b)$ for each $b \in \mathcal{O}^\flat_C \setminus \{0\}$ have no nonzero $\mathfrak{m}^\flat$-torsion.*

Proof. We may assume that $b \in \mathfrak{m}^\flat$, so, by using Frobenius, that $b \mid p^{1/p^\infty}$ in $\mathcal{O}^\flat_C$. Then Proposition 3.8 and the Δ-isomorphism $R^\flat_\infty/b \cong R_\infty/b^\sharp$ for some $b^\sharp \in \mathcal{O}_C$ give the claim. ☐

Lemma 3.13. *For any affinoid perfectoid $\mathrm{Spa}(R'_\infty[\frac{1}{p}], R'_\infty)$ over $\mathrm{Spa}(C, \mathcal{O}_C)$, the ring*

$$\mathbb{A}_{\mathrm{inf}}(R'_\infty) := W((R'_\infty)^\flat) \qquad (\textit{resp.,} \qquad \mathbb{A}_{\mathrm{inf}}(R'_\infty)/\mu)$$

is (p, μ)-adically complete (resp., p-adically complete). Moreover, for any $n, n' > 0$, the sequence $(p^n, \mu^{n'})$ is $\mathbb{A}_{\mathrm{inf}}(R'_\infty)$-regular and the $A_{\mathrm{inf}}/(p^n, \mu^{n'})$-algebra $\mathbb{A}_{\mathrm{inf}}(R'_\infty)/(p^n, \mu^{n'})$ is flat.

Proof. By its definition, the perfect $\mathcal{O}^\flat_C$-algebra $(R'_\infty)^\flat := \varprojlim_{y \mapsto y^p}(R'_\infty/p)$ has no nonzero p^{1/p^∞}-torsion (that is, it is $\mathcal{O}^\flat_C$-flat), so the regular sequence claim follows from [SP, 07DV]. The formal criterion of flatness [BouAC, Ch. III, §5.2, Thm. 1 (i)⇔(iv)] then implies the $A_{\mathrm{inf}}/(p^n, \mu^{n'})$-flatness of $\mathbb{A}_{\mathrm{inf}}(R'_\infty)/(p^n, \mu^{n'})$ (even with $n' = 0$). In addition, the short exact sequences (2.1.2) with $(R'_\infty)^\flat$ in place of $\mathcal{O}^\flat_C$ imply the p-adic completeness of $\mathbb{A}_{\mathrm{inf}}(R'_\infty)/\mu$.

Analogously to the case of A_{inf} discussed in §2.1, we use the Witt coordinate bijection and the μ-adic topology on $(R'_\infty)^\flat$ to topologize $\mathbb{A}_{\mathrm{inf}}(R'_\infty) \cong \prod_{n=1}^\infty (R'_\infty)^\flat$ and we see that this topology agrees with the (p, μ)-adic topology. Thus, $\mathbb{A}_{\mathrm{inf}}(R'_\infty)$ is (p, μ)-adically complete. ☐

3.14. The ring $\mathbb{A}_{\mathrm{inf}}(R_\infty)$. By [Sch13, 6.5 (i)], the value on $\mathfrak{X}^{\mathrm{ad}}_{C,\infty}$ of the sheaf $\mathbb{A}_{\mathrm{inf},\mathfrak{X}^{\mathrm{ad}}_C}$ is the ring

$$\mathbb{A}_{\mathrm{inf}}(R_\infty) := W(R^\flat_\infty).$$

By Lemma 3.13 and the formal criterion of flatness, $\mathbb{A}_{\mathrm{inf}}(R_\infty)$ is (p, μ)-adically formally flat as an A_{inf}-algebra and (p, μ)-adically formally étale as an $\mathbb{A}_{\mathrm{inf}}(R^\square_\infty)$-algebra. By using, in addition, Lemma 3.12, we see that each quotient

$$\mathbb{A}_{\mathrm{inf}}(R_\infty)/(p^n, \mu^{n'}), \qquad \text{so also} \qquad \mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu, \qquad \text{has no nonzero } W(\mathfrak{m}^\flat)\text{-torsion.} \tag{3.14.1}$$

In general, for a perfect $\mathbb{F}_p$-algebra A, the Witt ring $W(A)$ is the unique p-adically complete p-torsion free $\mathbb{Z}_p$-algebra $\widetilde{A}$ equipped with an isomorphism $\widetilde{A}/p \simeq A$ (see [Bha16, 2.5]). For an $a \in A$, the Teichmüller $[a] \in \widetilde{A}$ is $\lim_{n \to \infty}(\widetilde{a}_n^{p^n})$ where $\widetilde{a}_n \in \widetilde{A}$ is any lift of a^{1/p^n} (see [Bha16, 2.4]). Therefore,

$$\begin{aligned}\mathbb{A}_{\mathrm{inf}}(R^\square_\infty) &\cong \Big(\varinjlim_m A_{\mathrm{inf}}[X_0^{1/p^m}, \dots, X_r^{1/p^m}, X_{r+1}^{\pm 1/p^m}, \dots, X_d^{\pm 1/p^m}]/(\textstyle\prod_{i=0}^r X_i^{1/p^m} - [(p^{1/p^\infty})^{q/p^m}])\Big)^{\widehat{}} \\ &\cong \widehat{\bigoplus}_{\substack{(a_0, \dots, a_d) \in (\mathbb{Z}[\frac{1}{p}]_{\geq 0})^{\oplus(r+1)} \oplus (\mathbb{Z}[\frac{1}{p}])^{\oplus(d-r)}, \\ a_j = 0 \text{ for some } 0 \leq j \leq r}} A_{\mathrm{inf}} \cdot X_0^{a_0} \cdots X_d^{a_d},\end{aligned}$$

where the completions are (p,μ)-adic, the decomposition is as A_{inf}-modules, and, in terms of §3.11, we have $X_i^{1/p^m} = [x_i^{1/p^m}]$. The summands for which $a_i \in \mathbb{Z}$ for all i comprise a subring

$$A(R^\square) \cong A_{\mathrm{inf}}\{X_0, \dots, X_r, X_{r+1}^{\pm 1}, \dots, X_d^{\pm 1}\}/(X_0 \cdots X_r - [(p^{1/p^\infty})^q]) \quad \text{inside} \quad \mathbb{A}_{\mathrm{inf}}(R_\infty^\square), \quad (3.14.2)$$

where the convergence is (p,μ)-adic. The remaining summands, that is, those for which $a_i \notin \mathbb{Z}$ for some i, comprise an $A(R^\square)$-submodule $N_\infty^\square \subset \mathbb{A}_{\mathrm{inf}}(R_\infty^\square)$.

On sections over $\mathfrak{X}^{\mathrm{ad}}_{C,\infty}$, the map θ from (2.2.3) is identified with the unique ring homomorphism

$$\theta\colon \mathbb{A}_{\mathrm{inf}}(R_\infty) \twoheadrightarrow R_\infty \qquad \text{such that} \qquad [x] \mapsto x^{(0)},$$

is surjective with the kernel generated by the regular element ξ (see [BMS16, 3.10, 3.11]), and intertwines the Witt vector Frobenius of $\mathbb{A}_{\mathrm{inf}}(R_\infty)$ with the absolute Frobenius of R_∞/p. Thus,

$$\theta\colon A(R^\square) \twoheadrightarrow R^\square \qquad \text{is described by} \qquad X_i \mapsto t_i. \quad (3.14.3)$$

We use the surjection (3.14.3) to uniquely lift the étale $R^\square/p$-algebra R/p to a (p,μ)-adically complete, formally étale $A(R^\square)$-algebra $A(R)$. By construction, we have the identification

$$\mathbb{A}_{\mathrm{inf}}(R_\infty) \cong \mathbb{A}_{\mathrm{inf}}(R_\infty^\square)\widehat{\otimes}_{A(R^\square)} A(R), \quad (3.14.4)$$

where the completion is (p,μ)-adic. Therefore, by setting $N_\infty := N_\infty^\square \widehat{\otimes}_{A(R^\square)} A(R)$, we arrive at the decompositions of $\mathbb{A}_{\mathrm{inf}}(R_\infty^\square)$ and $\mathbb{A}_{\mathrm{inf}}(R_\infty)$ into "integral" and "nonintegral" parts:

$$\mathbb{A}_{\mathrm{inf}}(R_\infty^\square) \cong A(R^\square) \oplus N_\infty^\square \qquad \text{and} \qquad \mathbb{A}_{\mathrm{inf}}(R_\infty) \cong A(R) \oplus N_\infty. \quad (3.14.5)$$

Modulo $\operatorname{Ker}\theta$ (that is, modulo ξ), these decompositions reduce to the decompositions (3.2.2).

The Witt vector Frobenius of $\mathbb{A}_{\mathrm{inf}}(R_\infty^\square)$ preserves $A(R^\square)$; explicitly: it is semilinear with respect to the Frobenius of A_{inf} and raises each X_i^{1/p^m} to the p-th power. By construction, $A(R)$ inherits a Frobenius ring endomorphism from $A(R^\square)$, and the identification (3.14.4) is Frobenius-equivariant.

The natural Δ-action on $\mathbb{A}_{\mathrm{inf}}(R_\infty)$ is continuous and commutes with the Frobenius. Explicitly, Δ respects the completed direct sum decomposition and an $(\epsilon_0, \dots, \epsilon_d) \in \Delta$ scales $X_j^{a_j}$ by $[\epsilon_j^{a_j}] \in A_{\mathrm{inf}}$. The Δ-action on $A(R^\square)$ lifts uniquely to a necessarily Frobenius-equivariant Δ-action on $A(R)$. In particular, Δ acts trivially on $A(R)/\mu$. The identifications (3.14.4) and (3.14.5) are Δ-equivariant.

3.15. The cohomology of $\mathbb{A}_{\mathrm{inf}}$ and continuous group cohomology. Similarly to §3.3, the Čech complex of the sheaf $\mathbb{A}_{\mathrm{inf},\mathfrak{X}_C^{\mathrm{ad}}}$ with respect to the pro-(finite étale) affinoid perfectoid cover $\mathfrak{X}^{\mathrm{ad}}_{C,\infty} \to \mathfrak{X}^{\mathrm{ad}}_C$ is identified with the continuous cochain complex $R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty))$. Thus, by using [SP, 01GY], we obtain the edge map to the proétale cohomology of $\mathbb{A}_{\mathrm{inf},\mathfrak{X}_C^{\mathrm{ad}}}$:

$$e\colon R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)) \to R\Gamma_{\mathrm{pro\acute{e}t}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{A}_{\mathrm{inf}}). \quad (3.15.1)$$

By the almost purity theorem, more precisely, by [Sch13, 6.5 (ii)], the subset $[\mathfrak{m}^\flat] \subset A_{\mathrm{inf}}$ that consists of the Teichmüller lifts of the elements in the maximal ideal $\mathfrak{m}^\flat \subset \mathcal{O}_C^\flat$ kills all the cohomology groups of $\operatorname{Cone}(e)$. Since $\mu \in W(\mathfrak{m}^\flat)$ (see (2.3.3)), it will be useful to strengthen this annihilation as follows.

Lemma 3.16. *The ideal $W(\mathfrak{m}^\flat) \subset A_{\mathrm{inf}}$ defined in (2.3.3) kills each $H^i(\operatorname{Cone}(e))$.*

Proof. We argue similarly to [BMS16, proof of Thm. 5.6]. Both the source and the target of e are derived p-adically complete (see §1.7), so, by [BS15, 3.4.4 and 3.4.14], each $H^i(\operatorname{Cone}(e))$ is also derived p-adically complete. Thus, the desired conclusion follows from the following lemma. ☐

Lemma 3.17. *If $[\mathfrak{m}^\flat]A_{\mathrm{inf}}$ kills a derived p-adically complete A_{inf}-module H, then so does $W(\mathfrak{m}^\flat)$.*

Proof. By the derived p-adic completeness, any free A_{inf}-module resolution $F^\bullet$ of H satisfies

$$H \cong \mathrm{Coker}\left(\varprojlim_n (F^{-1}/p^n) \to \varprojlim_n (F^0/p^n)\right).$$

Moreover, for every $n \geq 1$ the ideals $[\mathfrak{m}^\flat] \cdot W_n(\mathcal{O}_C^\flat)$ and $W_n(\mathfrak{m}^\flat) := \mathrm{Ker}(W_n(\mathcal{O}_C^\flat) \to W_n(k))$ of $W_n(\mathcal{O}_C^\flat)$ agree. Thus, the $([\mathfrak{m}^\flat]A_{\mathrm{inf}})$-annihilation of H implies that $W_n(\mathfrak{m}^\flat)$ kills both

$$H/p^n \cong H^0(F^\bullet \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/p^n) \qquad \text{and} \qquad \mathrm{Tor}_1^{A_{\mathrm{inf}}}(H, A_{\mathrm{inf}}/p^n) \cong H^{-1}(F^\bullet \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/p^n).$$

Thus, since $[\mathfrak{m}^\flat]^2 = [\mathfrak{m}^\flat]$ and F_0/p^n has no nonzero m-torsion for every nonzero $m \in [\mathfrak{m}^\flat]$, any element $x \in W_{n+1}(\mathfrak{m}^\flat) \cdot (F_0/p^{n+1})$ may be lifted to $W_{n+1}(\mathfrak{m}^\flat) \cdot (F_{-1}/p^{n+1})$, compatibly with a specified lift of its image $\overline{x} \in W_n(\mathfrak{m}^\flat) \cdot (F_0/p^n)$ to $W_n(\mathfrak{m}^\flat) \cdot (F_{-1}/p^n)$. In particular, $W(\mathfrak{m}^\flat) \cdot (\varprojlim_n (F^0/p^n))$ lies in the image of $\varprojlim_n (F^{-1}/p^n)$, that is, $W(\mathfrak{m}^\flat)$ kills H, as desired. □

We will show in Theorem 3.20 that $L\eta_{(\mu)}(e)$ is an isomorphism, so that continuous group cohomology computes $L\eta_{(\mu)}(R\Gamma_{\text{proét}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{A}_{\mathrm{inf}}))$. For this, we will use the following variant of [Bha16, 6.14].

Lemma 3.18. *If $B \xrightarrow{b} B'$ is a morphism in $D(A_{\mathrm{inf}})$ such that each $H^i(B \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu)$ has no nonzero $W(\mathfrak{m}^\flat)$-torsion and $W(\mathfrak{m}^\flat)$ kills each $H^i(\mathrm{Cone}(b))$, then $L\eta_{(\mu)}(b)$ is an isomorphism.*

Proof. Since $L\eta$ is in general not a triangulated functor, the fact that $L\eta_{(\mu)}(\mathrm{Cone}(b)) \cong 0$ does not *a priori* suffice. Nevertheless, the argument used to prove [Bha16, 6.14] gives the claim. In more detail, $(W(\mathfrak{m}^\flat))^2$ kills the cohomology of $\mathrm{Cone}(b) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu$, so the sequences

$$0 \to H^i(B \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu) \to H^i(B' \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu) \to H^i(\mathrm{Cone}(b) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu) \to 0$$

are short exact. By the Bockstein construction (see [BMS16, 6.12]), as i varies, they comprise a short exact sequence whose terms are complexes that compute $L\eta_{(\mu)}(B) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu$, etc. Thus, the vanishing of $L\eta_{(\mu)}(\mathrm{Cone}(b))$ implies that $(L\eta_{(\mu)}(b)) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu$ is an isomorphism. It follows that $\mathrm{Cone}(L\eta_{(\mu)}(b)) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu \cong 0$, so μ acts invertibly on the cohomology of $\mathrm{Cone}(L\eta_{(\mu)}(b))$. But then, as we see after applying $- \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}[\frac{1}{\mu}]$, this cohomology vanishes. □

We now verify that the edge map e defined in (3.15.1) also meets the first assumption of Lemma 3.18.

Proposition 3.19. *For each $i \in \mathbb{Z}$, the A_{inf}-module $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu)$ is p-torsion free and p-adically complete; moreover, the following natural maps are isomorphisms:*

$$H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu) \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/p^n \xrightarrow{\sim} H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/(\mu, p^n)) \quad \textit{for} \quad n > 0 \tag{3.19.1}$$

and

$$H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu) \xrightarrow{\sim} \varprojlim_n \left(H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/(\mu, p^n))\right). \tag{3.19.2}$$

In addition, $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/(\mu, p^n))$ and $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu)$ have no nonzero $W(\mathfrak{m}^\flat)$-torsion.

Proof. Since $A(R)/\mu$ is p-adically complete and trivial as a Δ-module (see Lemma 3.13 and §3.14), Lemma 3.7 implies that $H^i_{\mathrm{cont}}(\Delta, A(R)/\mu)$ is a direct sum of copies of $A(R)/\mu$, and likewise for $H^i_{\mathrm{cont}}(\Delta, A(R)/(\mu, p^n))$. Consequently, since, by (3.14.1), the rings $A(R)/(\mu, p^n)$ and $A(R)/\mu$ have no nonzero $W(\mathfrak{m}^\flat)$-torsion, the analogues of all the claims with $A(R)$ in place of $\mathbb{A}_{\mathrm{inf}}(R_\infty)$ follow. Thus, due to (3.14.5), we only need to establish these analogues with N_∞ in place of $\mathbb{A}_{\mathrm{inf}}(R_\infty)$.

To prepare for treating N_∞, we start by building on the ideas of [Bha16, proof of Lem. 4.6] to analyze a single summand $S := A_{\mathrm{inf}} \cdot X_0^{a_0} \cdots X_d^{a_d}$ that, as in §3.14, contributes to $N_\infty^\square$. We set

$$b_j := a_j - a_0 \quad \text{for} \quad 1 \leq j \leq r \qquad \text{and} \qquad b_j := a_j \quad \text{for} \quad r+1 \leq j \leq d, \tag{3.19.3}$$

and let $m \in \mathbb{Z}_{>0}$ be the minimal such that $p^m b_j \in \mathbb{Z}$ for all j. Lemma 3.7 applied with the topological generators $\delta_1, \dots, \delta_d$ of Δ defined in §3.2 gives an A_{inf}-isomorphism $H^i_{\mathrm{cont}}(\Delta, S/\mu) \simeq H^i(C^\bullet)$, where $C^\bullet$ is the (A_{inf}/μ)-tensor product of the d complexes

$$[A_{\mathrm{inf}}/\mu \xrightarrow{[\epsilon^{b_j}]-1} A_{\mathrm{inf}}/\mu] \cong A_{\mathrm{inf}}/([\epsilon^{b_j}]-1) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu. \tag{3.19.4}$$

By reordering the b_j, we may assume that for all j we have $b_j/b_1 \in \mathbb{Z}_{(p)}$, so that $b_1 \notin \mathbb{Z}$ and both $[\epsilon^{b_1}]-1 \mid [\epsilon^{b_j}]-1$ and $[\epsilon^{b_1}]-1 \mid \mu$. Then, by resolving A_{inf}/μ in (3.19.4) with $j=1$, we see that $C^\bullet$ is quasi-isomorphic to a direct sum of shifts of $A_{\mathrm{inf}}/([\epsilon^{b_1}]-1) \cong A_{\mathrm{inf}}/\varphi^{-m}(\mu)$. Thus, for $i \in \mathbb{Z}$,

$$H^i_{\mathrm{cont}}(\Delta, S/\mu) \simeq \textstyle\bigoplus_I A_{\mathrm{inf}}/\varphi^{-m}(\mu) \quad \text{for some set } I, \quad \text{and hence} \quad H^i_{\mathrm{cont}}(\Delta, S/\mu)[p] = 0. \tag{3.19.5}$$

By Lemma 3.7 and [SP, 061Z, 0662], this implies that

$$H^i_{\mathrm{cont}}(\Delta, S/\mu) \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/p^n \xrightarrow{\sim} H^i_{\mathrm{cont}}(\Delta, S/(\mu, p^n)). \tag{3.19.6}$$

We now analyze $N^\square_\infty$. Since $\mathbb{A}_{\mathrm{inf}}(R^\square_\infty)/\mu$ is p-adically complete, §3.14 gives the Δ-decomposition

$$\mathbb{A}_{\mathrm{inf}}(R^\square_\infty)/\mu \cong \widehat{\bigoplus}_{\substack{(a_0, \dots, a_d) \in (\mathbb{Z}[\frac{1}{p}]_{\geq 0})^{\oplus(r+1)} \oplus (\mathbb{Z}[\frac{1}{p}])^{\oplus(d-r)}, \\ a_j = 0 \text{ for some } 0 \leq j \leq r}} A_{\mathrm{inf}}/\mu \cdot X_0^{a_0} \cdots X_d^{a_d} \tag{3.19.7}$$

in which the completion is p-adic. Lemma 3.6 (i) then combines with (3.19.5) to prove that

$$H^i_{\mathrm{cont}}(\Delta, N^\square_\infty/\mu)[p] = 0 \quad \text{for each} \quad i \in \mathbb{Z}. \tag{3.19.8}$$

Analogously to (3.19.6), this, in turn, implies that

$$H^i_{\mathrm{cont}}(\Delta, N^\square_\infty/\mu) \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/p^n \xrightarrow{\sim} H^i_{\mathrm{cont}}(\Delta, N^\square_\infty/(\mu, p^n)). \tag{3.19.9}$$

Finally, we analyze N_∞. The identification

$$N_\infty/(\mu, p^n) \cong N^\square_\infty/(\mu, p^n) \otimes_{A(R^\square)} A(R) \tag{3.19.10}$$

is Δ-equivariant and $A(R)/(\mu, p^n)$ is $(A(R^\square)/(\mu, p^n))$-flat, so Lemma 3.7 gives the identifications

$$H^i_{\mathrm{cont}}(\Delta, N_\infty/(\mu, p^n)) \cong H^i_{\mathrm{cont}}(\Delta, N^\square_\infty/(\mu, p^n)) \otimes_{A(R^\square)} A(R) \quad \text{for} \quad n \geq 1, \tag{3.19.11}$$

which are compatible as n varies. Consequently, for $n > 1$, the sequences

$$0 \to H^i_{\mathrm{cont}}(\Delta, N_\infty/(\mu, p^n))[p] \to H^i_{\mathrm{cont}}(\Delta, N_\infty/(\mu, p^n)) \to H^i_{\mathrm{cont}}(\Delta, N_\infty/(\mu, p^{n-1})) \to 0 \tag{3.19.12}$$

are short exact because, by (3.19.5) and (3.19.9), so are their analogues with $N^\square_\infty$ in place of N_∞. By taking the inverse limit of these sequences for varying n and using [SP, 08U5], we obtain

$$H^i_{\mathrm{cont}}(\Delta, N_\infty/\mu) \xrightarrow{\sim} \varprojlim_n \left(H^i_{\mathrm{cont}}(\Delta, N_\infty/(\mu, p^n))\right), \tag{3.19.13}$$

which is the sought analogue of (3.19.2). The p-torsion freeness of $H^i_{\mathrm{cont}}(\Delta, N_\infty/\mu)$ follows from (3.19.12)–(3.19.13) and, as in (3.19.6), it implies that

$$H^i_{\mathrm{cont}}(\Delta, N_\infty/\mu) \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/p^n \xrightarrow{\sim} H^i_{\mathrm{cont}}(\Delta, N_\infty/(\mu, p^n)).$$

It remains to show that each $H^i_{\mathrm{cont}}(\Delta, N_\infty/(\mu, p^n))$ has no nonzero $W(\mathfrak{m}^\flat)$-torsion.

The surjectivity aspect of the short exact sequences (3.19.12) implies that the sequences

$$0 \to N_\infty/(\mu, p) \xrightarrow{p^{n-1}} N_\infty/(\mu, p^n) \to N_\infty/(\mu, p^{n-1}) \to 0$$

stay short exact after applying $H^i_{\mathrm{cont}}(\Delta, -)$. Thus, $H^i_{\mathrm{cont}}(\Delta, N_\infty/(\mu, p^n))$ is a successive extension of copies of $H^i_{\mathrm{cont}}(\Delta, N_\infty/(\mu, p))$. Consequently, it has no nonzero $W(\mathfrak{m}^\flat)$-torsion because, by Lemma 3.12, neither does $H^i_{\mathrm{cont}}(\Delta, N_\infty/(\mu, p))$. □

Theorem 3.20. *The edge map e defined in* (3.15.1) *induces the isomorphism*

$$L\eta_{(\mu)}(e)\colon L\eta_{(\mu)}(R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty))) \xrightarrow{\sim} L\eta_{(\mu)}(R\Gamma_{\text{proét}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{A}_{\mathrm{inf},\, \mathfrak{X}_C^{\mathrm{ad}}})).$$

Proof. By the projection formula [SP, 0944],

$$R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu \cong R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu), \tag{3.20.1}$$

so Proposition 3.19 implies that the cohomology modules of $R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu$ have no nonzero $W(\mathfrak{m}^\flat)$-torsion. Thus, the claim follows from Lemmas 3.16 and 3.18. $\square$

Remark 3.21. Analogously to Remark 3.10, Theorem 3.20 extends as follows: for a pro-(finite étale) affinoid perfectoid Δ'-cover

$$\mathrm{Spa}(R'_\infty[\tfrac{1}{p}], R'_\infty) \to \mathrm{Spa}(R[\tfrac{1}{p}], R) \cong \mathfrak{X}_C^{\mathrm{ad}} \qquad \text{that refines} \qquad \mathfrak{X}_{C,\,\infty}^{\mathrm{ad}} \to \mathfrak{X}_C^{\mathrm{ad}}, \tag{3.21.1}$$

the edge map e' defined analogously to (3.15.1) induces the isomorphism

$$L\eta_{(\mu)}(e')\colon L\eta_{(\mu)}(R\Gamma_{\mathrm{cont}}(\Delta', \mathbb{A}_{\mathrm{inf}}(R'_\infty))) \xrightarrow{\sim} L\eta_{(\mu)}(R\Gamma_{\text{proét}}(\mathfrak{X}_C^{\mathrm{ad}}, \widehat{\mathcal{O}}^+)).$$

Indeed, like in Remark 3.10, by the almost purity theorem and the octahedral axiom, $[\mathfrak{m}^\flat]A_{\mathrm{inf}}$ kills the cohomology modules of the cone of the map $e_0\colon R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)) \to R\Gamma_{\mathrm{cont}}(\Delta', \mathbb{A}_{\mathrm{inf}}(R'_\infty))$ and, by [BS15, 3.4.4 and 3.4.14], these modules are derived p-adically complete; thus, by Lemma 3.17, even $W(\mathfrak{m}^\flat)$ kills them, to the effect that Lemma 3.18 applies to the map e_0 and proves the claim.

As a final goal of §3, we wish to show in Theorem 3.34 that even the maps $L\eta_{(\mu)}(e\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}^{(m)})$ are isomorphisms for A_{inf}-algebras $A_{\mathrm{cris}}^{(m)}$ reviewed in §3.26 below. This extension of Theorem 3.20 will be important for relating $A\Omega_{\mathfrak{X}}$ to logarithmic crystalline cohomology in §5. Our analysis of $L\eta_{(\mu)}(e\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}^{(m)})$ will use the following further consequences of the proof of Proposition 3.19.

3.22. The decomposition of N_∞. For $m \geq 0$, let $N_m^\square$ be the (p, μ)-adically completed direct sum of those summands $A_{\mathrm{inf}} \cdot X_0^{a_0} \cdots X_d^{a_d}$ that contribute to $\mathbb{A}_{\mathrm{inf}}(R_\infty^\square)$ in §3.14 for which m is the smallest nonnegative integer such that $p^m a_j \in \mathbb{Z}$ for all j (equivalently, in the notation of (3.19.3), such that $p^m b_j \in \mathbb{Z}$ for all j). For varying $m > 0$, the $A(R^\square)$-modules $N_m^\square$ and the $A(R)$-modules $N_m := N_m^\square \widehat{\otimes}_{A(R^\square)} A(R)$ comprise the (p, μ)-adically completed direct sum decompositions

$$N_\infty^\square \cong \widehat{\bigoplus}_{m>0} N_m^\square \qquad \text{and} \qquad N_\infty \cong \widehat{\bigoplus}_{m>0} N_m. \tag{3.22.1}$$

For a fixed i, Lemma 3.7 and (3.19.5)–(3.19.6) imply that

$$H^i_{\mathrm{cont}}(\Delta, N_m^\square/(\mu, p^n)) \simeq \bigoplus_{I'} A_{\mathrm{inf}}/(\varphi^{-m}(\mu), p^n) \quad \text{for some set } I' \text{ and every } n > 0. \tag{3.22.2}$$

Corollary 3.23. *For all i and $n, m \geq 0$,*

$$H^i_{\mathrm{cont}}(\Delta, N_m/(\mu, p^n)) \quad \textit{is killed by } \varphi^{-m}(\mu) \quad \textit{and} \quad \textit{is a flat } A_{\mathrm{inf}}/(\varphi^{-m}(\mu), p^n)\textit{-module.}$$

Proof. If $R = R^\square$, then (3.22.2) suffices. In addition, by Lazard's theorem, $A(R)/(\mu, p^n)$ is a filtered direct limit of finite free $A(R^\square)/(\mu, p^n)$-modules. Thus, the general case of the claim follows by using (3.19.11) and its analogue for N_0 and $N_0^\square$. $\square$

We wish to supplement Proposition 3.19 with Proposition 3.25 that analyzes the cohomology of N_∞ without reducing modulo μ. Its proof will use the following base change result for $L\eta$.

Lemma 3.24 ([Bha16, 5.14])**.** *For a ring A, a regular sequence $f, g \in A$, and a $K \in D(A)$, if the cohomology modules $H^i(K \otimes^{\mathbb{L}}_A A/f)$ have no nonzero g-torsion, then the natural map*

$$L\eta_{(f)}(K) \otimes^{\mathbb{L}}_A A/g \to L\eta_{(\overline{f})}(K \otimes^{\mathbb{L}}_A A/g), \qquad \text{where } \overline{f} \text{ denotes the image of } f \text{ in } A/g,$$

is an isomorphism. □

Proposition 3.25. *The element μ kills every $H^i_{\mathrm{cont}}(\Delta, N_\infty)$.*

Proof. Let $\delta_1, \dots, \delta_d$ be the free generators of Δ fixed in §3.2. By Lemma 3.7, we need to prove that

$$L\eta_{(\mu)}(K_{N_\infty}(\delta_1 - 1, \dots, \delta_d - 1)) \cong 0. \tag{3.25.1}$$

The key point, with which we start, is to prove the vanishing (3.25.1) modulo $\varphi(\xi)$. The isomorphism

$$K_{N_\infty}(\delta_1 - 1, \dots, \delta_d - 1) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu \cong K_{N_\infty/\mu}(\delta_1 - 1, \dots, \delta_d - 1),$$

Lemma 3.7, and Proposition 3.19 show that the cohomology of $K_{N_\infty}(\delta_1 - 1, \dots, \delta_d - 1) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu$ is p-torsion free. Therefore, Lemma 3.24 supplies the identification

$$L\eta_{(\mu)}(K_{N_\infty}(\delta_1 - 1, \dots, \delta_d - 1)) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\varphi(\xi) \cong L\eta_{(\zeta_p - 1)}(K_{N_\infty/\varphi(\xi)}(\delta_1 - 1, \dots, \delta_d - 1)). \tag{3.25.2}$$

The inverse Frobenius φ^{-1} maps $N^{\square}_\infty$ isomorphically onto a direct summand of $N^{\square}_\infty$, so it maps N_∞ isomorphically onto a direct summand of N_∞. Thus, φ^{-1} maps $N_\infty/\varphi(\xi)$ isomorphically onto a direct summand of $N_\infty/\xi \cong M_\infty$ (see (3.14.5)). In particular, by Lemma 3.7 and Proposition 3.8, $\zeta_p - 1$ kills the cohomology of $K_{N_\infty/\varphi(\xi)}(\delta_1 - 1, \dots, \delta_d - 1)$, so both sides of (3.25.2) are acyclic.

Since $K_{N_\infty}(\delta_1 - 1, \dots, \delta_d - 1)$ is derived $\varphi(\xi)$-adically complete (see [SP, 090T]), [BMS16, 6.19] implies the same for $L\eta_{(\mu)}(K_{N_\infty}(\delta_1 - 1, \dots, \delta_d - 1))$. The established acyclicity of the left side of (3.25.2) therefore implies the desired vanishing (3.25.1). □

3.26. The A_{inf}-algebras $A^{(m)}_{\mathrm{cris}}$. The ring $A^{(m)}_{\mathrm{cris}}$ for $m \in \mathbb{Z}_{\geq 1}$ is the p-adic completion of the A_{inf}-subalgebra $A^{0,(m)}_{\mathrm{cris}}$ of $A_{\mathrm{inf}}[\frac{1}{p}]$ generated by the elements $\frac{\xi^s}{s!}$ with $s \leq m$. In particular, $A^{(m)}_{\mathrm{cris}} \cong A_{\mathrm{inf}}$ for $m < p$. In contrast, if $m \geq p$, then, since $\frac{\mu^p}{p!} \in A^{(m)}_{\mathrm{cris}}$, the p-adic and (p, μ)-adic topologies of $A^{(m)}_{\mathrm{cris}}$ agree. By its definition, $A^{(m)}_{\mathrm{cris}}$ is p-torsion free; in fact, although we will not use this, Proposition 5.36 below implies that $A^{(m)}_{\mathrm{cris}}$ is even a domain. The map θ of (2.1.3) extends to $A^{(m)}_{\mathrm{cris}}$:

$$\theta\colon A^{(m)}_{\mathrm{cris}} \twoheadrightarrow \mathcal{O}_C. \tag{3.26.1}$$

Due to the "finite type nature" of the A_{inf}-algebra $A^{(m)}_{\mathrm{cris}}$, more precisely, due to [BMS16, 12.7 (ii)], the systems of ideals

$$(p^n A^{(m)}_{\mathrm{cris}})_{n \geq 1} \quad \text{and} \quad (\{x \in A^{(m)}_{\mathrm{cris}} \mid \mu x \in p^n A^{(m)}_{\mathrm{cris}}\})_{n \geq 1} \qquad \text{of } A^{(m)}_{\mathrm{cris}} \text{ are intertwined.} \tag{3.26.2}$$

Equivalently,

$$\text{for every} \quad n \geq 1, \quad \text{the map} \quad (A^{(m)}_{\mathrm{cris}}/p^{n'})[\mu] \to A^{(m)}_{\mathrm{cris}}/p^n \quad \text{vanishes for large} \quad n' > n. \tag{3.26.3}$$

Therefore, by taking the inverse limit over n of the sequences

$$0 \to (A^{(m)}_{\mathrm{cris}}/p^n)[\mu] \to A^{(m)}_{\mathrm{cris}}/p^n \xrightarrow{\mu} A^{(m)}_{\mathrm{cris}}/p^n \to A^{(m)}_{\mathrm{cris}}/(\mu, p^n) \to 0, \tag{3.26.4}$$

we conclude that

$$A^{(m)}_{\mathrm{cris}} \quad \text{is } \mu\text{-torsion free} \qquad \text{and} \qquad A^{(m)}_{\mathrm{cris}}/\mu \quad \text{is } p\text{-adically complete.} \tag{3.26.5}$$

The Frobenius automorphism of A_{inf} preserves the subring $A^{0,(m)}_{\mathrm{cris}} \subset A_{\mathrm{inf}}[\frac{1}{p}]$: indeed, for $m \geq p$, since $\xi = \sum_{i=0}^{p-1}[\epsilon^{i/p}]$ and $\xi^p \in pA^{0,(m)}_{\mathrm{cris}}$, we have $\varphi(\xi) = \sum_{i=0}^{p-1}[\epsilon^i]$ and $\varphi(\xi) \in pA^{0,(m)}_{\mathrm{cris}}$. Thus, the Frobenius induces a ring endomorphism

$$\varphi\colon A^{(m)}_{\mathrm{cris}} \to A^{(m)}_{\mathrm{cris}}, \tag{3.26.6}$$

which, via the map θ, intertwines the absolute Frobenius of $\mathcal{O}_C/p$ (compare with (2.1.3)).

3.27. The $A(R)$-algebras $A^{(m)}_{\mathrm{cris}}(R)$. We define the "relative version" of the ring $A^{(m)}_{\mathrm{cris}}$ and its "highly ramified cover" by

$$A^{(m)}_{\mathrm{cris}}(R) := A(R)\widehat{\otimes}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}} \qquad \text{and} \qquad \mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty) := \mathbb{A}_{\mathrm{inf}}(R_\infty)\widehat{\otimes}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}},$$

respectively, where the completion is (p,μ)-adic (equivalently, p-adic if $m \geq p$). In the case $m < p$, one has the identifications $A^{(m)}_{\mathrm{cris}}(R) \cong A(R)$ and $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty) \cong \mathbb{A}_{\mathrm{inf}}(R_\infty)$. Due to the decomposition (3.14.5), the $A(R)$-algebra $A^{(m)}_{\mathrm{cris}}(R)$ is an $A^{(m)}_{\mathrm{cris}}(R)$-module direct summand of $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)$.

The completed direct sum decomposition of $\mathbb{A}_{\mathrm{inf}}(R^\square_\infty)$ (see §3.14) gives the decomposition

$$\mathbb{A}^{(m)}_{\mathrm{cris}}(R^\square_\infty) \cong \widehat{\bigoplus}_{\substack{(a_0,\dots,a_d)\in \mathbb{Z}[\frac{1}{p}]^{\oplus(r+1)}_{\geq 0}\oplus\mathbb{Z}[\frac{1}{p}]^{\oplus(d-r)},\\ a_j = 0 \text{ for some } 0\leq j\leq r}} A^{(m)}_{\mathrm{cris}}\cdot X_0^{a_0}\cdots X_d^{a_d}, \tag{3.27.1}$$

where the completion is (p,μ)-adic (equivalently, p-adic if $m \geq p$), and, by Lemma 3.13, the algebra $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)$ is (p,μ)-adically formally étale over $\mathbb{A}^{(m)}_{\mathrm{cris}}(R^\square_\infty)$. In particular, (3.26.3) holds with $\mathbb{A}^{(m)}_{\mathrm{cris}}(R^\square_\infty)$ in place of $A^{(m)}_{\mathrm{cris}}$, and hence also with $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)$ in place of $A^{(m)}_{\mathrm{cris}}$. Consequently, the generalization of (3.26.5) holds, too:

$$\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty) \quad \text{is } \mu\text{-torsion free} \qquad \text{and} \qquad \mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)/\mu \quad \text{is } p\text{-adically complete.} \tag{3.27.2}$$

In addition, by (3.27.1) and the formal étaleness, each $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)$ is p-torsion free. By §3.14 and §3.26, the rings $A^{(m)}_{\mathrm{cris}}(R)$ and $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)$ come equipped with $A^{(m)}_{\mathrm{cris}}$-semilinear Frobenius endomorphisms that are compatible as m varies.

The group Δ acts continuously, Frobenius-equivariantly, and $A^{(m)}_{\mathrm{cris}}$-linearly on $A^{(m)}_{\mathrm{cris}}(R)$ and $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)$. For each $\delta \in \Delta$, the abelian group endomorphism $\frac{\delta-1}{\mu}$ of $A(R)$ induces the endomorphism $\frac{\delta-1}{\mu}$ of $A^{(m)}_{\mathrm{cris}}(R)$ that satisfies $\delta = 1+\mu\cdot\frac{\delta-1}{\mu}$, so, in particular, the induced Δ-action on $A^{(m)}_{\mathrm{cris}}(R)/\mu$ is trivial.

3.28. The $A^{(m)}_{\mathrm{cris}}$-base change of the edge map. Since $A^{(m)}_{\mathrm{cris}} \cong A_{\mathrm{inf}}$ for $m < p$ (see §3.26), for the sake of analyzing the map $e\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}$, let us suppose that $m \geq p$. Then for each $A^{(m)}_{\mathrm{cris}}/p^n$, we have $A^{(m)}_{\mathrm{cris}}/p^n \cong A^{(m)}_{\mathrm{cris}}/(p^n,\mu^{n'})$ for every large enough $n' > 0$ (see §3.26). Consequently, since each sequence $(p^n,\mu^{n'})$ is $\mathbb{A}_{\mathrm{inf}}(R_\infty)$-regular with $\mathbb{A}_{\mathrm{inf}}(R_\infty)/(p^n,\mu^{n'})$ flat over $A_{\mathrm{inf}}/(p^n,\mu^{n'})$ (see Lemma 3.13), the projection formula [SP, 0944] and Lemma 3.7 imply that

$$R\Gamma_{\mathrm{cont}}(\Delta,\mathbb{A}_{\mathrm{inf}}(R_\infty))\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}} \cong R\Gamma_{\mathrm{cont}}(\Delta,\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)).$$

Consequently, the edge map e defined in (3.15.1) gives rise to the map

$$e\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}\colon R\Gamma_{\mathrm{cont}}(\Delta,\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)) \to R\Gamma_{\mathrm{pro\acute{e}t}}(\mathfrak{X}^{\mathrm{ad}}_C,\mathbb{A}_{\mathrm{inf}})\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}. \tag{3.28.1}$$

Since $[\mathfrak{m}^\flat]$ kills each $H^i(\mathrm{Cone}(e))$ (see §3.15) and $\mathrm{Cone}(e)$ is bounded, a spectral sequence (see [SP, 0662]) shows that $[\mathfrak{m}^\flat]$ also kills each $H^i(\mathrm{Cone}(e)\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}/p^n)$. Consequently, by [SP, 08U5], the ideal $[\mathfrak{m}^\flat]A_{\mathrm{inf}}$ kills each $H^i(\mathrm{Cone}(e)\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}})$. In fact, since $H^i(\mathrm{Cone}(e)\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}})$ is derived p-adically complete (see [BS15, 3.4.4 and 3.4.14]), by Lemma 3.17, even $W(\mathfrak{m}^\flat)$ kills it. In conclusion,

$$W(\mathfrak{m}^\flat) \quad \text{kills all the cohomology groups of} \quad \mathrm{Cone}(e\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}) \cong \mathrm{Cone}(e)\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}. \tag{3.28.2}$$

We will show in Theorem 3.34 that $L\eta_{(\mu)}(e\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}})$ is an isomorphism by applying Lemma 3.18. Thus, we need to know that the A_{inf}-modules $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)/\mu)$ have no nonzero $W(\mathfrak{m}^\flat)$-torsion (compare with Proposition 3.19 for $\mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu$). The following result is a step in that direction:

Proposition 3.29. *The rings $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)/\mu$ and $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)/(\mu, p^n)$ have no nonzero $W(\mathfrak{m}^\flat)$-torsion.*

Proof. Due to the p-adic completeness of $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)/\mu$, it suffices to establish the claim about $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)/(\mu, p^n)$ for every $n \in \mathbb{Z}_{\geq 1}$. The argument for the latter is similar to that of [BMS16, 12.7 (iii)] and uses approximation by Noetherian rings. Namely, due to the μ-adic completeness of A_{inf}, the assignment

$$T \mapsto [\epsilon]^{1/p} - 1 \qquad \text{defines a } \mathbb{Z}_p\text{-algebra morphism} \qquad \mathbb{Z}_p[\![T]\!] \to A_{\mathrm{inf}}. \tag{3.29.1}$$

By [BMS16, 4.31], this turns A_{inf} into a faithfully flat $\mathbb{Z}_p[\![T]\!]$-module, so, letting M be the mod $(p^n, (T+1)^p - 1)$ reduction of the $\mathbb{Z}_p[\![T]\!]$-subalgebra of $\mathbb{Z}_p[\![T]\!][\frac{1}{p}]$ generated by the $\frac{1}{s!}(\sum_{i=0}^{p-1}(T+1)^i)^s$ with $s \leq m$, we have the identification

$$\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)/(\mu, p^n) \cong M \otimes_{\mathbb{Z}_p[\![T]\!]/(p^n,\,(T+1)^p-1)} \mathbb{A}_{\mathrm{inf}}(R_\infty)/(p^n, \mu).$$

The $(\mathbb{Z}_p[\![T]\!]/(p^n,\, (T+1)^p - 1))$-flatness of $\mathbb{A}_{\mathrm{inf}}(R_\infty)/(p^n, \mu)$ ensures that the $\varphi^{-1}(\mu)$-torsion submodule of $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)/(\mu, p^n)$ is the base change of the T-torsion submodule $M[T] \subset M$. Consequently, since $\varphi^{-1}(\mu) \in W(\mathfrak{m}^\flat)$, the consideration of the p-adic filtration of $M[T]$ reduces us to proving that

$$\mathbb{F}_p \otimes_{\mathbb{Z}_p[\![T]\!]/(p^n,\,(T+1)^p-1)} \mathbb{A}_{\mathrm{inf}}(R_\infty)/(p^n, \mu) \cong R^\flat_\infty/\varphi^{-1}(\mu) \qquad \text{has no nonzero } \mathfrak{m}^\flat\text{-torsion},$$

which follows from Lemma 3.12. □

To relate $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)/\mu)$ to $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu)$, we will use the following generality about the exactness properties of p-adically completed tensor products (that has little to do with the particular A_{inf}-algebra $A^{(m)}_{\mathrm{cris}}$, or even with A_{inf} itself).

Lemma 3.30. *For a fixed $m \in \mathbb{Z}_{\geq 1}$, consider the following condition on an A_{inf}-module L:*

$$\textit{for } j > 0, \quad \{\mathrm{Tor}^{A_{\mathrm{inf}}}_j(L, A^{(m)}_{\mathrm{cris}}/p^n)\}_{n>0} \quad \textit{is Mittag–Leffler with vanishing eventual images,} \tag{$\star$}$$

which means concretely that for every j, n, the map $\mathrm{Tor}^{A_{\mathrm{inf}}}_j(L, A^{(m)}_{\mathrm{cris}}/p^{n'}) \to \mathrm{Tor}^{A_{\mathrm{inf}}}_j(L, A^{(m)}_{\mathrm{cris}}/p^n)$ vanishes for some $n' > n$. For a bounded complex

$$M^\bullet = \ldots \to M^i \xrightarrow{d^i} M^{i+1} \to \ldots$$

of A_{inf}-modules, if each M^i and each $H^i(M^\bullet)$ satisfy ($\star$), then, for every $i \in \mathbb{Z}$, we have

$$H^i(M^\bullet\widehat{\otimes}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}) \cong \varprojlim_n H^i(M^\bullet \otimes_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}/p^n) \cong H^i(M^\bullet)\widehat{\otimes}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}. \tag{3.30.1}$$

Proof. For an inverse system $\{0 \to I'_n \to I_n \to I''_n \to 0\}_{n>0}$ of short exact sequences of abelian groups, $\{I_n\}_{n>0}$ is Mittag–Leffler with vanishing eventual images if and only if so are both $\{I'_n\}_{n>0}$ and $\{I''_n\}_{n>0}$. Therefore, the short exact sequences

$$0 \to \mathrm{Ker}\, d^i \to M^i \to \mathrm{Im}\, d^i \to 0 \qquad \text{and} \qquad 0 \to \mathrm{Im}\, d^{i-1} \to \mathrm{Ker}\, d^i \to H^i(M^\bullet) \to 0 \tag{3.30.2}$$

imply, by descending induction on i, that each $\operatorname{Ker} d^i$ and each $\operatorname{Im} d^i$ satisfies ($\star$). Consequently, these sequences stay short exact after applying $-\widehat{\otimes}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}^{(m)}$, to the effect that the flanking terms of (3.30.1) get identified. By construction, this identification is compatible with the canonical maps to $\varprojlim_n H^i(M^\bullet \otimes_{A_{\mathrm{inf}}} A_{\mathrm{cris}}^{(m)}/p^n)$, so it remains to establish the second identification in (3.30.1).

By [SP, 0662], spectral sequences associated to a double complex give the following spectral sequences that converge to $H^{i+j}(M^\bullet \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}^{(m)}/p^n)$:

$$ {}^{(n)}E_2^{ij} = H^i(H^j(M^\bullet) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}^{(m)}/p^n) \qquad \text{and} \qquad {}^{(n)\prime}E_1^{ij} = H^j(M^i \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}^{(m)}/p^n), $$

where the differential on the ${}^{(n)\prime}E_1$-page is $H^j(d^i \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}^{(m)}/p^n)$. As n varies, both families of spectral sequences form an inverse system. Moreover, by assumption, when $j \neq 0$, the systems $\{{}^{(n)}E_2^{ij}\}_{n>0}$ and $\{{}^{(n)\prime}E_1^{ij}\}_{n>0}$ are Mittag–Leffler with vanishing eventual images. Thus, by the first sentence of the proof, when $j \neq 0$, the same holds for the systems $\{{}^{(n)}E_s^{ij}\}_{n>0}$ and $\{{}^{(n)\prime}E_s^{ij}\}_{n>0}$ for any $s \leq \infty$. Consequently, for $i \in \mathbb{Z}$, the edge maps

$$ H^i(M^\bullet) \otimes A_{\mathrm{cris}}^{(m)}/p^n \to H^i(M^\bullet \otimes^{\mathbb{L}} A_{\mathrm{cris}}^{(m)}/p^n) \quad \text{and} \quad H^i(M^\bullet \otimes^{\mathbb{L}} A_{\mathrm{cris}}^{(m)}/p^n) \to H^i(M^\bullet \otimes A_{\mathrm{cris}}^{(m)}/p^n) $$

become isomorphisms after applying the functor $\varprojlim_n$. It remains to note that then so does their composition, which is the canonical map $H^i(M^\bullet) \otimes_{A_{\mathrm{inf}}} A_{\mathrm{cris}}^{(m)}/p^n \to H^i(M^\bullet \otimes_{A_{\mathrm{inf}}} A_{\mathrm{cris}}^{(m)}/p^n)$. ☐

To make Lemma 3.30 practical to use, we now establish its condition ($\star$) in several key cases.

Lemma 3.31. *For a fixed $m \in \mathbb{Z}_{\geq 1}$, the condition ($\star$) of Lemma 3.30 holds for an A_{inf}-module L in any of the following cases*:

(i) *$m < p$ and L has no nonzero p-torsion*;

(ii) *for any $n, n' > 0$, the sequence $(p^n, \mu^{n'})$ is regular on L and $L/(p^n, \mu^{n'})$ is $A_{\mathrm{inf}}/(p^n, \mu^{n'})$-flat*;

(iii) *the module L has no nonzero p-torsion and for every $n > 0$, the quotient L/p^n is a filtered direct limit of direct sums of modules of the form $A_{\mathrm{inf}}/(\varphi^{-s}(\mu), p^n)$ for variable $s \in \mathbb{Z}_{\geq 0}$.*

Thus, ($\star$) holds for $\mathbb{A}_{\mathrm{inf}}(R_\infty)$ and $\mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu$, and for each $H^i_{\mathrm{cont}}(\Delta, N_\infty)$ and $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu)$.

Proof. If (i) holds, then $A_{\mathrm{cris}}^{(m)} \cong A_{\mathrm{inf}}$ and $\operatorname{Tor}_j^{A_{\mathrm{inf}}}(L, A_{\mathrm{cris}}^{(m)}/p^n) = 0$ for $j > 0$, so ($\star$) holds. Therefore, when arguing the assertions about (ii) and (iii), we may assume that $m \geq p$, so that each $A_{\mathrm{cris}}^{(m)}/p^n$ is an $A_{\mathrm{inf}}/(p^n, \mu^{n'})$-algebra for some $n' > 0$ (see §3.26).

If (ii) holds, then the regular sequence aspect ensures that $L \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/(p^n, \mu^{n'}) \cong L/(p^n, \mu^{n'})$. Thus, the flatness aspect implies that $L \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}^{(m)}/p^n$ is concentrated in degree 0, i.e., that the systems in ($\star$) vanish termwise, so that ($\star$) holds in this case.

For $s \in \mathbb{Z}_{\geq 0}$, one has $\varphi^{-s}(\mu) \mid \mu$, so (3.26.3) ensures that for every $n \in \mathbb{Z}_{\geq 0}$ there is an $n' > n$ such that the reduction modulo p^n map

$$ \operatorname{Tor}_1^{A_{\mathrm{inf}}/p^{n'}}(A_{\mathrm{inf}}/(\varphi^{-s}(\mu), p^{n'}), A_{\mathrm{cris}}^{(m)}/p^{n'}) \cong (A_{\mathrm{cris}}^{(m)}/p^{n'})[\varphi^{-s}(\mu)] \to A_{\mathrm{cris}}^{(m)}/p^n $$

vanishes for every $s \geq 0$. Thus, if (iii) holds, then for every $j > 0$ the transition map from position n' to position n vanishes in the projective system

$$ \{\operatorname{Tor}_j^{A_{\mathrm{inf}}/p^n}(L/p^n, A_{\mathrm{cris}}^{(m)}/p^n)\}_{n>0} \cong \{\operatorname{Tor}_j^{A_{\mathrm{inf}}}(L, A_{\mathrm{cris}}^{(m)}/p^n)\}_{n>0}, \tag{3.31.1} $$

where the termwise identification follows from the p-torsion freeness of L, more precisely, from the fact that $L \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/p^n$ is concentrated in degree 0. Consequently, (iii) implies ($\star$).

By §3.14 and Lemma 3.13, (ii) holds for $\mathbb{A}_{\mathrm{inf}}(R_\infty)$ and, by additionally using Lazard's theorem, (iii) holds for $\mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu$. Likewise, Proposition 3.19, Corollary 3.23, and Lazard's theorem imply that (iii) holds for each $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu)$. By Lemma 3.7, $H^i_{\mathrm{cont}}(\Delta, N_\infty)$ vanishes for large i. Therefore, similarly to the proof of Lemma 3.30, the following short exact sequences that result from Proposition 3.25:

$$0 \to H^i_{\mathrm{cont}}(\Delta, N_\infty) \to H^i_{\mathrm{cont}}(\Delta, N_\infty/\mu) \to H^{i+1}_{\mathrm{cont}}(\Delta, N_\infty) \to 0 \tag{3.31.2}$$

show, by induction on i, that ($\star$) for $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu)$ implies ($\star$) for $H^i_{\mathrm{cont}}(\Delta, N_\infty)$. ∎

Thanks to Lemma 3.31, we may apply Lemma 3.30 to draw the following concrete consequences.

Proposition 3.32. *For every $m \in \mathbb{Z}_{\geq 1}$ and $i \in \mathbb{Z}$, we have the identifications*

$$H^i_{\mathrm{cont}}(\Delta, N_\infty \widehat{\otimes}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}) \cong \varprojlim_n H^i_{\mathrm{cont}}(\Delta, N_\infty \otimes_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}/p^n) \cong H^i_{\mathrm{cont}}(\Delta, N_\infty) \widehat{\otimes}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}. \tag{3.32.1}$$

In particular, μ kills every $H^i_{\mathrm{cont}}(\Delta, N_\infty \widehat{\otimes}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}})$.

Proof. By Lemma 3.7, a Koszul complex $M^\bullet$ of N_∞ with respect to Δ satisfies

$$H^i(M^\bullet) \cong H^i_{\mathrm{cont}}(\Delta, N_\infty) \qquad \text{and} \qquad H^i(M^\bullet \widehat{\otimes} A^{(m)}_{\mathrm{cris}}) \cong H^i_{\mathrm{cont}}(\Delta, N_\infty \widehat{\otimes}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}),$$

as well as $H^i(M^\bullet \otimes A^{(m)}_{\mathrm{cris}}/p^n) \cong H^i_{\mathrm{cont}}(\Delta, N_\infty \otimes A^{(m)}_{\mathrm{cris}}/p^n)$ for every $n > 0$. Moreover, by Lemma 3.31, each M^i and each $H^i(M^\bullet)$ satisfy ($\star$). Thus, (3.32.1) is a special case of (3.30.1). Finally, by Proposition 3.25, μ kills every $H^i_{\mathrm{cont}}(\Delta, N_\infty)$, so, by (3.32.1), it also kills every $H^i_{\mathrm{cont}}(\Delta, N_\infty \widehat{\otimes}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}})$. ∎

Proposition 3.33. *For every $m \in \mathbb{Z}_{\geq 1}$ and $i \in \mathbb{Z}$, we have the identifications*

$$H^i_{\mathrm{cont}}(\Delta, \mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)/\mu) \cong \varprojlim_n H^i_{\mathrm{cont}}(\Delta, \mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)/(\mu, p^n)) \cong H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu) \widehat{\otimes}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}.$$

Moreover, the A_{inf}-module $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)/\mu)$ has no nonzero $W(\mathfrak{m}^\flat)$-torsion.

Proof. Similarly to the proof of Proposition 3.32, Lemma 3.30 applied to the Koszul complex of $\mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu$ proves the identifications. Thus, for the claim about the $W(\mathfrak{m}^\flat)$-torsion, it suffices to prove that each

$$H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu) \otimes_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}/p^n \overset{(3.19.1)}{\cong} H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/(\mu, p^n)) \otimes_{A_{\mathrm{inf}}/p^n} A^{(m)}_{\mathrm{cris}}/p^n$$

has no nonzero $W(\mathfrak{m}^\flat)$-torsion. Since Δ acts trivially on $A(R)/(\mu, p^n)$, Lemma 3.7 and Proposition 3.29 imply that each $H^i_{\mathrm{cont}}(\Delta, A(R)/(\mu, p^n)) \otimes_{A_{\mathrm{inf}}/p^n} A^{(m)}_{\mathrm{cris}}/p^n$ has no nonzero $W(\mathfrak{m}^\flat)$-torsion. Consequently, due to the decomposition (3.22.1), it suffices to prove that each

$$H^i_{\mathrm{cont}}(\Delta, N_j/(\mu, p^n)) \otimes_{A_{\mathrm{inf}}/p^n} A^{(m)}_{\mathrm{cris}}/p^n \overset{3.23}{\cong} H^i_{\mathrm{cont}}(\Delta, N_j/(\mu, p^n)) \otimes_{A_{\mathrm{inf}}/(\varphi^{-j}(\mu), p^n)} A^{(m)}_{\mathrm{cris}}/(\varphi^{-j}(\mu), p^n)$$

with $j > 0$ has no nonzero $W(\mathfrak{m}^\flat)$-torsion. For this, similarly to the proof of Proposition 3.29, we will approximate by Noetherian rings. More precisely, similarly to (3.29.1), the assignment

$$T \mapsto [\epsilon]^{1/p^j} - 1 \qquad \text{defines a } \mathbb{Z}_p\text{-algebra morphism} \qquad \mathbb{Z}_p[\![T]\!] \to A_{\mathrm{inf}},$$

with respect to which A_{inf} is $\mathbb{Z}[\![T]\!]$-flat. The A_{inf}-algebra $A^{(m)}_{\mathrm{cris}}/(\varphi^{-j}(\mu), p^n)$ is then identified with the $A_{\mathrm{inf}}/(\varphi^{-j}(\mu), p^n)$-base change of the mod (T, p^n) reduction M of the $\mathbb{Z}_p[\![T]\!]$-subalgebra of

$\mathbb{Z}_p[\![T]\!][\frac{1}{p}]$ generated by the elements $\frac{1}{s!}\sum_{i=0}^{p-1}(T+1)^{p^{j-1}\cdot i}$ with $s \le m$. Consequently, we need to prove that

$$H^i_{\mathrm{cont}}(\Delta, N_j/(\mu, p^n)) \otimes_{\mathbb{Z}_p[\![T]\!]/(T,p^n)} M$$

has no nonzero $W(\mathfrak{m}^\flat)$-torsion. In fact, since, by Corollary 3.23, the module $H^i_{\mathrm{cont}}(\Delta, N_j/(\mu, p^n))$ is $\mathbb{Z}_p[\![T]\!]/(T, p^n)$-flat and M is a successive extension of direct sums of $\mathbb{F}_p$, it suffices to prove that $H^i_{\mathrm{cont}}(\Delta, N_j/(\mu, p^n))/p$ has no nonzero $W(\mathfrak{m}^\flat)$-torsion. This, in turn, follows from Proposition 3.19 and Lemma 3.12. $\square$

With Proposition 3.33 in hand, we are ready for the promised claim about $L\eta_{(\mu)}(e\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}})$:

Theorem 3.34. *For each $m \ge p$, the map $e\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}$ from (3.28.1) induces the isomorphism*

$$L\eta_{(\mu)}(e\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}})\colon L\eta_{(\mu)}(R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty))) \xrightarrow{\sim} L\eta_{(\mu)}(R\Gamma_{\mathrm{pro\acute{e}t}}(X^{\mathrm{ad}}_{\mathrm{pro\acute{e}t}}, \mathbb{A}_{\mathrm{inf},X})\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}).$$

Proof. By (3.28.2), the ideal $W(\mathfrak{m}^\flat) \subset A_{\mathrm{inf}}$ kills the cohomology of $\mathrm{Cone}(e\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}})$. By Proposition 3.33 (and the projection formula [SP, 0944] with (3.27.2)), the cohomology of

$$R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu$$

has no nonzero $W(\mathfrak{m}^\flat)$-torsion. Thus, Lemma 3.18 applies and gives the desired conclusion. $\square$

Remark 3.35. Analogously to Remark 3.21, we may extend Theorem 3.34 to any affinoid perfectoid Δ'-cover that refines $\mathfrak{X}^{\mathrm{ad}}_{C,\infty} \to \mathfrak{X}^{\mathrm{ad}}_C$: more precisely, in the notation used there, we have

$$L\eta_{(\mu)}(e'\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}})\colon L\eta_{(\mu)}(R\Gamma_{\mathrm{cont}}(\Delta', \mathbb{A}^{(m)}_{\mathrm{cris}}(R'_\infty))) \xrightarrow{\sim} L\eta_{(\mu)}(R\Gamma_{\mathrm{pro\acute{e}t}}(X^{\mathrm{ad}}_{\mathrm{pro\acute{e}t}}, \mathbb{A}_{\mathrm{inf},X})\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}),$$

where $\mathbb{A}^{(m)}_{\mathrm{cris}}(R'_\infty) := \mathbb{A}_{\mathrm{inf}}(R'_\infty)\widehat{\otimes}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}$. Indeed, as there (see also Lemma 3.13 and §3.28), the ideal $W(\mathfrak{m}^\flat)$ kills the cohomology of the cone of the map $R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)) \to R\Gamma_{\mathrm{cont}}(\Delta', \mathbb{A}^{(m)}_{\mathrm{cris}}(R'_\infty))$, so Lemma 3.18 applies to it and gives the claim.

4. The de Rham specialization of $A\Omega_{\mathfrak{X}}$

The main goal of this section is to identify the de Rham specialization of $A\Omega_{\mathfrak{X}}$ with the logarithmic de Rham complex of $\mathfrak{X}$ over $\mathcal{O}_C$ (see Theorem 4.16). The key steps for this are the identification and the analysis of the Hodge–Tate specialization of $A\Omega_{\mathfrak{X}}$ in Theorems 4.2 and 4.11. These steps were also used in the smooth case in [BMS16, §8 and §9] but, due to the difficulties mentioned in the beginning of §3, we carry them out differently. Namely, we rely on the analysis of group cohomology from §3 and, in the identification step, we use Lemma 3.24 (which comes from [Bha16]).

4.1. The presheaf version $A\Omega^{\mathrm{psh}}_{\mathfrak{X}}$. In addition to the étale site $\mathfrak{X}_{\acute{e}t}$, we consider the site $\mathfrak{X}^{\mathrm{psh}}_{\acute{e}t}$ whose objects are those connected affine opens of $\mathfrak{X}_{\acute{e}t}$ that have an étale coordinate map (3.1.1) and coverings are isomorphisms over $\mathfrak{X}$. Thus, the topology of $\mathfrak{X}^{\mathrm{psh}}_{\acute{e}t}$ is the coarsest one possible and any presheaf is already a sheaf. Since $\mathfrak{X}^{\mathrm{psh}}_{\acute{e}t}$ is a subcategory of $\mathfrak{X}_{\acute{e}t}$, there is an evident morphism of sites

$$\phi\colon \mathfrak{X}^{\mathrm{shv}}_{\acute{e}t} \to \mathfrak{X}^{\mathrm{psh}}_{\acute{e}t}, \tag{4.1.1}$$

for which the pushforward ϕ_* is given by restricting and, since the objects of $\mathfrak{X}^{\mathrm{psh}}_{\acute{e}t}$ form a basis of $\mathfrak{X}_{\acute{e}t}$, the pullback ϕ^{-1} is given by sheafifying. In particular, (ϕ^{-1}, ϕ_*) constitutes a morphism of topoi, and, in addition, since any sheaf is the sheafification of its associated presheaf, $\phi^{-1} \circ \phi_* \cong \mathrm{id}$.

We set

$$\nu^{\mathrm{psh}} := \phi \circ \nu \colon (\mathfrak{X}_C^{\mathrm{ad}})_{\mathrm{pro\acute{e}t}} \to \mathfrak{X}_{\mathrm{\acute{e}t}}^{\mathrm{psh}} \qquad \text{and} \qquad A\Omega_{\mathfrak{X}}^{\mathrm{psh}} := L\eta_{(\mu)}(R\nu_*^{\mathrm{psh}}(\mathbb{A}_{\mathrm{inf},\, \mathfrak{X}_C^{\mathrm{ad}}})), \tag{4.1.2}$$

so that, explicitly, for every object $\mathfrak{U}$ of $\mathfrak{X}_{\mathrm{\acute{e}t}}^{\mathrm{psh}}$, we have

$$R\Gamma(\mathfrak{U}, A\Omega_{\mathfrak{X}}^{\mathrm{psh}}) \cong L\eta_{(\mu)}(R\Gamma((\mathfrak{U}_C^{\mathrm{ad}})_{\mathrm{pro\acute{e}t}}, \mathbb{A}_{\mathrm{inf},\, \mathfrak{U}_C^{\mathrm{ad}}})). \tag{4.1.3}$$

Since, by [BMS16, 6.19], the functor $L\eta$ preserves derived completeness when used in the context of a *replete* topos (such as that of sets), the identification (4.1.3) shows in particular that the object $A\Omega_{\mathfrak{X}}^{\mathrm{psh}}$ is derived ξ-adically (and also $\varphi(\xi)$-adically) complete (compare with Corollary 4.6 below).

Since the functor of $L\eta$ commutes with pullback under flat morphisms of ringed topoi (see [BMS16, 6.14]) and any sheaf on $\mathfrak{X}_{\mathrm{\acute{e}t}}$ is the sheafification of its restriction to $\mathfrak{X}_{\mathrm{\acute{e}t}}^{\mathrm{psh}}$, we have

$$\phi^{-1}(A\Omega_{\mathfrak{X}}^{\mathrm{psh}}) \cong A\Omega_{\mathfrak{X}}. \tag{4.1.4}$$

Armed with this formalism, we now identify the Hodge–Tate specialization of $A\Omega_{\mathfrak{X}}$.

Theorem 4.2. *We have the identification*

$$A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\, \theta\circ\varphi^{-1}} \mathcal{O}_C \xrightarrow{\sim} L\eta_{(\zeta_p - 1)}(R\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}})), \tag{4.2.1}$$

where on the right side $L\eta$ is formed with respect to the ideal sheaf $(\zeta_p - 1)\mathcal{O}_{\mathfrak{X},\, \mathrm{\acute{e}t}} \subset \mathcal{O}_{\mathfrak{X},\, \mathrm{\acute{e}t}}$. If the étale morphisms (1.5.1) exist Zariski locally on $\mathfrak{X}$, then (4.2.1) also holds for $A\Omega_{\mathfrak{X}_{\mathrm{Zar}}}$ (see (2.2.5)).

Proof. The kernel of $\theta_{\mathfrak{X}_C^{\mathrm{ad}}} \circ \varphi^{-1} \colon \mathbb{A}_{\mathrm{inf},\, \mathfrak{X}_C^{\mathrm{ad}}} \twoheadrightarrow \widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}}$ is generated by the nonzero divisor $\varphi(\xi)$ (see §2.2), so the projection formula [SP, 0944] provides the identification

$$R\nu_*(\mathbb{A}_{\mathrm{inf},\, \mathfrak{X}_C^{\mathrm{ad}}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\, \theta\circ\varphi^{-1}} \mathcal{O}_C \cong R\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}}). \tag{4.2.2}$$

Since $(\theta \circ \varphi^{-1})(\mu) = \zeta_p - 1$, this identification induces the map (4.2.1), and likewise we also obtain the presheaf version:

$$A\Omega_{\mathfrak{X}}^{\mathrm{psh}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\, \theta\circ\varphi^{-1}} \mathcal{O}_C \to L\eta_{(\zeta_p - 1)}(R\phi_*(R\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}}))). \tag{4.2.3}$$

The functor ϕ^{-1} brings (4.2.3) to (4.2.1) (compare with (4.1.4)), so it remains to show that (4.2.3) is an isomorphism.

For every object $\mathfrak{U} = \mathrm{Spf}(R)$ of $\mathfrak{X}_{\mathrm{\acute{e}t}}^{\mathrm{psh}}$ equipped with an étale morphism as in (3.1.1), Proposition 3.19 and (3.20.1) ensure that the cohomology of $R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu$ is p-torsion free. Thus, since $\varphi(\xi) \equiv p \bmod (\mu)$ (see §2.1), [Bha16, 5.14 and its proof] imply that

$$L\eta_{(\mu)}(R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty))) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\, \theta\circ\varphi^{-1}} \mathcal{O}_C \xrightarrow{\sim} L\eta_{(\zeta_p - 1)}(R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\, \theta\circ\varphi^{-1}} \mathcal{O}_C).$$

Since the maps (3.3.1) and (3.15.1) are compatible, Theorem 3.9 and Theorem 3.20 then imply that

$$L\eta_{(\mu)}(R\Gamma((\mathfrak{U}_C^{\mathrm{ad}})_{\mathrm{pro\acute{e}t}}, \mathbb{A}_{\mathrm{inf}})) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\, \theta\circ\varphi^{-1}} \mathcal{O}_C \xrightarrow{\sim} L\eta_{(\zeta_p - 1)}(R\Gamma((\mathfrak{U}_C^{\mathrm{ad}})_{\mathrm{pro\acute{e}t}}, \mathbb{A}_{\mathrm{inf}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\, \theta\circ\varphi^{-1}} \mathcal{O}_C).$$

This shows that (4.2.3) is indeed an isomorphism on every $\mathfrak{U}$, as desired. ∎

4.3. The object $\widetilde{\Omega}_{\mathfrak{X}}$. To proceed further, we need to analyze the right side of (4.2.1). For the sake of brevity, we denote it by

$$\widetilde{\Omega}_{\mathfrak{X}} := L\eta_{(\zeta_p - 1)}(R\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}})) \in D^{\geq 0}(\mathcal{O}_{\mathfrak{X},\, \mathrm{\acute{e}t}}), \tag{4.3.1}$$

where, as in Theorem 4.2, the functor $L\eta$ is formed with respect to the ideal sheaf $(\zeta_p - 1)\mathcal{O}_{\mathfrak{X},\, \mathrm{\acute{e}t}}$.

Proposition 4.4. *For each $i \geq 0$, the $\mathcal{O}_{\mathfrak{X},\, \text{ét}}$-module $H^i(\widetilde{\Omega}_{\mathfrak{X}})$ is locally free and its rank at a closed point x of $\mathfrak{X}_k$ is equal to $\binom{\dim_x(\mathfrak{X}_k)}{i}$; in particular, each $H^i(\widetilde{\Omega}_{\mathfrak{X}})/p^n$ is a quasi-coherent $\mathcal{O}_{\mathfrak{X},\, \text{ét}}/p^n$-module. In addition,*

$$\nu^\sharp \colon \mathcal{O}_{\mathfrak{X},\, \text{ét}} \xrightarrow{\sim} \nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\text{ad}}}), \qquad \textit{so that} \qquad H^0(\widetilde{\Omega}_{\mathfrak{X}}) \cong \mathcal{O}_{\mathfrak{X},\, \text{ét}}. \tag{4.4.1}$$

Proof. The claims are étale local (see [SP, 058S]), so we assume that $\mathfrak{X} = \operatorname{Spf} R$, that $\mathfrak{X}$ is connected, and that for some $q \in \mathbb{Q}_{>0}$, there is an étale $\operatorname{Spf}(\mathcal{O}_C)$-morphism as in (1.5.1):

$$\mathfrak{X} = \operatorname{Spf} R \to \operatorname{Spf} R^\square =: \mathfrak{X}^\square \quad \text{with} \quad R^\square := \mathcal{O}_C\{t_0, \dots, t_r, t_{r+1}^{\pm 1}, \dots, t_d^{\pm 1}\}/(t_0 \cdots t_r - p^q). \tag{4.4.2}$$

In particular, this places us in the local setup of §3.1, so the results of §3 apply.

Since R is $R^\square$-flat and Δ acts trivially on $R^\square$ and R, Lemma 3.7 and Proposition 3.8 imply that

$$R^{\oplus \binom{d}{i}} \cong H^i_{\text{cont}}(\Delta, R^\square) \otimes_{R^\square} R \cong \frac{H^i_{\text{cont}}(\Delta, R^\square_\infty)}{H^i_{\text{cont}}(\Delta, R^\square_\infty)[\zeta_p - 1]} \otimes_{R^\square} R \xrightarrow{\sim} \frac{H^i_{\text{cont}}(\Delta, R_\infty)}{H^i_{\text{cont}}(\Delta, R_\infty)[\zeta_p - 1]}. \tag{4.4.3}$$

Therefore, since the map e of (3.3.1) is compatible with its analogue $e^\square$ for $R^\square$, Theorem 3.9 shows that the base change morphism

$$\frac{H^i((\mathfrak{X}^\square)^{\text{ad}}_C, \widehat{\mathcal{O}}^+)}{H^i((\mathfrak{X}^\square)^{\text{ad}}_C, \widehat{\mathcal{O}}^+)[\zeta_p - 1]} \otimes_{R^\square} R \to \frac{H^i(\mathfrak{X}^{\text{ad}}_C, \widehat{\mathcal{O}}^+)}{H^i(\mathfrak{X}^{\text{ad}}_C, \widehat{\mathcal{O}}^+)[\zeta_p - 1]} \tag{4.4.4}$$

is an isomorphism of free R-modules of rank $\binom{d}{i}$. Since the connected affine $\mathfrak{X}$ is arbitrary (subject to (4.4.2)), we conclude that

$$\frac{H^i((\mathfrak{X}^\square)^{\text{ad}}_C, \widehat{\mathcal{O}}^+)}{H^i((\mathfrak{X}^\square)^{\text{ad}}_C, \widehat{\mathcal{O}}^+)[\zeta_p - 1]} \otimes_{R^\square} \mathcal{O}_{\operatorname{Spf} R,\, \text{ét}} \xrightarrow{\sim} \frac{R^i \nu_*(\widehat{\mathcal{O}}^+)}{(R^i \nu_*(\widehat{\mathcal{O}}^+))[\zeta_p - 1]} \overset{\text{[BMS16, 6.4]}}{\cong} H^i(\widetilde{\Omega}_{\mathfrak{X}}) \tag{4.4.5}$$

and that $H^i(\widetilde{\Omega}_{\mathfrak{X}})$ is free of rank $\binom{d}{i}$, as desired.

For (4.4.1), due to the discussion in §3.3, we need to show that $R \xrightarrow{\sim} (R_\infty)^\Delta$. However, this map is an inclusion of a direct summand whose complementary summand M_∞^Δ is both p-torsion free and, by Proposition 3.8, killed by $\zeta_p - 1$, so the claim follows. $\square$

Remark 4.5. The proof of Proposition 4.4, specifically, (4.4.4) and (4.4.5), shows that if $\mathfrak{X}$ is affine, connected, and admits a coordinate map (4.4.2), then the presheaf which to a variable $\mathfrak{X}$-étale affine $\mathfrak{X}'$ assigns $\frac{H^i(\mathfrak{X}'^{\text{ad}}_C, \widehat{\mathcal{O}}^+)}{H^i(\mathfrak{X}'^{\text{ad}}_C, \widehat{\mathcal{O}}^+)[\zeta_p - 1]}$ is already a sheaf and that

$$H^i(\widetilde{\Omega}_{\mathfrak{X}^\square}) \otimes_{R^\square} R \xrightarrow{\sim} H^i(\widetilde{\Omega}_{\mathfrak{X}}). \tag{4.5.1}$$

In particular, if the coordinate maps (4.4.2) exist Zariski locally on $\mathfrak{X}$ (for instance, if $\mathfrak{X}$ is $\mathcal{O}_C$-smooth or arises as in (1.5.3) from a strictly semistable $\mathcal{X}$), then the sheaves $H^i(\widetilde{\Omega}_{\mathfrak{X}})$ may be computed using the Zariski topology: more precisely, then the object $\widetilde{\Omega}_{\mathfrak{X}_{\text{Zar}}}$ defined by the formula (4.3.1) using the Zariski topology of $\mathfrak{X}$ satisfies

$$H^i(\widetilde{\Omega}_{\mathfrak{X}_{\text{Zar}}}) \xrightarrow{\sim} (H^i(\widetilde{\Omega}_{\mathfrak{X}}))|_{\mathfrak{X}_{\text{Zar}}} \qquad \text{for every } i \in \mathbb{Z}_{\geq 0}. \tag{4.5.2}$$

Corollary 4.6. *The following adjunction map is an isomorphism:*

$$A\Omega^{\text{psh}}_{\mathfrak{X}} \xrightarrow{\sim} R\phi_*(A\Omega_{\mathfrak{X}}) \cong R\phi_*(\phi^{-1}(A\Omega^{\text{psh}}_{\mathfrak{X}})) \tag{4.6.1}$$

(see §4.1 for the identification) and $A\Omega_{\mathfrak{X}}$ is derived ξ-adically complete.

Proof. Since $\phi^{-1} \circ R\phi_* \cong \mathrm{id}$ and $A\Omega_{\mathfrak{X}}^{\mathrm{psh}}$ is derived ξ-adically complete (see §4.1), the second assertion follows from the first: indeed, for the derived ξ-adic completeness, it suffices to check that the map

$$A\Omega_{\mathfrak{X}} \to R\lim_n (A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\xi^n)$$

becomes an isomorphism after applying $R\phi_*$.

As for the first assertion, namely, (4.6.1), we may assume that $\mathfrak{X}$ is connected and admits an étale morphism (1.5.1). In addition, since $A\Omega_{\mathfrak{X}}^{\mathrm{psh}}$ is derived $\varphi(\xi)$-adically complete (see §4.1), the $\mathfrak{X}_{\mathrm{\acute{e}t}}^{\mathrm{psh}}$-analogue of [BMS16, 9.15] allows us to replace $A\Omega_{\mathfrak{X}}^{\mathrm{psh}}$ in (4.6.1) by $A\Omega_{\mathfrak{X}}^{\mathrm{psh}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/(\varphi(\xi)^n)$. Then, due to the five lemma, it suffices to establish (4.6.1) for $A\Omega_{\mathfrak{X}}^{\mathrm{psh}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/(\varphi(\xi))$ in place of $A\Omega_{\mathfrak{X}}^{\mathrm{psh}}$. However, by (4.2.3), the object $A\Omega_{\mathfrak{X}}^{\mathrm{psh}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/(\varphi(\xi))$ is identified with the presheaf analogue $\Omega_{\mathfrak{X}}^{\mathrm{psh}}$ of $\Omega_{\mathfrak{X}}$ defined as the right side of (4.2.3). By Remark 4.5, the cohomology modules of this presheaf analogue are already sheaves, so the desired (4.6.1) holds for $\Omega_{\mathfrak{X}}^{\mathrm{psh}}$. ☐

Our next task is to identify the vector bundles $H^i(\widetilde{\Omega}_{\mathfrak{X}})$ with the twists of bundles given by the logarithmic differentials (see Theorem 4.11). For this, we first express $H^i(\widetilde{\Omega}_{\mathfrak{X}})$ as $\bigwedge^i H^1(\widetilde{\Omega}_{\mathfrak{X}})$ in Proposition 4.8, and then construct a map (4.10.2) that relates $H^1(\widetilde{\Omega}_{\mathfrak{X}})$ to Kähler differentials.

4.7. The cup product maps. By [SP, 0B6C],[4] there is a cup product map

$$R\nu_*(\widehat{\mathcal{O}}^+) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X},\mathrm{\acute{e}t}}} R\nu_*(\widehat{\mathcal{O}}^+) \to R\nu_*(\widehat{\mathcal{O}}^+). \tag{4.7.1}$$

Moreover, arguments analogous to those used to construct the map [SP, 068H] give product maps

$$R^j\nu_*(\widehat{\mathcal{O}}^+) \otimes_{\mathcal{O}_{\mathfrak{X},\mathrm{\acute{e}t}}} R^{j'}\nu_*(\widehat{\mathcal{O}}^+) \xrightarrow{-\cup-} H^{j+j'}(R\nu_*(\widehat{\mathcal{O}}^+) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X},\mathrm{\acute{e}t}}} R\nu_*(\widehat{\mathcal{O}}^+)), \tag{4.7.2}$$

which satisfy $x \cup y = (-1)^{jj'} y \cup x$ (see [SP, 0BYI]) and combine with (4.7.1) to give the map

$$\textstyle\bigotimes_{s=1}^{i} R^1\nu_*(\widehat{\mathcal{O}}^+) \to R^i\nu_*(\widehat{\mathcal{O}}^+) \qquad \text{for} \quad i \in \mathbb{Z}_{>0}. \tag{4.7.3}$$

Proposition 4.8. *For each $i > 0$, the map (4.7.3) induces the isomorphism*

$$\textstyle\bigwedge^i \left(\frac{R^1\nu_*(\widehat{\mathcal{O}}^+)}{R^1\nu_*(\widehat{\mathcal{O}}^+)[\zeta_p - 1]}\right) \cong \bigwedge^i H^1(\widetilde{\Omega}_{\mathfrak{X}}) \xrightarrow{\sim} H^i(\widetilde{\Omega}_{\mathfrak{X}}) \cong \frac{R^i\nu_*(\widehat{\mathcal{O}}^+)}{R^i\nu_*(\widehat{\mathcal{O}}^+)[\zeta_p - 1]}. \tag{4.8.1}$$

Proof. By Proposition 4.4, each $H^i(\widetilde{\Omega}_{\mathfrak{X}})$ has no nontrivial 2-torsion, so the "antisymmetric in each pair of variables" map (4.7.3) indeed induces the $\mathcal{O}_{\mathfrak{X},\mathrm{\acute{e}t}}$-module map (4.8.1). For the isomorphism claim, we may work étale locally, so we put ourselves in the situation (4.4.2). The edge maps

$$e\colon H^i_{\mathrm{cont}}(\Delta, R_\infty) \to H^i(\mathfrak{X}_C^{\mathrm{ad}}, \widehat{\mathcal{O}}^+)$$

of (3.3.1) are compatible with cup products: in order to check this one identifies $H^i(\mathfrak{X}_C^{\mathrm{ad}}, \widehat{\mathcal{O}}^+)$ with the direct limit of the i^{th} Čech cohomology groups of $\widehat{\mathcal{O}}^+$ with respect to a variable proétale hypercovering of $\mathfrak{X}_C^{\mathrm{ad}}$ (see [SP, 01H0]) and uses the hypercovering construction of the cup product (see [SP, 01FP]). Due to Theorem 3.9 and (4.4.3), it then remains to argue that the identification

$$H^1_{\mathrm{cont}}(\Delta, R^\square) \overset{3.7}{\cong} (R^\square)^d \qquad \text{induces} \qquad H^i_{\mathrm{cont}}(\Delta, R^\square) \overset{3.7}{\cong} \textstyle\bigwedge^i (R^\square)^d$$

via the cup product, which follows from [BMS16, 7.3 and 7.5]. ☐

To relate $H^1(\widetilde{\Omega}_{\mathfrak{X}})$ to Kähler differentials, we now review the needed material on cotangent complexes.

[4] *Loc. cit.* applies in its present form because $(\mathfrak{X}_C^{\mathrm{ad}})_{\mathrm{pro\acute{e}t}}$ has enough points by [Sch13e, (2)].

4.9. The completed cotangent complex $\widehat{\mathbb{L}}_{\widehat{\mathcal{O}}^+/\mathbb{Z}_p}$. Affinoid perfectoids form a basis of the proétale topology of $\mathfrak{X}^{\mathrm{ad}}_C$ (see [Sch13, 4.7]). Therefore, [BMS16, 3.14] ensures that for the sheaf of rings $\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{ad}}_C}$ defined in (2.2.1), the cotangent complex $\mathbb{L}_{\widehat{\mathcal{O}}^+/\mathcal{O}_C} \in D^{\leq 0}(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{ad}}_C})$ (whose levelwise terms are $\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{ad}}_C}$-flat) satisfies

$$\mathbb{L}_{\widehat{\mathcal{O}}^+/\mathcal{O}_C} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \cong 0, \qquad \text{and hence also} \qquad \widehat{\mathbb{L}}_{\widehat{\mathcal{O}}^+/\mathcal{O}_C} \cong 0.$$

Consequently, derived p-adic completion turns the canonical morphism

$$\mathbb{L}_{\mathcal{O}_C/\mathbb{Z}_p} \otimes_{\mathcal{O}_C} \widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{ad}}_C} \to \mathbb{L}_{\widehat{\mathcal{O}}^+/\mathbb{Z}_p} \qquad \text{into an isomorphism} \qquad (\mathbb{L}_{\mathcal{O}_C/\mathbb{Z}_p} \otimes_{\mathcal{O}_C} \widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{ad}}_C})\widehat{\ } \xrightarrow{\sim} \widehat{\mathbb{L}}_{\widehat{\mathcal{O}}^+/\mathbb{Z}_p}$$

in the derived category. By [GR03, 6.5.12 (ii)], the complex $\mathbb{L}_{\mathcal{O}_C/\mathbb{Z}_p}$ is quasi-isomorphic to $\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}$ placed in degree 0. The p-divisibility of $\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}$ then ensures that

$$\mathbb{L}_{\mathcal{O}_C/\mathbb{Z}_p} \otimes^{\mathbb{L}}_{\mathcal{O}_C} (\widehat{\mathcal{O}}^+/p^n\widehat{\mathcal{O}}^+) \cong (\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}[p^n] \otimes_{\mathcal{O}_C} \widehat{\mathcal{O}}^+)[1] \overset{\text{[Sch13, 4.2 (iii)]}}{\cong} (\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}[p^n] \otimes_{\mathcal{O}_C} (\mathcal{O}^+/p^n\mathcal{O}^+))[1],$$

where $\mathcal{O}^+$ abbreviates the integral structure sheaf $\mathcal{O}^+_{\mathfrak{X}^{\mathrm{ad}}_C}$. Moreover, by [Fon82, Thm. 1′ (ii)],[5]

$$\mathcal{O}_C\{1\} := \varprojlim_n \left(\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}[p^n]\right) \qquad \text{is a free } \mathcal{O}_C\text{-module of rank 1.}$$

In conclusion, letting $\{1\}$ abbreviate the $\mathcal{O}_C$-tensor product with $\mathcal{O}_C\{1\}$, we obtain an isomorphism

$$(\mathbb{L}_{\mathcal{O}_C/\mathbb{Z}_p} \otimes_{\mathcal{O}_C} \widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{ad}}_C})\widehat{\ } \cong \widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{ad}}_C}\{1\}[1], \quad \text{and hence also } \widehat{\mathbb{L}}_{\widehat{\mathcal{O}}^+/\mathbb{Z}_p} \cong \widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{ad}}_C}\{1\}[1], \quad \text{in } D(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{ad}}_C}). \tag{4.9.1}$$

4.10. The relation between $\widetilde{\Omega}_{\mathfrak{X}}$ and Kähler differentials. We equip the étale site $\mathfrak{X}_{\text{ét}}$ with the sheaf of rings $\mathcal{O}_{\mathfrak{X},\,\text{ét}}$ and the proétale site $(\mathfrak{X}^{\mathrm{ad}}_C)_{\text{proét}}$ with $\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{ad}}_C}$, so that (ν^{-1}, ν_*) from §1.5 becomes a morphism of ringed of topoi (see [Hub96, 1.9.1 b)]). In particular, we obtain the pullback morphism

$$\widehat{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X},\,\text{ét}}/\mathbb{Z}_p} \to R\nu_*(\widehat{\mathbb{L}}_{\widehat{\mathcal{O}}^+/\mathbb{Z}_p}) \overset{(4.9.1)}{\cong} R\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{ad}}_C}\{1\})[1]. \tag{4.10.1}$$

To explicate its source, we note that an argument analogous to that of §4.9 gives

$$(\mathbb{L}_{\mathcal{O}_C/\mathbb{Z}_p} \otimes_{\mathcal{O}_C} \mathcal{O}_{\mathfrak{X},\,\text{ét}})\widehat{\ } \cong \mathcal{O}_{\mathfrak{X},\,\text{ét}}\{1\}[1], \quad \text{so} \quad H^0(\widehat{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X},\,\text{ét}}/\mathbb{Z}_p}) \cong H^0(\widehat{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X},\,\text{ét}}/\mathcal{O}_C}) \overset{\text{[GR03, 7.2.4, 7.2.8]}}{\cong} \Omega^1_{\mathfrak{X}/\mathcal{O}_C}.$$

Moreover, by [GR03, 7.2.10 (iii)], the last identification induces a quasi-isomorphism between

$$\widehat{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}^{\mathrm{sm}},\,\text{ét}}/\mathcal{O}_C} \qquad \text{and} \qquad \Omega^1_{\mathfrak{X}^{\mathrm{sm}}/\mathcal{O}_C} \text{ placed in degree 0.}$$

Consequently, by applying $H^0(-)$ to the map (4.10.1) and twisting by $\mathcal{O}_C\{-1\}$ we obtain the first map in the following composition of $\mathcal{O}_{\mathfrak{X},\,\text{ét}}$-module morphisms:

$$\Omega^1_{\mathfrak{X}/\mathcal{O}_C}\{-1\} \to R^1\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{ad}}_C}) \twoheadrightarrow \frac{R^1\nu_*(\widehat{\mathcal{O}}^+)}{(R^1\nu_*(\widehat{\mathcal{O}}^+))[\zeta_p - 1]} \cong H^1(\widetilde{\Omega}_{\mathfrak{X}}). \tag{4.10.2}$$

By [BMS16, proof of Prop. 8.15], the restriction of this composition to $\mathfrak{X}^{\mathrm{sm}}$ is an isomorphism onto $((\zeta_p - 1) \cdot H^1(\widetilde{\Omega}_{\mathfrak{X}}))|_{\mathfrak{X}^{\mathrm{sm}}}$. Moreover, by Proposition 4.4, $H^1(\widetilde{\Omega}_{\mathfrak{X}})$ is a vector bundle, so it is $(\zeta_p - 1)$-torsion free and has no nonzero local sections that vanish on $\mathfrak{X}^{\mathrm{sm}}$ (as may be seen using (1.5.1)). In conclusion, we may divide the composition (4.10.2) by $\zeta_p - 1$ to obtain a map

$$\Omega^1_{\mathfrak{X}/\mathcal{O}_C}\{-1\} \to H^1(\widetilde{\Omega}_{\mathfrak{X}}) \qquad \text{that is an isomorphism over } \mathfrak{X}^{\mathrm{sm}}. \tag{4.10.3}$$

We are ready for the promised relation between $H^i(\widetilde{\Omega}_{\mathfrak{X}})$ and $\Omega^i_{\mathfrak{X}/\mathcal{O}_C,\,\log}$.

[5] For passage from $\Omega^1_{\overline{\mathbb{Z}}_p/\mathbb{Z}_p}$ of *loc. cit.* to $\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}$, one may use [GR03, 6.5.20 (i)] to conclude that $\Omega^1_{\mathcal{O}_C/\overline{\mathbb{Z}}_p}[p] = 0$.

Theorem 4.11. *The restriction of* (4.10.3) *to* $\mathfrak{X}^{\mathrm{sm}}$ *extends uniquely to an* $\mathcal{O}_{\mathfrak{X},\,\mathrm{\acute{e}t}}$*-module isomorphism*

$$H^1(\widetilde{\Omega}_{\mathfrak{X}}) \cong \Omega^1_{\mathfrak{X}/\mathcal{O}_C,\,\log}\{-1\}; \tag{4.11.1}$$

by (4.4.1) *and Proposition* 4.8*, for every* $i \geq 0$*, it induces an* $\mathcal{O}_{\mathfrak{X},\,\mathrm{\acute{e}t}}$*-module identification*

$$H^i(\widetilde{\Omega}_{\mathfrak{X}}) \cong \Omega^i_{\mathfrak{X}/\mathcal{O}_C,\,\log}\{-i\}. \tag{4.11.2}$$

In the proof of Theorem 4.11 we will use the formal GAGA and Grothendieck existence theorems. The Noetherian cases of these theorems proved in [EGA III_1, §5] have been extended to suitable non-Noetherian settings by K. Fujiwara and F. Kato (with important inputs due to O. Gabber). The relevant to our aims special case of this extension is summarized in the following theorem.

Theorem 4.12 (Fujiwara–Kato)**.** *For a complete valuation ring* V *of height* 1*, a nonzero nonunit* $a \in V$*, and a proper, finitely presented* V*-scheme* Y*, the category of finitely presented* $\mathcal{O}_Y$*-modules is equivalent to that of sequences* $(\mathcal{F}_n)_{n\in\mathbb{Z}_{>0}}$ *of finitely presented* $\mathcal{O}_{Y_{V/a^n}}$*-modules* $\mathcal{F}_n$ *equipped with isomorphisms* $\mathcal{F}_{n+1}|_{Y_{V/a^n}} \simeq \mathcal{F}_n$ *via the functor*

$$\mathcal{F} \mapsto (\mathcal{F}/a^n\mathcal{F})_{n\in\mathbb{Z}_{>0}}. \tag{4.12.1}$$

Proof. The claim is a special case of [FK17, I.10.1.2]. In order to explain why *loc. cit.* applies, we first reinterpret our source and target categories.

By a result of Gabber [FK17, 0.9.2.7], the ring V is "a-adically topologically universally adhesive," so, by [FK17, 0.8.5.25 (2)], it is also "topologically universally coherent with respect to (a)." In particular, by [FK17, 0.8.5.24], every finitely presented V-algebra is a coherent ring, and hence, by [FK17, 0.5.1.2], the $\mathcal{O}_Y$-module $\mathcal{O}_Y$ is coherent. In particular, by [FK17, 0.4.1.8], an $\mathcal{O}_Y$-module $\mathcal{F}$ is finitely presented if and only if $\mathcal{F}$ is coherent, and likewise for $\mathcal{O}_{Y_{V/a^n}}$-modules for $n \in \mathbb{Z}_{>0}$.

By [FK17, 0.8.5.19 (3) and 0.8.4.2], the formal a-adic completion $\widehat{Y}$ of Y may be covered by open affines whose coordinate rings are "topologically universally adhesive" so also, by [FK17, 0.8.5.18], "topologically universally Noetherian outside (a)." In particular, by [FK17, I.2.1.7 and I.2.1.1 (1)], the formal scheme $\widehat{Y}$ is "universally rigid-Noetherian." In addition, by [FK17, 0.8.4.5], it is locally of finite presentation over $\mathrm{Spf}\, V$, so [FK17, I.7.2.2] applied with $A = V$ and [FK17, I.7.2.1] imply that $\widehat{Y}$ is "universally cohesive." Then, by [FK17, I.7.2.4 and I.3.4.1], the functor $(\mathcal{F}_n) \mapsto \varprojlim \mathcal{F}_n$ is an equivalence from the target category of (4.12.1) to the category of coherent $\mathcal{O}_{\widehat{Y}}$-modules.

In conclusion, our claim is that the quasi-coherent pullback i^* along the morphism $i\colon \widehat{Y} \to Y$ of locally ringed spaces induces an equivalence between the category of coherent $\mathcal{O}_Y$-modules and that of coherent $\mathcal{O}_{\widehat{Y}}$-modules. This is a special case of [FK17, I.10.1.2] (see also [FK17, I.§9.1]). ☐

Remarks.

4.13. In Theorem 4.12, if each $\mathcal{F}_n$ is locally free, then the $\mathcal{O}_Y$-module $\mathcal{F}$ that algebraizes the sequence $(\mathcal{F}_n)$ is also locally free. Indeed, it is enough to argue that the stalks of $\mathcal{F}$ at the points of $Y_{V/a}$ are flat, so, since, by [FK17, I.1.4.7 (2)], the morphism i is flat, it suffices to note that the $\mathcal{O}_{\widehat{Y}}$-module $i^*\mathcal{F} \cong \varprojlim \mathcal{F}_n$ is locally free because the Nakayama lemma ensures that $\mathcal{F}_{n+1}$ is locally trivialized by any lifts of local sections that trivialize $\mathcal{F}_n$.

4.14. Remark 4.13 and the proof of Theorem 4.12 also show that i is flat and that the functor $(\mathcal{F}_n) \mapsto \varprojlim \mathcal{F}_n$ is an equivalence to the category of finitely presented $\mathcal{O}_{\widehat{Y}}$-modules.

Proof of Theorem 4.11. As may be checked with the help of étale local semistable coordinates (4.4.2), no nonzero local section of $\mathcal{O}_{\mathfrak{X}}$ vanishes on $\mathfrak{X}^{\mathrm{sm}}$. Thus, the same holds for any vector bundle in place of $\mathcal{O}_{\mathfrak{X}}$, to the effect that the desired isomorphism (4.11.1) is unique if it exists.

Thanks to the uniqueness, we may assume the local setup of §3.1. Remark 4.5 then reduces us further to the case when $\mathfrak{X} = \mathfrak{X}^{\square}$. In this case, there exists a discrete valuation subring $\mathcal{O} \subset \mathcal{O}_C$ and a proper, flat $\mathcal{O}$-scheme $\mathcal{X}$ which étale locally has étale "coordinate morphisms" (1.5.2) and such that $\mathfrak{X}$ is an open subscheme of the formal p-adic completion of $\overline{\mathcal{X}} := \mathcal{X}_{\mathcal{O}_C}$. Thus, finally, we may drop the previous assumptions and assume instead that $\mathfrak{X} = \widehat{\overline{\mathcal{X}}}$ with $\mathcal{X}$ and $\overline{\mathcal{X}}$ as above. We equip $\overline{\mathcal{X}}$ with the log structure $\mathcal{O}_{\overline{\mathcal{X}}} \cap (\mathcal{O}_{\overline{\mathcal{X}}}[\frac{1}{p}])^{\times}$, so that $\overline{\mathcal{X}}$ is log smooth over $\mathcal{O}_C$ (see Claim 1.6.1) and the map $\mathfrak{X} \to \mathcal{X}$ of log ringed étale sites is strict (see Claim 1.6.3).

By Theorem 4.12, the map (4.10.3) algebraizes to an $\mathcal{O}_{\overline{\mathcal{X}}}$-module map

$$f \colon \Omega^1_{\overline{\mathcal{X}}/\mathcal{O}_C}\{-1\} \to \mathcal{H}$$

where, by Proposition 4.4 and Remark 4.13, $\mathcal{H}$ is a vector bundle on $\overline{\mathcal{X}}$ of rank equal to the relative dimension of $\overline{\mathcal{X}}$ over $\mathcal{O}_C$. Moreover, by (4.10.3) and the Nakayama lemma, f is surjective at every point of $\overline{\mathcal{X}}_k^{\mathrm{sm}}$.

Claim 4.14.2. *There is an isomorphism $\mathcal{H}[\frac{1}{p}] \simeq \Omega^1_{\overline{\mathcal{X}}_C/C}$.*

Proof. By the adic GAGA (see [Sch13, 9.1 (i)]), it suffices to find an analogous isomorphism after pullback to $(\overline{\mathcal{X}}_C)^{\mathrm{ad}} \cong \mathfrak{X}_C^{\mathrm{ad}}$. Such a pullback of $\mathcal{H}[\frac{1}{p}]$ is isomorphic to $(R^1\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}}))[\frac{1}{p}]$, and [Sch13, 6.19] supplies an isomorphism between $(R^1\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}}))[\frac{1}{p}]$ and the pullback of $\Omega^1_{\overline{\mathcal{X}}_C/C}$ to $(\overline{\mathcal{X}}_C)^{\mathrm{ad}}$. ☐

Claim 4.14.2 ensures that $f[\frac{1}{p}]$ is a generically surjective morphism between isomorphic vector bundles on $\overline{\mathcal{X}}_C$. Since $\overline{\mathcal{X}}_C$ is proper and smooth, every global section of the structure sheaf of each connected component of $\overline{\mathcal{X}}_C$ is constant, so $\det(f[\frac{1}{p}])$ is an isomorphism, and hence $f[\frac{1}{p}]$ is surjective on the entire $\overline{\mathcal{X}}_C$. In conclusion, $f|_{\overline{\mathcal{X}}^{\mathrm{sm}}}$ is a surjection between vector bundles of the same rank, so

$$f|_{\overline{\mathcal{X}}^{\mathrm{sm}}} \colon \Omega^1_{\overline{\mathcal{X}}^{\mathrm{sm}}/\mathcal{O}_C}\{-1\} \xrightarrow{\sim} \mathcal{H}|_{\overline{\mathcal{X}}^{\mathrm{sm}}}. \tag{4.14.3}$$

Since $\mathcal{X}$ is Cohen–Macaulay and $\mathcal{X} \backslash \mathcal{X}^{\mathrm{sm}}$ is of codimension ≥ 2 in $\mathcal{X}$, limit arguments and [EGA IV_2, 5.10.5] ensure that $\mathcal{H}$ is the unique vector bundle extension of $\mathcal{H}|_{\overline{\mathcal{X}}^{\mathrm{sm}}}$ to $\overline{\mathcal{X}}$. The isomorphism (4.14.3) then leads to an isomorphism $\mathcal{H} \simeq \Omega^1_{\overline{\mathcal{X}}/\mathcal{O}_C,\,\log}\{-1\}$ whose formal p-adic completion gives the desired isomorphism (4.11.1). ☐

Remark 4.15. If the coordinate morphisms (1.5.1) exist Zariski locally on $\mathfrak{X}$, then, by Remark 4.5, the identifications of Theorem 4.11 hold already for the Zariski topology; more precisely, then

$$H^i(\widetilde{\Omega}_{\mathfrak{X}_{\mathrm{Zar}}}) \cong \Omega^i_{\mathfrak{X}/\mathcal{O}_C,\,\log}\{-i\} \qquad \text{as } \mathcal{O}_{\mathfrak{X}_{\mathrm{Zar}}}\text{-modules for every } i \geq 0.$$

We are ready to relate the de Rham specialization of $A\Omega_{\mathfrak{X}}$ to differential forms.

Theorem 4.16. *There is an identification*

$$A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\,\theta} \mathcal{O}_C \cong \Omega^{\bullet}_{\mathfrak{X}/\mathcal{O}_C,\,\log}; \tag{4.16.1}$$

If the étale morphisms (1.5.1) exist Zariski locally on $\mathfrak{X}$, then (4.16.1) also holds for $A\Omega_{\mathfrak{X}_{\mathrm{Zar}}}$.

Proof. Similarly to [BMS16, proof of Thm. 14.1], since $\varphi(\mu) = \varphi(\xi)\mu$ (see §2.1), [BMS16, 6.11] gives

$$A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta} \mathcal{O}_C \cong A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\varphi} A_{\mathrm{inf}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta\circ\varphi^{-1}} \mathcal{O}_C \cong (L\eta_{(\varphi(\xi))}(A\Omega_{\mathfrak{X}})) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta\circ\varphi^{-1}} \mathcal{O}_C. \quad (4.16.2)$$

By [BMS16, 6.12], since $A_{\mathrm{inf}}/(\varphi(\xi)) \cong \mathcal{O}_C$ via $\theta \circ \varphi^{-1}$, the object $(L\eta_{(\varphi(\xi))}(A\Omega_{\mathfrak{X}})) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta\circ\varphi^{-1}} \mathcal{O}_C$ is identified in the derived category with the complex whose i^{th} degree term is

$$H^i(A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta\circ\varphi^{-1}} \mathcal{O}_C) \otimes_{\mathcal{O}_C} \left(\tfrac{\mathrm{Ker}(\theta\circ\varphi^{-1})}{(\mathrm{Ker}(\theta\circ\varphi^{-1}))^2}\right)^{\otimes i} \overset{(4.2.1)}{\cong} H^i(\widetilde{\Omega}_{\mathfrak{X}}) \otimes_{\mathcal{O}_C} \left(\tfrac{\mathrm{Ker}(\theta\circ\varphi^{-1})}{(\mathrm{Ker}(\theta\circ\varphi^{-1}))^2}\right)^{\otimes i}$$

and the differentials are given by Bockstein homomorphisms.

The perfectness of $\mathcal{O}_C^\flat$ implies that $\widehat{\mathbb{L}}_{A_{\mathrm{inf}}/\mathbb{Z}_p} \cong 0$ and (4.9.1) (applied with $\mathfrak{X} = \mathrm{Spf}\,\mathcal{O}_C$) implies that $\widehat{\mathbb{L}}_{\mathcal{O}_C/\mathbb{Z}_p} \cong \mathcal{O}_C\{1\}[1]$, so $\widehat{\mathbb{L}}_{\mathcal{O}_C/A_{\mathrm{inf}}} \cong \mathcal{O}_C\{1\}[1]$ where $\mathcal{O}_C$ is regarded as an A_{inf}-algebra via $\theta \circ \varphi^{-1}$. This combines with [Ill71, III.3.2.4 (iii)] to supply an isomorphism $\frac{\mathrm{Ker}(\theta\circ\varphi^{-1})}{(\mathrm{Ker}(\theta\circ\varphi^{-1}))^2} \cong \mathcal{O}_C\{1\}$. In conclusion, due to Theorem 4.11 and the previous paragraph, $A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta} \mathcal{O}_C$ is identified with the complex whose i^{th} degree term is $\Omega^i_{\mathfrak{X}/\mathcal{O}_C,\log}$ and whose differentials are certain Bockstein homomorphisms. Since each $\Omega^i_{\mathfrak{X}/\mathcal{O}_C,\log}$ is a vector bundle, the agreement of these differentials with those of $\Omega^\bullet_{\mathfrak{X}/\mathcal{O}_C,\log}$ may be checked over $\mathfrak{X}^{\mathrm{sm}}$ (compare with the argument for (4.10.3)), where it follows from [BMS16, 14.1 (ii)] (or from [Bha16, proof of Prop. 7.9]).

Due to Remark 4.15, the proof for $A\Omega_{\mathfrak{X}_{\mathrm{Zar}}}$ is the same. □

Corollary 4.17. *The de Rham specialization of $R\Gamma(\mathfrak{X}_{\mathrm{\acute{e}t}}, A\Omega_{\mathfrak{X}})$ may be identified as follows:*

$$R\Gamma(\mathfrak{X}_{\mathrm{\acute{e}t}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta} \mathcal{O}_C \cong R\Gamma_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C). \quad (4.17.1)$$

Proof. The claim follows from the projection formula [SP, 0944] and Theorem 4.16. □

Remark 4.18. In the case when $\mathfrak{X} \cong \widehat{\mathcal{X}}$ for a proper and flat $\mathcal{O}_C$-scheme $\mathcal{X}$ that étale locally has étale morphisms (1.5.2) (with $\mathcal{O}_C$ there replaced by $\mathcal{O}$), we have a further identification

$$R\Gamma(\mathcal{X}_{\mathrm{\acute{e}t}}, \Omega^\bullet_{\mathcal{X}/\mathcal{O}_C,\log}) \xrightarrow{\sim} R\Gamma(\mathfrak{X}_{\mathrm{\acute{e}t}}, \Omega^\bullet_{\mathfrak{X}/\mathcal{O}_C,\log})$$

granted that $\mathcal{X}$ is endowed with the log structure $\mathcal{O}_{\mathcal{X}} \cap (\mathcal{O}_{\mathcal{X}}[\frac{1}{p}])^\times$ (whose pullback to $\mathfrak{X}$ is the log structure of $\mathfrak{X}$, see Claim 1.6.3). Indeed, the natural pullback map between the E_1-spectral sequences

$$H^j(\mathcal{X}, \Omega^i_{\mathcal{X}/\mathcal{O}_C,\log}) \Rightarrow H^{i+j}(R\Gamma(\mathcal{X}_{\mathrm{\acute{e}t}}, \Omega^\bullet_{\mathcal{X},\log})) \quad \text{and} \quad H^j(\mathfrak{X}, \Omega^i_{\mathfrak{X}/\mathcal{O}_C,\log}) \Rightarrow H^{i+j}(R\Gamma(\mathfrak{X}_{\mathrm{\acute{e}t}}, \Omega^\bullet_{\mathfrak{X}/\mathcal{O}_C,\log}))$$

is an isomorphism because, by the Grothendieck finiteness and comparison theorems [EGA III_1, 3.2.1 and 4.1.7] (combined with limit arguments; or, by [FK17, I.9.2.1] directly),

$$H^j(\mathcal{X}, \Omega^i_{\mathcal{X}/\mathcal{O}_C,\log}) \xrightarrow{\sim} H^j(\mathfrak{X}, \Omega^i_{\mathfrak{X}/\mathcal{O}_C,\log}) \qquad \text{for all } i, j.$$

Corollary 4.19. *If $\mathfrak{X}$ is proper over $\mathcal{O}_C$, then $R\Gamma(\mathfrak{X}_{\mathrm{\acute{e}t}}, A\Omega_{\mathfrak{X}})$ is a perfect object of $D^{\geq 0}(A_{\mathrm{inf}})$; in other words, then $R\Gamma(\mathfrak{X}_{\mathrm{\acute{e}t}}, A\Omega_{\mathfrak{X}})$ is quasi-isomorphic to a bounded complex of finite free A_{inf}-modules.*

Proof. By the Grothendieck finiteness theorem [Ull95, 5.3] and the spectral sequence as in Remark 4.18, the $\mathcal{O}_C$-modules $H^j(R\Gamma(\mathfrak{X}_{\mathrm{\acute{e}t}}, \Omega^\bullet_{\mathfrak{X}/\mathcal{O}_C,\log}))$ are finitely presented, and hence also perfect (see [SP, 0ASP]). Thus, by Corollary 4.17 and [SP, 066U], the object $R\Gamma(\mathfrak{X}_{\mathrm{\acute{e}t}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/(\xi)$ of $D^{\geq 0}(\mathcal{O}_C)$ is perfect. Moreover, by Corollary 4.6, the object $R\Gamma(\mathfrak{X}_{\mathrm{\acute{e}t}}, A\Omega_{\mathfrak{X}})$ is derived ξ-adically complete. Therefore, by [SP, 09AW], it is also perfect, as desired. □

We close the section by comparing $R\Gamma(\mathfrak{X}_{\mathrm{\acute{e}t}}, A\Omega_{\mathfrak{X}})$ to its analogue defined using the Zariski topology.

Corollary 4.20. *If the coordinate maps* (1.5.1) *exist Zariski locally on* $\mathfrak{X}$*, then* $R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}})$ *may be computed using the Zariski topology of* $\mathfrak{X}$*; more precisely, then*

$$R\Gamma(\mathfrak{X}_{\text{Zar}}, A\Omega_{\mathfrak{X}_{\text{Zar}}}) \xrightarrow{\sim} R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}). \tag{4.20.1}$$

Proof. By Theorem 4.16 and its corollary 4.17, the reduction of (4.20.1) modulo ξ is identified with

$$R\Gamma(\mathfrak{X}_{\text{Zar}}, \Omega^\bullet_{\mathfrak{X}/\mathcal{O}_C,\,\log}) \xrightarrow{\sim} R\Gamma(\mathfrak{X}_{\text{ét}}, \Omega^\bullet_{\mathfrak{X}/\mathcal{O}_C,\,\log}),$$

and hence is an isomorphism. Thus, due to the derived ξ-adic completeness of $R\Gamma(\mathfrak{X}_{\text{Zar}}, A\Omega_{\mathfrak{X}_{\text{Zar}}})$ and $R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}})$ ensured by Corollary 4.6 (and its analogue for the Zariski topology), the morphism (4.20.1) is also an isomorphism. □

Example 4.21. By §1.5, Corollary 4.20 applies to any $\mathcal{O}_C$-smooth $\mathfrak{X}$ and, more generally, to any $\mathfrak{X}$ that Zariski locally arises from a strictly semistable scheme defined over a discrete valuation ring.

5. The absolute crystalline comparison isomorphism

In Theorem 4.16, we have identified the $\mathcal{O}_C$-base change (along θ) of the object $A\Omega_{\mathfrak{X}}$ with $\Omega^\bullet_{\mathfrak{X}/\mathcal{O}_C,\,\log}$. The goal of the present section is to similarly identify the A_{cris}-base change of $A\Omega_{\mathfrak{X}}$ with an object that computes the logarithmic crystalline (that is, Hyodo–Kato) cohomology of $\mathfrak{X}_{\mathcal{O}_C/p}$ over A_{cris} (see Theorem 5.4). This is more general because, on the one hand, θ factors through the map $A_{\text{inf}} \to A_{\text{cris}}$, while, on the other, $\Omega^\bullet_{\mathfrak{X}/\mathcal{O}_C,\,\log}$ computes the log crystalline cohomology of $\mathfrak{X}_{\mathcal{O}_C/p}$ over $\mathcal{O}_C$. In fact, even the map $A_{\text{inf}} \to A_{\text{inf}}/\mu$ factors through $A_{\text{inf}} \to A_{\text{cris}}$ (see [BMS16, proof of Lemma 4.19]), so the identification of the A_{cris}-base change of $A\Omega_{\mathfrak{X}}$ will capture the entire $\mu = 0$ locus of A_{inf} (in contrast, the comparison with the p-adic étale cohomology captured the $\mu \neq 0$ locus, see Theorem 2.3).

In comparison to the case when $\mathfrak{X}$ is smooth treated in [BMS16, §12], it seems more subtle to control the interaction of the functor $L\eta_{(\mu)}$ with the relevant base changes. To overcome this, we resort to the analysis of continuous group cohomology carried out in §3. Another major complication is the presence of log structures. Specifically, not knowing the existence of logarithmic divided power envelopes of certain (nonexact) logarithmic closed immersions in mixed characteristic, we are forced to devise slightly indirect arguments when analyzing the relevant divided power envelopes. For this, we rely on the results and arguments from [Kat89] and [Bei13b]; the latter reference is especially useful for us because some log structures that we use are not coherent (only quasi-coherent).

5.1. The ring A_{cris}. With the generator ξ of the kernel of $\theta\colon A_{\text{inf}} \twoheadrightarrow \mathcal{O}_C$ in hand (see (2.1.4)), we let A^0_{cris} be the A_{inf}-subalgebra of $A_{\text{inf}}[\frac{1}{p}]$ generated by the divided powers $\frac{\xi^n}{n!}$ for $n \in \mathbb{Z}_{\geq 1}$. The induced map $\theta\colon A^0_{\text{cris}} \twoheadrightarrow \mathcal{O}_C$ identifies A^0_{cris} with the divided power envelope of $\theta\colon A_{\text{inf}} \twoheadrightarrow \mathcal{O}_C/p$ over $(\mathbb{Z}_p, p\mathbb{Z}_p)$ (where $p\mathbb{Z}_p$ is equipped with its canonical divided powers), see [Tsu99, A2.8].

We let A_{cris} be the p-adic completion of A^0_{cris}. The map $A^0_{\text{cris}} \to A_{\text{cris}}$ is injective (see [Tsu99, A2.13] or Proposition 5.36 below), and the induced map $\theta\colon A_{\text{cris}} \twoheadrightarrow \mathcal{O}_C$ identifies A_{cris} with the initial p-adically complete divided power thickening of $\mathcal{O}_C$ over $\mathbb{Z}_p$ (see [Tsu99, A1.3 and A1.5]). Moreover, since $\theta(\mu) = 0$ (see (2.1.3) and (2.1.1)), we have $\mu^p \in pA^0_{\text{cris}}$, so the p-adic topology of A^0_{cris} agrees with the (p,μ)-adic topology, and hence A_{cris} is complete for these topologies (see [SP, 05GG]). By [Bri06, 2.33] (or Proposition 5.36 below), the A_{inf}-algebra A_{cris} is a domain of characteristic 0.

Analogously to §3.26, the ring A_{cris} comes equipped with the Frobenius endomorphism φ that intertwines the absolute Frobenius endomorphism of $\mathcal{O}_C/p$ via the map θ. The identification

$$A_{\text{cris}} \cong (\varinjlim_m A^{(m)}_{\text{cris}})^\wedge, \qquad \text{which results from the evident} \qquad A^0_{\text{cris}} \cong \varinjlim_m A^{0,\,(m)}_{\text{cris}} \tag{5.1.1}$$

(see §3.26), is Frobenius equivariant and compatible with the maps θ.

5.2. The log structure on A_{cris}. For each $n \in \mathbb{Z}_{\geq 1}$, the ring A_{cris}/p^n is a divided power thickening of $\mathcal{O}_C/p$ over $\mathbb{Z}/p^n$. Therefore, by [Bei13b, §1.17, Lemma], every quasi-coherent integral log structure $\mathcal{N}$ on $\mathcal{O}_C/p$ for which the multiplication by p map is an automorphism of $\mathcal{N}/(\mathcal{O}_C/p)^\times$ lifts uniquely to a quasi-coherent integral log structure on A_{cris}/p^n. Thus, letting $\mathcal{N}$ be the "default" log structure on $\mathcal{O}_C/p$ (see §1.6 (1)), for which $\mathcal{N}/(\mathcal{O}_C/p)^\times \cong \mathbb{Q}_{\geq 0}$, we obtain compatible quasi-coherent integral log structures on the rings A_{cris}/p^n, to the effect that each A_{cris}/p^n becomes a log PD thickening of $\mathcal{O}_C/p$. Explicitly, these log structures are the pullbacks of the log structure on A_{cris} associated to the prelog structure

$$\mathcal{O}_C^\flat \setminus \{0\} \to A_{\mathrm{cris}}, \qquad x \mapsto [x]. \tag{5.2.1}$$

In what follows, we always equip

- each A_{cris}/p^n, as well as A_{cris}, with the log structures described above;
- each $\mathbb{Z}/p^n\mathbb{Z}$ with the standard divided powers on $p\mathbb{Z}/p^n\mathbb{Z}$ and the trivial log structure.

By, for instance, [Tsu99, A1.5], for every $\mathcal{O}_C/p$-scheme Z and every divided power thickening $\widetilde{Z}$ of Z over $\mathbb{Z}/p^n\mathbb{Z}$, the map $z\colon Z \to \mathrm{Spec}(\mathcal{O}_C/p)$ extends uniquely to a PD map $\widetilde{z}\colon \widetilde{Z} \to \mathrm{Spec}(A_{\mathrm{cris}}/p^n)$. If, in addition, $\widetilde{Z}$ is equipped with a quasi-coherent integral log structure for which z is enhanced to a map $z^\sharp$ of log schemes, then, by [Bei13b, §1.17, Exercise], the map $z^\sharp$ extends uniquely to a PD map $\widetilde{z}^\sharp\colon \widetilde{Z} \to \mathrm{Spec}(A_{\mathrm{cris}}/p^n)$ of log schemes.

5.3. The absolute crystalline cohomology of $\mathfrak{X}_{\mathcal{O}_C/p}$. We let

$$(\mathfrak{X}_{\mathcal{O}_C/p}/\mathbb{Z}_p)_{\log \mathrm{cris}}$$

be the log crystalline site of $\mathfrak{X}_{\mathcal{O}_C/p}$ over $\mathbb{Z}_p$ defined as in [Bei13b, §1.12]; each object of this site is an étale $\mathfrak{X}_{\mathcal{O}_C/p}$-scheme Z equipped with a divided power thickening $\widetilde{Z}$ over some $\mathbb{Z}/p^n\mathbb{Z}$ such that $\widetilde{Z}$ is, in turn, equipped with a quasi-coherent integral log structure whose pullback to Z is identified with the pullback of the log structure of $\mathfrak{X}_{\mathcal{O}_C/p}$ (which is defined in §1.6). The universal property of A_{cris} reviewed in the last paragraph of §5.2 gives the following identification of sites:

$$(\mathfrak{X}_{\mathcal{O}_C/p}/\mathbb{Z}_p)_{\log \mathrm{cris}} \cong (\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}})_{\log \mathrm{cris}}.$$

The *absolute logarithmic crystalline cohomology* of $\mathfrak{X}_{\mathcal{O}_C/p}$ is the cohomology of the structure sheaf:

$$R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) := R\Gamma((\mathfrak{X}_{\mathcal{O}_C/p}/\mathbb{Z}_p)_{\log \mathrm{cris}}, \mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/\mathbb{Z}_p});$$

equivalently (see also [Bei13b, (1.18.1)]),

$$R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) \cong R\Gamma((\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}})_{\log \mathrm{cris}}, \mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}).$$

Letting

$$u\colon (\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}})_{\log \mathrm{cris}} \to (\mathfrak{X}_{\mathcal{O}_C/p})_{\text{ét}} \cong \mathfrak{X}_{\text{ét}}$$

be the morphism of topoi that forgets the thickenings $\widetilde{Z}$ (see [Bei13b, §1.5]), we get the identification

$$R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) \cong R\Gamma(\mathfrak{X}_{\text{ét}}, Ru_*(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}})). \tag{5.3.1}$$

By functoriality, the absolute Frobenius (which is the multiplication by p on the log structures) induces the "Frobenius" endomorphisms of $Ru_*(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}})$ and $R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}})$ that are semilinear with respect to the Frobenius endomorphism of A_{cris} (see §5.1).

The main goal of this section is to establish the following A_{cris}-comparison isomorphism.

Theorem 5.4. *There is a Frobenius-equivariant* (*see* §2.1*,* §2.2*,* §5.1*,* §5.3) *identification*

$$A\Omega_{\mathfrak{X}}\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}A_{\mathrm{cris}} \cong Ru_*(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}), \tag{5.4.1}$$

where, consistently with the definition (1.7.1)*, we have* $A\Omega_{\mathfrak{X}}\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}A_{\mathrm{cris}} = R\lim_n(A\Omega_{\mathfrak{X}}\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}A_{\mathrm{cris}}/p^n)$.

We will first prove a version of Theorem 5.4 in a local setting, that is, in the presence of semistable coordinates. We will then complete the proof by using "all possible coordinates" to globalize the argument. This overall strategy is similar to the one used in [BMS16, §12] in the smooth case.

5.5. The local setup. For the local argument, we assume until §5.17 that $\mathfrak{X} = \mathrm{Spf}(R)$, that $\mathfrak{X}$ is connected, and that for some $0 \le r \le d$ and $q \in \mathbb{Q}_{>0}$ there is an étale $\mathcal{O}_C$-morphism

$$\mathfrak{X} = \mathrm{Spf}(R) \to \mathrm{Spf}(R^{\square}) \qquad \text{with} \qquad R^{\square} = \mathcal{O}_C\{t_0,\dots,t_r,t_{r+1}^{\pm1},\dots,t_d^{\pm1}\}/(t_0\cdots t_r - p^q). \tag{5.5.1}$$

We use the rings $R^{\square}_\infty$ and R_∞ and the group Δ introduced in §3.2, as well as the rings $\mathbb{A}_{\mathrm{inf}}(R^{\square}_\infty)$, $\mathbb{A}_{\mathrm{inf}}(R_\infty)$, $A(R^{\square})$, and $A(R)$, and the modules $N^{\square}_\infty$ and N_∞, introduced in §3.14.

Roughly speaking, in the local case we will access the right side of (5.4.1) through the de Rham cohomology of an explicitly constructed log smooth lift $\mathrm{Spf}(A_{\mathrm{cris}}(R))$ of $\mathfrak{X}_{\mathcal{O}_C/p}$ to $\mathrm{Spf}\,A_{\mathrm{cris}}$. The de Rham complex that computes this cohomology can be made explicit by expressing its differentials in terms of the Δ-action on $A_{\mathrm{cris}}(R)$ (see Lemma 5.15). On the other hand, results from §3, namely (3.25.1) and Theorem 3.20, make the left side of (5.4.1) explicit. Once both sides of (5.4.1) are explicit, it is possible to identify them, and hence to establish (the presheaf version of) the local case of Theorem 5.4.

However, this relatively short local proof, whose detailed version in the good reduction case is given in [BMS16, 12.4], is ill-suited for globalizing. This is so because it appears difficult to extend the implicit exchange of the order of the functors $L\eta_{(\mu)}$ and $-\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}A_{\mathrm{cris}}$ in this argument to general perfectoid covers that appear in the "all possible coordinates" technique. For instance, one may attempt to use the almost purity theorem and Lemma 3.18 to reduce such commutation to the "base case" of R_∞, but this requires understanding the $W(\mathfrak{m}^\flat)$-torsion in the groups $H^i_{\mathrm{cont}}(\Delta, (\mathbb{A}_{\mathrm{inf}}(R_\infty)\widehat{\otimes}_{A_{\mathrm{inf}}}A_{\mathrm{cris}})/\mu)$ that seem difficult to access due to pathologies of the ring A_{cris}/μ.

Similarly to [BMS16, §12.2], to overcome this difficulty we will use the A_{inf}-algebras $A^{(m)}_{\mathrm{cris}}$ reviewed in §3.26 that retain better finite type properties over A_{inf} than A_{cris}. In particular, we commute the functors $L\eta_{(\mu)}$ and $-\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}A^{(m)}_{\mathrm{cris}}$ in the following proposition:

Proposition 5.6. *In the local setting of* (5.5.1)*, for every* $m \ge p$*, we have*

$$L\eta_{(\mu)}(R\Gamma(X^{\mathrm{ad}}_{\mathrm{pro\acute{e}t}}, \mathbb{A}_{\mathrm{inf},X}))\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}A^{(m)}_{\mathrm{cris}} \xrightarrow{\sim} L\eta_{(\mu)}(R\Gamma(X^{\mathrm{ad}}_{\mathrm{pro\acute{e}t}}, \mathbb{A}_{\mathrm{inf},X})\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}A^{(m)}_{\mathrm{cris}}). \tag{5.6.1}$$

Proof. The map (5.6.1) exists because its target is derived p-adically complete (see [BMS16, 6.19]). Moreover, by Theorems 3.20 and 3.34, it suffices to prove that

$$L\eta_{(\mu)}(R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)))\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}A^{(m)}_{\mathrm{cris}} \xrightarrow{\sim} L\eta_{(\mu)}(R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty))).$$

By Propositions 3.25 and 3.32, the "nonintegral" part N_∞ does not contribute, which reduces to

$$L\eta_{(\mu)}(R\Gamma_{\mathrm{cont}}(\Delta, A(R)))\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}A^{(m)}_{\mathrm{cris}} \xrightarrow{\sim} L\eta_{(\mu)}(R\Gamma_{\mathrm{cont}}(\Delta, A^{(m)}_{\mathrm{cris}}(R))). \tag{5.6.2}$$

In turn, (5.6.2) follows from the triviality of the Δ-action on $A(R)/\mu$ and $A^{(m)}_{\mathrm{cris}}(R)/\mu$ (see §3.14 and §3.27): namely, due to Lemma 3.7 and this triviality, the left (resp., right) side of (5.6.2) becomes

$$K_{A(R)}(\tfrac{\delta_1-1}{\mu},\dots,\tfrac{\delta_d-1}{\mu})\widehat{\otimes}_{A_{\mathrm{inf}}}A^{(m)}_{\mathrm{cris}} \qquad (\text{resp.,} \qquad K_{A^{(m)}_{\mathrm{cris}}(R)}(\tfrac{\delta_1-1}{\mu},\dots,\tfrac{\delta_d-1}{\mu})). \qquad \square$$

Continuing to work in the local setting, we now express the (presheaf version of the) left side of (5.4.1) in the form that will be convenient for the "all possible coordinates" technique.

Corollary 5.7. *In the local setting of* (5.5.1)*, there is a natural Frobenius-equivariant identification*

$$R\Gamma(\mathfrak{X}_{\text{ét}}^{\text{psh}}, A\Omega_{\mathfrak{X}}^{\text{psh}})\widehat{\otimes}^{\mathbb{L}}_{A_{\text{inf}}} A_{\text{cris}} \cong \left(\varinjlim_m \left(\eta_{(\mu)}\left(K_{\mathbb{A}^{(m)}_{\text{cris}}(R_\infty)}(\delta_1 - 1, \dots, \delta_d - 1)\right)\right)\right)^{\widehat{}}$$

(*see* (4.1.2) *for* $A\Omega_{\mathfrak{X}}^{\text{psh}}$) *where, on the right side, the direct limit and the p-adic completion are termwise.*

Proof. The Δ-equivariant Frobenius action on each $\mathbb{A}^{(m)}_{\text{cris}}(R_\infty)$ that is compatible as m varies (see §3.27) and the divisibility $\mu \mid \varphi(\mu)$ supply the Frobenius action on the right side. The proof of Proposition 5.6 gives the Frobenius-equivariant identification

$$R\Gamma(\mathfrak{X}_{\text{ét}}^{\text{psh}}, A\Omega_{\mathfrak{X}}^{\text{psh}})\widehat{\otimes}^{\mathbb{L}}_{A_{\text{inf}}} A^{(m)}_{\text{cris}} \cong \eta_{(\mu)}\left(K_{\mathbb{A}^{(m)}_{\text{cris}}(R_\infty)}(\delta_1 - 1, \dots, \delta_d - 1)\right),$$

so it remains to pass to the direct limit and to form the p-adic completion. □

We now turn our attention to the right side of (5.4.1) in a local setting and begin by constructing the ring $A_{\text{cris}}(R)$ that underlies a log smooth lift of R/p to A_{cris}.

5.8. The ring $A_{\text{cris}}(R)$. The relative version of A_{cris} and the corresponding variant that models a "highly ramified cover" of the relative version are

$$A_{\text{cris}}(R) := A(R)\widehat{\otimes}_{A_{\text{inf}}} A_{\text{cris}} \qquad \text{and} \qquad \mathbb{A}_{\text{cris}}(R_\infty) := \mathbb{A}_{\text{inf}}(R_\infty)\widehat{\otimes}_{A_{\text{inf}}} A_{\text{cris}},$$

respectively, where the completions are p-adic (equivalently, (p, μ)-adic, see §5.1). Due to (3.14.5), the ring $A_{\text{cris}}(R)$ is an $A_{\text{cris}}(R)$-module direct summand of $\mathbb{A}_{\text{cris}}(R_\infty)$. The maps θ from §3.14 and §5.1 induce compatible surjections (which we abusively also call θ):

$$\theta\colon A_{\text{cris}}(R) \twoheadrightarrow R \qquad \text{and} \qquad \theta\colon \mathbb{A}_{\text{cris}}(R_\infty) \to R_\infty.$$

Let $\mathbb{A}^0_{\text{cris}}(R_\infty)$ be the $\mathbb{A}_{\text{inf}}(R_\infty)$-subalgebra of $\mathbb{A}_{\text{inf}}(R_\infty)[\frac{1}{p}]$ generated by the $\frac{\xi^n}{n!}$ for $n \in \mathbb{Z}_{\geq 1}$. By [Tsu99, proof of A2.8], letting $\mathbb{A}_{\text{inf}}(R_\infty)[\frac{T^n}{n!}]_{n\geq 1}$ denote the divided power polynomial algebra over $\mathbb{A}_{\text{inf}}(R_\infty)$ in one variable, we have

$$\mathbb{A}^0_{\text{cris}}(R_\infty) \cong (\mathbb{A}_{\text{inf}}(R_\infty)[\tfrac{T^n}{n!}]_{n\geq 1})/(T - \xi), \qquad \text{so also} \qquad \mathbb{A}^0_{\text{cris}}(R_\infty) \cong \mathbb{A}_{\text{inf}}(R_\infty) \otimes_{A_{\text{inf}}} A^0_{\text{cris}}.$$

Consequently, since ξ generates $\text{Ker}(\theta) \subset \mathbb{A}_{\text{inf}}(R_\infty)$, the ring $\mathbb{A}^0_{\text{cris}}(R_\infty)$ is identified with the divided power envelope of $(\mathbb{A}_{\text{inf}}(R_\infty), \text{Ker}(\theta) + p\mathbb{A}_{\text{inf}}(R_\infty))$ over $(\mathbb{Z}_p, p\mathbb{Z}_p)$. Therefore, by §5.1 and base change for divided power envelopes (see [BO78, 3.20 1)] and [SP, 07HB, 07HD]),

$$\mathbb{A}_{\text{cris}}(R_\infty) \cong (\mathbb{A}^0_{\text{cris}}(R_\infty))^{\widehat{}}.$$

Due to Lemma 3.13, $\mathbb{A}_{\text{cris}}(R_\infty)$ (resp., $A_{\text{cris}}(R)$) is p-adically formally étale as an $\mathbb{A}_{\text{cris}}(R^{\square}_\infty)$-algebra (resp., as an $A(R)$-algebra) and p-adically formally flat as an A_{cris}-algebra. In particular, $\mathbb{A}_{\text{cris}}(R_\infty)$ inherits p-torsion freeness from A_{cris}. Moreover, even though we will not use this, $\mathbb{A}_{\text{cris}}(R_\infty)$ is also μ-torsion free, as follows from Proposition 5.36 below (contrast with (3.26.2) and (3.27.2)).

Analogously to §3.27, the rings $A_{\text{cris}}(R)$ and $\mathbb{A}_{\text{cris}}(R_\infty)$ come equipped with A_{cris}-semilinear Frobenius endomorphisms that are compatible with their counterparts on $A^{(m)}_{\text{cris}}(R)$ and $\mathbb{A}^{(m)}_{\text{cris}}(R_\infty)$.

The profinite group Δ acts continuously, Frobenius-equivariantly, and A_{cris}-linearly on $A_{\text{cris}}(R)$ and $\mathbb{A}_{\text{cris}}(R_\infty)$. As in §3.27, the induced Δ-action on $A_{\text{cris}}(R)/\mu$ is trivial.

5.9. The log structure on $A(R)$. Provisionally, we consider the (fine) log structures on $A(R)$ and A_{inf} associated to the prelog structures

$$\mathbb{N}_{\geq 0}^{r+1} \xrightarrow{(a_i)\mapsto \prod X_i^{a_i}} A(R) \qquad \text{and} \qquad \mathbb{N}_{\geq 0} \xrightarrow{a \mapsto [(p^{1/p^\infty})^q]^a} A_{\mathrm{inf}}.$$

Moreover, we map $\mathbb{N}_{\geq 0} \to \mathbb{N}_{\geq 0}^{r+1}$ diagonally, so that $A(R)$ becomes (p, μ)-adically formally log smooth over A_{inf} (see (3.14.2) and [Kat89, 3.5]). To eliminate the dependence on q, we always, unless noted otherwise, equip A_{inf} with the log structure associated to the prelog structure

$$\mathcal{O}_C^\flat \setminus \{0\} \to A_{\mathrm{inf}}, \qquad x \mapsto [x].$$

Likewise, we always, unless noted otherwise, equip $A(R)$ with the log structure that is the base change of the fine log structure on $A(R)$ described above along the "change of log structure" self-map of A_{inf} determined by $\mathbb{N}_{\geq 0} \xrightarrow{i \mapsto [(p^{1/p^\infty})^q]^i} \mathcal{O}_C^\flat \setminus \{0\}$. Explicitly, this log structure is associated to the prelog structure

$$\mathbb{N}_{\geq 0}^{r+1} \textstyle\bigsqcup_{\mathbb{N}_{\geq 0}} (\mathcal{O}_C^\flat \setminus \{0\}) \to A(R)$$

that embeds $\mathbb{N}_{\geq 0}$ diagonally into $\mathbb{N}_{\geq 0}^{r+1}$, sends an $i \in \mathbb{N}_{\geq 0}$ to $[(p^{1/p^\infty})^q]^i$, and sends the i^{th} standard basis vector of $\mathbb{N}_{\geq 0}^{r+1}$ (resp., an $x \in \mathcal{O}_C^\flat \setminus \{0\}$) to X_i (resp., to $[x]$).

These latter "default" log structures on $A(R)$ and A_{inf} are quasi-coherent and integral and, by base change, with them $A(R)$ is log smooth over A_{inf}. Moreover, via the map θ, the ring $A(R)$ becomes a (p, μ)-adically formally log smooth thickening of R/p over $\mathrm{Spf}(A_{\mathrm{inf}})$ (where R/p is endowed with the log structure discussed in §5.3).

The Frobenius on A_{inf} and $A(R)$ extends to the log structures: we may let it act as multiplication by p on $\mathbb{N}_{\geq 0}^{r+1}$ and $\mathbb{N}_{\geq 0}$ and as the p^{th} power map on $\mathcal{O}_C^\flat \setminus \{0\}$. Consequently, the Frobenius of the log A_{inf}-algebra $A(R)$ lifts the absolute Frobenius of the log $\mathcal{O}_C/p$-algebra R/p.

The Frobenius-equivariant Δ-action on the A_{inf}-algebra $A(R)$ (see §3.14) extends to a Frobenius-equivariant Δ-action on the log A_{inf}-scheme $\mathrm{Spec}(A(R))$: indeed, a $\delta \in \Delta$ sends each X_i with $0 \leq i \leq r$ to $u_{\delta, i} \cdot X_i$ for some unit $u_{\delta, i} \in A(R)^\times$ that is a Teichmüller element (see §3.14) and the prelog structures

$$\mathbb{N}_{\geq 0}^{r+1} \xrightarrow{(a_i)\mapsto \prod X_i^{a_i}} A(R) \qquad \text{and} \qquad \mathbb{N}_{\geq 0}^{r+1} \xrightarrow{(a_i)\mapsto \prod (u_{\delta, i}\cdot X_i)^{a_i}} A(R)$$

define the same log structure on $\mathrm{Spec}(A(R))$, namely, the one defined by the prelog structure

$$\mathbb{Z}^{r+1} \times \mathbb{N}_{\geq 0}^{r+1} \xrightarrow{((z_i),(a_i))\mapsto \prod u_{\delta, i}^{z_i} \cdot \prod X_i^{a_i}} A(R).$$

5.10. The logarithmic de Rham complex. We let

$$\Omega^\bullet_{A(R)/A_{\mathrm{inf}}, \log}$$

be the (global section complex of the) logarithmic de Rham complex of $\mathrm{Spf}(A(R))$ over $\mathrm{Spf}(A_{\mathrm{inf}})$; more precisely, $\Omega^\bullet_{A(R)/A_{\mathrm{inf}}, \log}$ is the (termwise) inverse limit over $n, n' > 0$ of the logarithmic de Rham complexes of $A(R)/(p^n, \mu^{n'})$ over $A_{\mathrm{inf}}/(p^n, \mu^{n'})$. Due to the formal log smoothness of $A(R)$ over A_{inf}, each $\Omega^i_{A(R)/A_{\mathrm{inf}}, \log}$ is a free $A(R)$-module: indeed, the logarithmic differentials $d\log(X_1), \dots, d\log(X_d)$ form a basis of $\Omega^1_{A(R)/A_{\mathrm{inf}}, \log}$. We let

$$\tfrac{\partial}{\partial \log(X_i)} \colon A(R) \to A(R) \qquad \text{for} \quad i = 1, \dots, d \tag{5.10.1}$$

denote the dual basis of log A_{inf}-derivations (we do not notationally explicate the accompanying homomorphisms from the log structure to $A(R)$). These derivations satisfy the following explicit formulas derived using the relation $d\log(X_0) + \cdots + d\log(X_r) = 0$:

$$\tfrac{\partial}{\partial \log(X_i)}(X_j) = \begin{cases} 0, & \text{if } 0 < j \neq i, \\ X_i, & \text{if } j = i, \end{cases} \qquad \text{and} \qquad \tfrac{\partial}{\partial \log(X_i)}(X_0) = \begin{cases} -X_0, & \text{if } 0 < i \leq r, \\ 0, & \text{if } r < i. \end{cases} \tag{5.10.2}$$

They also define an isomorphism $\Omega^1_{A(R)/A_{\mathrm{inf}},\log} \cong A(R)^{\oplus d}$, which extends to an isomorphism

$$\Omega^\bullet_{A(R)/A_{\mathrm{inf}},\log} \cong K_{A(R)}\left(\tfrac{\partial}{\partial \log(X_1)}, \dots, \tfrac{\partial}{\partial \log(X_d)}\right) \tag{5.10.3}$$

that may be regarded to be canonical because its construction uses only data determined by the local coordinate map (5.5.1).

The endomorphism induced by the Frobenius of the log A_{inf}-algebra $A(R)$ multiplies each $d\log(X_i)$ by p, so its effect on the right side of (5.10.3) is given in each degree j by p^j times the endomorphism induced by the Frobenius of $A(R)$.

5.11. The log structure on $A_{\mathrm{cris}}(R)$. We always, unless noted otherwise, equip the A_{inf}-algebras A_{cris} and $A^{(m)}_{\mathrm{cris}}$ for $m > 0$, as well as A_{cris}/p^n and $A^{(m)}_{\mathrm{cris}}/p^n$ for $n > 0$, with the base changes of the "default" log structure on A_{inf} described in §5.9. In the case of A_{cris}, this agrees with the log structure defined in §5.2. Likewise, we always, unless noted otherwise, equip the $A(R)$-algebras $A_{\mathrm{cris}}(R)$ and $A^{(m)}_{\mathrm{cris}}(R)$ for $m > 0$, as well as $A_{\mathrm{cris}}(R)/p^n$ and $A^{(m)}_{\mathrm{cris}}(R)/p^n$ for $n > 0$, with the base changes of the "default" log structure on $A(R)$, so that $A_{\mathrm{cris}}(R)$ and $A^{(m)}_{\mathrm{cris}}(R)$ are log smooth over A_{cris} and $A^{(m)}_{\mathrm{cris}}$, respectively. We set

$$\Omega^\bullet_{A_{\mathrm{cris}}(R)/A_{\mathrm{cris}},\log} := \Omega^\bullet_{A(R)/A_{\mathrm{inf}},\log}\widehat{\otimes}_{A_{\mathrm{inf}}} A_{\mathrm{cris}} \quad \text{and} \quad \Omega^\bullet_{A^{(m)}_{\mathrm{cris}}(R)/A^{(m)}_{\mathrm{cris}},\log} := \Omega^\bullet_{A(R)/A_{\mathrm{inf}},\log}\widehat{\otimes}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}.$$

These complexes are identified with the (global section complexes of the) logarithmic de Rham complexes of $\mathrm{Spf}(A_{\mathrm{cris}}(R))$ and $\mathrm{Spf}(A^{(m)}_{\mathrm{cris}}(R))$ over $\mathrm{Spf}(A_{\mathrm{cris}})$ and $\mathrm{Spf}(A^{(m)}_{\mathrm{cris}})$, respectively.

We use the p-adic completeness of $A_{\mathrm{cris}}(R)$ and its p-adic formal flatness over A_{cris} to extend the divided power structure of A_{cris} to $A_{\mathrm{cris}}(R)$ (see §5.8 and [SP, 07H1]). In effect, $\mathrm{Spf}(A_{\mathrm{cris}}(R))$ becomes a log PD thickening of $\mathrm{Spec}(R/p)$ that is log smooth over $\mathrm{Spf}(A_{\mathrm{cris}})$.

Through the results of [Bei13b], the following lemma will be key for relating the right side of (5.4.1) to the logarithmic de Rham cohomology of $\mathrm{Spf}(A_{\mathrm{cris}}(R))$ over $\mathrm{Spf}(A_{\mathrm{cris}})$.

Lemma 5.12. *For each $n \in \mathbb{Z}_{\geq 1}$, the log smooth log PD thickening $A_{\mathrm{cris}}(R)/p^n$ of R/p over A_{cris}/p^n is PD smooth in the sense of* [Bei13b, §1.4] (*see the proof for the definition*).

Proof. The PD smoothness is the claim that the indicated lift exists in every commutative square

$$\begin{tikzcd} U \arrow[r] \arrow[d, hook] & \mathrm{Spec}(R/p) \arrow[r, hook] & \mathrm{Spec}(A_{\mathrm{cris}}(R)/p^n) \arrow[d] \\ \widetilde{U} \arrow[urr, dashed] \arrow[rr] & & \mathrm{Spec}(A_{\mathrm{cris}}/p^n) \end{tikzcd}$$

of log schemes subject to the requirements that U is affine, $U \hookrightarrow \widetilde{U}$ is an (exact) log PD thickening, the log structure on $\widetilde{U}$ is integral and quasi-coherent, and the lift is a log PD morphism (see *loc. cit.*).

This sought property of $A_{\mathrm{cris}}(R)/p^n$ is invariant under base change that changes the log structure on A_{cris}/p^n, so we may assume that A_{cris}/p^n and $A_{\mathrm{cris}}(R)/p^n$ are instead equipped with the base changes of the "provisional" fine log structures defined in §5.9. Moreover, since the PD structure of $A_{\mathrm{cris}}(R)/p^n$ is extended from A_{cris}/p^n, the log PD thickening $\mathrm{Spec}(R/p) \hookrightarrow \mathrm{Spec}(A_{\mathrm{cris}}(R)/p^n)$ over A_{cris}/p^n is its own log PD-envelope (in the sense of [Bei13b, §1.3]). Thus, the claimed PD smoothness follows from [Bei13b, §1.4, Remarks (ii)] and the log smoothness of $A_{\mathrm{cris}}(R)/p^n$ over A_{cris}/p^n. ☐

In a local setting, we are ready to express the (presheaf version of the) right side of (5.4.1) in the form that will be convenient for the "all possible coordinates" technique.

Proposition 5.13. *In the local setting of* (5.5.1), *there are Frobenius-equivariant identifications*

$$R\Gamma_{\log\mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}) \cong \Omega^\bullet_{A_{\mathrm{cris}}(R)/A_{\mathrm{cris}},\log} \overset{(5.10.3)}{\cong} \Big(\varinjlim_{m>0} K_{A^{(m)}_{\mathrm{cris}}(R)}\Big(\tfrac{\partial}{\partial\log(X_1)}, \dots, \tfrac{\partial}{\partial\log(X_d)}\Big)\Big)^\wedge$$

(*see* §5.10 *for the description of the Frobenius action on the last term*).

Proof. By Lemma 5.12, each $A_{\mathrm{cris}}(R)/p^n$ is PD smooth over A_{cris}/p^n, so [Bei13b, (1.8.1)] gives the Frobenius-equivariant identification[6]

$$R\Gamma_{\log\mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}) \cong R\Gamma(\mathrm{Spf}(A_{\mathrm{cris}}(R))_{\mathrm{\acute{e}t}}, \Omega^\bullet_{\mathrm{Spf}(A_{\mathrm{cris}}(R))/\mathrm{Spf}(A_{\mathrm{cris}}),\log}).$$

On the other hand, since the sheaves $\Omega^i_{\mathrm{Spf}(A_{\mathrm{cris}}(R))/\mathrm{Spf}(A_{\mathrm{cris}}),\log}$ are locally free and, in particular, coherent, they are acyclic for $\Gamma(\mathrm{Spf}(A_{\mathrm{cris}}(R))_{\mathrm{\acute{e}t}}, -)$ (see [Ull95, 5.1]), so

$$R\Gamma(\mathrm{Spf}(A_{\mathrm{cris}}(R))_{\mathrm{\acute{e}t}}, \Omega^\bullet_{\mathrm{Spf}(A_{\mathrm{cris}}(R))/\mathrm{Spf}(A_{\mathrm{cris}}),\log}) \cong \Gamma(\mathrm{Spf}(A_{\mathrm{cris}}(R))_{\mathrm{\acute{e}t}}, \Omega^\bullet_{\mathrm{Spf}(A_{\mathrm{cris}}(R))/\mathrm{Spf}(A_{\mathrm{cris}}),\log}).$$

It remains to observe that the latter complex is identified with $\Omega^\bullet_{A_{\mathrm{cris}}(R)/A_{\mathrm{cris}},\log}$. ☐

Having rewritten both the left side of (5.4.1) in Corollary 5.7 and the right side in Proposition 5.13 in the desired forms, we would now like to exhibit an isomorphism between them. We will achieve this in Proposition 5.16 after the following preparations.

5.14. The element $\log([\epsilon])$**.** Let us fix an $m \geq p^2$. The elements $\frac{\mu^n}{(n+1)!} \in A^{(m)}_{\mathrm{cris}}[\frac{1}{p}]$ lie in $A^{(m)}_{\mathrm{cris}}$, are topologically nilpotent in $A^{(m)}_{\mathrm{cris}}$ if $n > 1$, and tend to 0 in the p-adic topology of $A^{(m)}_{\mathrm{cris}}$ as $n \to \infty$ (see the proof of [BMS16, 12.2]).[7] Consequently, recalling that $\mu = [\epsilon] - 1$, we may define

$$\log([\epsilon]) := \mu - \tfrac{\mu^2}{2} + \tfrac{\mu^3}{3} - \dots \qquad \text{in } A^{(m)}_{\mathrm{cris}}.$$

By *loc. cit.*,[8] the elements $\log([\epsilon])$ and μ are unit multiples of each other in $A^{(m)}_{\mathrm{cris}}$, so $\frac{(\log([\epsilon]))^n}{\mu\cdot n!}$ lies in $A^{(m)}_{\mathrm{cris}}$, is topologically nilpotent if $n > 1$, and tends to 0 in the p-adic topology of $A^{(m)}_{\mathrm{cris}}$ as $n \to \infty$. The Frobenius of $A^{(m)}_{\mathrm{cris}}$ maps $\log([\epsilon])$ to $p \cdot \log([\epsilon])$.

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[6] *Loc. cit.* uses the logarithmic PD de Rham complex, that is, the quotient of $\Omega^\bullet_{\mathrm{Spf}(A_{\mathrm{cris}}(R))/\mathrm{Spf}(A_{\mathrm{cris}}),\log}$ by the PD relations $d(u^{[m]}) = u^{[m-1]}du$, see [Bei13b, §1.7]. In our situation, there is no difference: since the PD structure of $A_{\mathrm{cris}}(R)/p^n$ is extended from the base A_{cris}/p^n, the PD relations hold already in $\Omega^\bullet_{\mathrm{Spf}(A_{\mathrm{cris}}(R))/\mathrm{Spf}(A_{\mathrm{cris}}),\log}$.

[7] For completeness, we review an argument that gives these claims. Since $p, \mu\xi$ is an A_{inf}-regular sequence, $\mu^p - \mu\xi^p \in p\xi\mu A_{\mathrm{inf}}$, so $\frac{\mu^{p-1}}{p} = \frac{\xi^p}{p} + \xi a$ for some $a \in A_{\mathrm{inf}}$. Thus, since $\frac{(p^2)!}{p^p} \in p\mathbb{Z}$, we have $\left(\frac{\mu^{p-1}}{p}\right)^p \in pA^{(m)}_{\mathrm{cris}}$, so $\frac{\mu^{p-1}}{p}$ is topologically nilpotent in $A^{(m)}_{\mathrm{cris}}$. In effect, since $\frac{1}{(n+1)!}p^{\lfloor\frac{n}{p-1}\rfloor} \in \mathbb{Z}$, the elements $\frac{\mu^n}{(n+1)!}$ tend to 0 in the p-adic topology of $A^{(m)}_{\mathrm{cris}}$ and are topologically nilpotent.

[8] The argument is as follows. By the previous footnote, $\sum_{n\geq p}\frac{\mu^n}{n+1}$ lies in $pA^{(m)}_{\mathrm{cris}}$. Thus, since each $\frac{\mu^n}{n+1}$ with $0 < n < p$ is topologically nilpotent in $A^{(m)}_{\mathrm{cris}}$, so is $\sum_{n\geq 1}\frac{\mu^n}{n+1}$. In conclusion, $\frac{\log([\epsilon])}{\mu}$ is a unit in $A^{(m)}_{\mathrm{cris}}$.

The following lemma uses the element $\log([\epsilon])$ and the Δ-action on $A^{(m)}_{\mathrm{cris}}(R)$ to describe the derivations $\frac{\partial}{\partial \log(X_i)} \colon A^{(m)}_{\mathrm{cris}}(R) \to A^{(m)}_{\mathrm{cris}}(R)$ that are induced from those in (5.10.1).

Lemma 5.15. *For every $m \geq p^2$, the element $\delta_i \in \Delta$ with $i = 1, \dots, d$ (see §3.2) acts on $A^{(m)}_{\mathrm{cris}}(R)$ as the series*

$$\exp(\log([\epsilon]) \cdot \tfrac{\partial}{\partial \log(X_i)}) := \textstyle\sum_{n \geq 0} \frac{(\log([\epsilon]))^n}{n!} (\frac{\partial}{\partial \log(X_i)})^n. \tag{5.15.1}$$

In particular, for such m and i, we have the following description of the "q-derivative" $\frac{\delta_i - 1}{\mu}$:

$$\tfrac{\delta_i - 1}{\mu} = \tfrac{\partial}{\partial \log(X_i)} \cdot \left(\textstyle\sum_{n \geq 1} \frac{(\log([\epsilon]))^n}{\mu \cdot n!} (\frac{\partial}{\partial \log(X_i)})^{n-1} \right) \qquad \text{as maps} \qquad A^{(m)}_{\mathrm{cris}}(R) \to A^{(m)}_{\mathrm{cris}}(R), \tag{5.15.2}$$

where the parenthetical factor defines an $A^{(m)}_{\mathrm{cris}}$-linear additive automorphism of $A^{(m)}_{\mathrm{cris}}(R)$.

Proof. The argument is similar to that of [BMS16, 12.3]. Firstly, $\frac{(\log([\epsilon]))^n}{n!}$ tends to 0 in $A^{(m)}_{\mathrm{cris}}$ for the p-adic topology (see §5.14), so the series (5.15.1) does define an $A^{(m)}_{\mathrm{cris}}$-linear additive endomorphism of $A^{(m)}_{\mathrm{cris}}(R)$. This endomorphism is actually also multiplicative because, by the Leibniz rule,

$$\tfrac{(\log([\epsilon]))^n}{n!} (\tfrac{\partial}{\partial \log(X_i)})^n (ab) = \textstyle\sum_{j=0}^n \frac{(\log([\epsilon]))^j}{j!} (\frac{\partial}{\partial \log(X_i)})^j (a) \cdot \frac{(\log([\epsilon]))^{n-j}}{(n-j)!} (\frac{\partial}{\partial \log(X_i)})^{n-j} (b).$$

Therefore, in the case $R = R^{\square}$ the desired equality

$$\delta_i = \exp(\log([\epsilon]) \cdot \tfrac{\partial}{\partial \log(X_i)}) \qquad \text{of endomorphisms} \qquad A^{(m)}_{\mathrm{cris}}(R^{\square}) \to A^{(m)}_{\mathrm{cris}}(R^{\square}) \tag{5.15.3}$$

follows by noting that, due to the formulas (5.10.2), both of its sides send X_i to $[\epsilon]X_i$, fix each X_j with $0 < j \neq i$, and send X_0 to $[\epsilon^{-1}]X_0$ if $i \leq r$ and to X_0 if $r < i$.

In the general case, since μ, and hence also ξ, divides each $\frac{(\log([\epsilon]))^n}{n!}$ with $n \geq 1$ (see §5.14), both sides of the equality (5.15.3) induce the trivial action modulo (p, ξ). Therefore, due to the formal étaleness of $A^{(m)}_{\mathrm{cris}}(R)$ over $A^{(m)}_{\mathrm{cris}}(R^{\square})$ and the settled $R = R^{\square}$ case, the sides agree.

Since $A^{(m)}_{\mathrm{cris}}(R)$ is μ-torsion free (see (3.27.2)) and $\mu \mid \frac{(\log([\epsilon]))^n}{n!}$ in $A^{(m)}_{\mathrm{cris}}$ (see §5.14), the equality (5.15.2) follows from (5.15.3). Since $\frac{(\log([\epsilon]))^n}{\mu \cdot n!}$ is a unit for $n = 1$, is topologically nilpotent if $n > 1$ (see §5.14), and p-adically tends to 0 as $n \to \infty$, the parenthetical factor of (5.15.2) is indeed an automorphism, as desired. $\square$

We are ready to settle the (presheaf version of the) local case of Theorem 5.4.

Proposition 5.16. *In the local setting of (5.5.1), for every $m \geq p^2$ and $i = 1, \dots, d$, the morphism*

$$\left(A^{(m)}_{\mathrm{cris}}(R) \xrightarrow{\frac{\partial}{\partial \log(X_i)}} A^{(m)}_{\mathrm{cris}}(R) \right) \xrightarrow{\left(\mathrm{id},\, \sum_{n \geq 1} \frac{(\log([\epsilon]))^n}{n!} (\frac{\partial}{\partial \log(X_i)})^{n-1}\right)} \left(A^{(m)}_{\mathrm{cris}}(R) \xrightarrow{\delta_i - 1} A^{(m)}_{\mathrm{cris}}(R) \right) \tag{5.16.1}$$

of complexes concentrated in degrees 0 and 1 is Frobenius equivariant, where the Frobenius action on the $A^{(m)}_{\mathrm{cris}}(R)$ in degree 1 of the source is multiplied by p (compare with §5.10). For every $m \geq p^2$, these morphisms induce a Frobenius-equivariant quasi-isomorphism

$$K_{A^{(m)}_{\mathrm{cris}}(R)} \left(\tfrac{\partial}{\partial \log(X_1)}, \dots, \tfrac{\partial}{\partial \log(X_d)} \right) \xrightarrow{\sim} \eta_{(\mu)} \left(K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)} (\delta_1 - 1, \dots, \delta_d - 1) \right), \tag{5.16.2}$$

which, as m varies, induces a Frobenius-equivariant identification that is a local version of (5.4.1):

$$R\Gamma_{\log \mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}) \cong R\Gamma(\mathfrak{X}^{\mathrm{psh}}_{\mathrm{\acute{e}t}}, A\Omega^{\mathrm{psh}}_{\mathfrak{X}}) \widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}. \tag{5.16.3}$$

Proof. The Frobenius-equivariance of (5.16.1) follows from the equations $\varphi(\log([\epsilon])) = p \cdot \log([\epsilon])$ (see §5.14) and $\frac{\partial}{\partial \log(X_i)} \circ \varphi = p \cdot \left(\varphi \circ \frac{\partial}{\partial \log(X_i)}\right)$ (see §5.10). Since Δ acts trivially on $A_{\mathrm{cris}}^{(m)}(R)/\mu$, the subcomplex

$$\eta_{(\mu)}\left(K_{A_{\mathrm{cris}}^{(m)}(R)}(\delta_1 - 1, \dots, \delta_d - 1)\right) \subset K_{A_{\mathrm{cris}}^{(m)}(R)}(\delta_1 - 1, \dots, \delta_d - 1)$$

is obtained by letting its j^{th} term for $j \geq 0$ be the submodule of μ^j-multiples inside the j^{th} term of $K_{A_{\mathrm{cris}}^{(m)}(R)}(\delta_1 - 1, \dots, \delta_d - 1)$, see (1.7.2) and (1.7.3). In particular, since $\mu \mid \varphi(\mu)$, this subcomplex is Frobenius-stable. Thus, Lemma 5.15 implies that the morphisms (5.16.1) induce an isomorphism

$$K_{A_{\mathrm{cris}}^{(m)}(R)}\left(\frac{\partial}{\partial \log(X_1)}, \dots, \frac{\partial}{\partial \log(X_d)}\right) \xrightarrow{\sim} \eta_{(\mu)}\left(K_{A_{\mathrm{cris}}^{(m)}(R)}(\delta_1 - 1, \dots, \delta_d - 1)\right). \tag{5.16.4}$$

Proposition 3.32 then implies that the natural inclusion of the target of (5.16.4) into the target of (5.16.2) is a quasi-isomorphism, and (5.16.2) follows. The maps (5.16.2) are compatible as m varies, so, by passing to the limit over m, forming the termwise p-adic completions, and applying Corollary 5.7 and Proposition 5.13, we obtain the desired identification (5.16.3). ☐

We now turn to the "all possible coordinates" technique that will globalize the arguments and eventually prove Theorem 5.4. For globalizing, the key point is to build, for a small enough affine $\mathfrak{X}$, a *functorial* in $\mathfrak{X}$ explicit complex that computes the presheaf version of the left side of (5.4.1) (see §5.21), to then also build such complex for the right side of (5.4.1) (see §5.32), and, finally, to build a natural isomorphism between the two complexes (see §5.38 and Proposition 5.39). Virtually every step of this process will rely on our work in the local case (5.5.1) discussed so far.

5.17. More general coordinates. Continuing to work locally, we now assume until the proof of Theorem 5.4 given in §5.40 that $\mathfrak{X}$ is affine, that is, $\mathfrak{X} = \mathrm{Spf}\, R$, and connected, and that we are given

- a finite set Σ that indexes the coordinates of a formal torus
$$R_\Sigma^\square := \mathcal{O}_C\{t_\sigma^{\pm 1} \mid \sigma \in \Sigma\};$$
- a nonempty finite set Λ and for each $\lambda \in \Lambda$ a $q_\lambda \in \mathbb{Q}_{>0}$ and an $\mathcal{O}_C$-algebra
$$R_\lambda^\square := \mathcal{O}_C\{t_{\lambda,0}, \dots, t_{\lambda,r_\lambda}, t_{\lambda,r_\lambda+1}^{\pm 1}, \dots, t_{\lambda,d}^{\pm 1}\}/(t_{\lambda,0} \cdots t_{\lambda,r_\lambda} - p^{q_\lambda});$$
- a closed immersion
$$\mathfrak{X} = \mathrm{Spf}\, R \to \mathrm{Spf}\, R_\Sigma^\square \times \textstyle\prod_{\lambda \in \Lambda} \mathrm{Spf}\, R_\lambda^\square \tag{5.17.1}$$
where the products are formed over $\mathrm{Spf}\, \mathcal{O}_C$, subject to the requirements that already
$$\mathfrak{X} = \mathrm{Spf}\, R \to \mathrm{Spf}\, R_\Sigma^\square \qquad \text{is a closed immersion,} \tag{5.17.2}$$
the induced map
$$\mathfrak{X} = \mathrm{Spf}\, R \to \mathrm{Spf}\, R_\lambda^\square \qquad \text{is étale for each } \lambda \in \Lambda, \tag{5.17.3}$$
and for some $\lambda \in \Lambda$, each irreducible component of $\mathrm{Spec}(R \otimes k)$ is cut out by a (unique) $t_{\lambda,i}$ with $0 \leq i \leq r_\lambda$ (which is equivalent to the intersection of any two irreducible components of $\mathrm{Spec}(R \otimes k)$ being nonempty, and hence implies this condition for every $\lambda \in \Lambda$: indeed, by (5.17.3), the irreducible components of $\mathrm{Spec}(R \otimes k)$ are *a priori* identified with the connected components of $\bigsqcup_i \mathrm{Spec}((R \otimes k)/(t_{\lambda,i}))$ for any $\lambda \in \Lambda$).

By §1.5, if R/p is not $\mathcal{O}_C/p$-smooth, then q_λ is determined by R and does not depend on λ. On the other hand, if R/p is $\mathcal{O}_C/p$-smooth, then the q_λ may differ; this and also the possibility that $r_\lambda > 0$ complicate matters in the "simpler" smooth case but are crucial to allow in order for the eventual "all possible coordinates" constructions to be functorial in R.

For any $\mathfrak{X}$, the data above exists on a basis for $\mathfrak{X}_{\text{ét}}$: to see this, first fix a geometric point x of $\mathfrak{X}$. If x lies in $\mathfrak{X}^{\text{sm}}$, then it has a required basis of étale neighborhoods because any $\mathfrak{X}$ is étale locally the formal spectrum of the p-adic completion of a finite type $\mathcal{O}_C$-algebra, the spectrum of which Zariski locally embeds into some $\mathbb{G}_m^\Sigma$. If x does not lie in $\mathfrak{X}^{\text{sm}}$, then it has a basis of affine étale neighborhoods $\operatorname{Spf} R$ that admit semistable local coordinates (5.5.1) for which $(t_0, \dots, t_r)$ cuts out x; further Zariski localization at x then ensures the existence of a closed immersion into some $\widehat{\mathbb{G}}_m^\Sigma$.

Each (5.17.3) is an instance of the local setup (5.5.1), so the local discussion between §5.5 and the present section applies to it. Another instance (with $r = 0$ and $d = \#\Sigma$) is the identity map $\operatorname{Spf} R_\Sigma^\square \xrightarrow{=} \operatorname{Spf} R_\Sigma^\square$, so the indicated discussion also applies to the ring $R_\Sigma^\square$ in place of $R^\square$.

Our first aim in this setup is to reexpress the (presheaf version of the) left side of (5.4.1) in §5.21.

5.18. The perfectoid cover $R_{\Sigma,\Lambda,\infty}$. For each $\lambda \in \Lambda$, we set

$$\Delta_\lambda := \left\{ (\epsilon_0, \dots, \epsilon_d) \in \left(\varprojlim_{m \geq 0} \mu_{p^m}(\mathcal{O}_C) \right)^{\oplus(d+1)} \,\middle|\, \epsilon_0 \cdots \epsilon_{r_\lambda} = 1 \right\} \simeq \mathbb{Z}_p^{\oplus d}$$

and let

$$\operatorname{Spa}(R_{\lambda,\infty}[\tfrac{1}{p}], R_{\lambda,\infty}) \to \operatorname{Spa}(R[\tfrac{1}{p}], R) \quad \text{and} \quad \operatorname{Spa}(R^\square_{\lambda,\infty}[\tfrac{1}{p}], R^\square_{\lambda,\infty}) \to \operatorname{Spa}(R^\square_\lambda[\tfrac{1}{p}], R^\square_\lambda) \tag{5.18.1}$$

be the affinoid perfectoid pro-(finite étale) Δ_λ-covers defined as in §3.2 using the coordinate map (5.17.3). Similarly, we set

$$\Delta_\Sigma := \left(\varprojlim_{m \geq 0} \mu_{p^m}(\mathcal{O}_C) \right)^{\Sigma} \simeq \mathbb{Z}_p^\Sigma$$

and let

$$\operatorname{Spa}(R^\square_{\Sigma,\infty}[\tfrac{1}{p}], R^\square_{\Sigma,\infty}) \to \operatorname{Spa}(R^\square_\Sigma[\tfrac{1}{p}], R^\square_\Sigma)$$

be the affinoid perfectoid pro-(finite étale) Δ_Σ-cover defined as in §3.2 using the coordinate map $\operatorname{Spf} R_\Sigma^\square \xrightarrow{=} \operatorname{Spf} R_\Sigma^\square$. Explicitly,

$$R^\square_{\Sigma,\infty} := \left(\varinjlim_{m \geq 0} \mathcal{O}_C\{ t_\sigma^{\pm 1/p^m} \,|\, \sigma \in \Sigma \} \right)^{\wedge}.$$

By taking products over $\operatorname{Spa}(\mathcal{O}_C[\frac{1}{p}], \mathcal{O}_C)$ and setting

$$\Delta_{\Sigma,\Lambda} := \Delta_\Sigma \times \textstyle\prod_{\lambda \in \Lambda} \Delta_\lambda, \tag{5.18.2}$$

we obtain the affinoid perfectoid pro-(finite étale) $\Delta_{\Sigma,\Lambda}$-cover

$$\operatorname{Spa}(R^\square_{\Sigma,\infty}[\tfrac{1}{p}], R^\square_{\Sigma,\infty}) \times \textstyle\prod_{\lambda\in\Lambda} \operatorname{Spa}(R^\square_{\lambda,\infty}[\tfrac{1}{p}], R^\square_{\lambda,\infty}) \to \operatorname{Spa}(R^\square_\Sigma[\tfrac{1}{p}], R^\square_\Sigma) \times \textstyle\prod_{\lambda\in\Lambda} \operatorname{Spa}(R^\square_\lambda[\tfrac{1}{p}], R^\square_\lambda),$$

which we abbreviate as

$$\operatorname{Spa}(R^\square_{\Sigma,\Lambda,\infty}[\tfrac{1}{p}], R^\square_{\Sigma,\Lambda,\infty}) \to \operatorname{Spa}(R^\square_{\Sigma,\Lambda}[\tfrac{1}{p}], R^\square_{\Sigma,\Lambda}).$$

Its base change along the generic fiber of the closed immersion (5.17.1) gives the pro-(finite étale) $\Delta_{\Sigma,\Lambda}$-cover

$$\operatorname{Spa}(R_{\Sigma,\Lambda,\infty}[\tfrac{1}{p}], R_{\Sigma,\Lambda,\infty}) \to \operatorname{Spa}(R[\tfrac{1}{p}], R), \tag{5.18.3}$$

which contains each $\operatorname{Spa}(R_{\lambda,\infty}[\frac{1}{p}], R_{\lambda,\infty}) \to \operatorname{Spa}(R[\frac{1}{p}], R)$ as a subcover. Thus, by the almost purity theorem [Sch12, 7.9 (iii)], the $\mathcal{O}_C$-algebra $R_{\Sigma,\Lambda,\infty}$ defined by (5.18.3) is perfectoid (by [BMS16, 3.20], the notions of 'perfectoid' used in [Sch12] and here agree).

5.19. The rings $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$ and $\mathbb{A}_{\mathrm{cris}}^{(m)}(R_{\Sigma,\Lambda,\infty})$. Similarly to §3.14, we set

$$\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty}) := W(R_{\Sigma,\Lambda,\infty}^{\flat}).$$

By Lemma 3.13, for each $n, n' > 0$, the sequence $(p^n, \mu^{n'})$ is regular on $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$, the $A_{\mathrm{inf}}/(p^n, \mu^{n'})$-algebra $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})/(p^n, \mu^{n'})$ is flat, and $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})/\mu$ is p-adically complete. As in §3.14, we have the surjection

$$\theta\colon \mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty}) \twoheadrightarrow R_{\Sigma,\Lambda,\infty}. \tag{5.19.1}$$

To fix the notation for the coordinates, we write the isomorphism (3.14.2) for $R_{\Sigma}^{\square}$ and $R_{\lambda}^{\square}$ as follows:

$$\begin{aligned} &A(R_{\Sigma}^{\square}) \cong A_{\mathrm{inf}}\{X_{\sigma}^{\pm 1}\}_{\sigma\in\Sigma}, \\ &A(R_{\lambda}^{\square}) \cong A_{\mathrm{inf}}\{X_{\lambda,0}, \dots, X_{\lambda,r_\lambda}, X_{\lambda,r_\lambda+1}^{\pm 1}, \dots, X_{\lambda,d}^{\pm 1}\}/(X_{\lambda,0}\cdots X_{\lambda,r_\lambda} - [(p^{1/p^\infty})^{q_\lambda}]). \end{aligned} \tag{5.19.2}$$

Likewise, similarly to §3.27, for an $m \in \mathbb{Z}_{\geq 1}$, we set

$$\mathbb{A}_{\mathrm{cris}}^{(m)}(R_{\Sigma,\Lambda,\infty}) := \mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})\widehat{\otimes}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}^{(m)},$$

where the completion is (p,μ)-adic (equivalently, p-adic if $m \geq p$). Since $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$ is (p,μ)-adically formally flat over A_{inf}, the ring $\mathbb{A}_{\mathrm{cris}}^{(m)}(R_{\Sigma,\Lambda,\infty})$ inherits p-torsion freeness from $A_{\mathrm{cris}}^{(m)}$. By using, in addition, the short exact sequences (3.26.4) together with the vanishing (3.26.3), we also see that $\mathbb{A}_{\mathrm{cris}}^{(m)}(R_{\Sigma,\Lambda,\infty})$ is μ-torsion free and $\mathbb{A}_{\mathrm{cris}}^{(m)}(R_{\Sigma,\Lambda,\infty})/\mu$ is p-adically complete. The map (5.19.1) gives rise to the compatible map

$$\theta\colon \mathbb{A}_{\mathrm{cris}}^{(m)}(R_{\Sigma,\Lambda,\infty}) \twoheadrightarrow R_{\Sigma,\Lambda,\infty}. \tag{5.19.3}$$

As in §3.14, the map θ intertwines the Witt vector Frobenius of $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$ with the absolute Frobenius of $R_{\Sigma,\Lambda,\infty}/p$. Likewise, as in §3.27, each $\mathbb{A}_{\mathrm{cris}}^{(m)}(R_{\Sigma,\Lambda,\infty})$ comes equipped with an $A_{\mathrm{cris}}^{(m)}$-semilinear Frobenius, and these Frobenii are compatible as m varies.

The profinite group $\Delta_{\Sigma,\Lambda}$ acts continuously and Frobenius-equivariantly on the rings above. To analyze this action, we use the compatible system ϵ of p-power roots of unity chosen in §2.1 and define elements $\delta_\sigma \in \Delta_\Sigma$ by

$$\delta_\sigma := (1, \dots, 1, \epsilon, 1, \dots, 1) \quad \text{for } \sigma \in \Sigma, \qquad \text{where the } \sigma^{\text{th}} \text{ entry is nonidentity,}$$

as well as, for every $\lambda \in \Lambda$, the elements $\delta_{\lambda,i} \in \Delta_\lambda$ by

$$\delta_{\lambda,i} := (\epsilon^{-1}, 1, \dots, 1, \epsilon, 1, \dots, 1) \quad \text{for } i = 1, \dots, r_\lambda, \quad \text{where the } 0^{\text{th}} \text{ and } i^{\text{th}} \text{ entries are nonidentity;}$$

$$\delta_{\lambda,i} := (1, \dots, 1, \epsilon, 1, \dots, 1) \quad \text{for } i = r_\lambda + 1, \dots, d, \quad \text{where the } i^{\text{th}} \text{ entry is nonidentity.}$$

Jointly, the δ_σ and the $\delta_{\lambda,i}$ form a system of free topological generators for $\Delta_{\Sigma,\Lambda}$.

Using the following consequence of Theorem 3.34, in §5.21 we will build a functorial complex that locally computes the left side of the desired identification (5.4.1).

Proposition 5.20. *In the local setup of* §5.17, *for every $m \geq 1$, the edge map* (*see* §3.15 *and* §3.28)

$$\eta_{(\mu)}\left(K_{\mathbb{A}_{\mathrm{cris}}^{(m)}(R_{\Sigma,\Lambda,\infty})}((\delta_\sigma - 1)_{\sigma\in\Sigma}, (\delta_{\lambda,i} - 1)_{\lambda\in\Lambda,\, 1\leq i\leq d})\right) \xrightarrow{\sim} R\Gamma(\mathfrak{X}_{\mathrm{\acute{e}t}}^{\mathrm{psh}}, A\Omega_{\mathfrak{X}}^{\mathrm{psh}})\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}^{(m)} \tag{5.20.1}$$

is a Frobenius-equivariant quasi-isomorphism. In particular, we have the following Frobenius-equivariant identification in the derived category:

$$\left(\varinjlim_m \left(\eta_{(\mu)}\left(K_{\mathbb{A}_{\mathrm{cris}}^{(m)}(R_{\Sigma,\Lambda,\infty})}((\delta_\sigma - 1)_{\sigma\in\Sigma}, (\delta_{\lambda,i} - 1)_{\lambda\in\Lambda,\, 1\leq i\leq d})\right)\right)\right)^{\wedge} \xrightarrow{\sim} R\Gamma(\mathfrak{X}_{\mathrm{\acute{e}t}}^{\mathrm{psh}}, A\Omega_{\mathfrak{X}}^{\mathrm{psh}})\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}},$$

where the direct limit and the p-adic completion of the complexes in the source are formed termwise.

Proof. The termwise p-adic completion of the source in the last display agrees with the derived p-adic completion because each $\mathbb{A}_{\mathrm{cris}}^{(m)}(R_{\Sigma,\,\Lambda,\,\infty})$ is p-torsion free. Moreover, the Frobenius-equivariance aspects follow from the Frobenius-equivariance of the edge map used to construct (5.20.1) and that of the identification (5.1.1). Thus, since the pro-(finite étale) affinoid perfectoid $\Delta_{\Sigma,\,\Lambda}$-cover $\mathrm{Spa}(R_{\Sigma,\,\Lambda,\,\infty}[\frac{1}{p}], R_{\Sigma,\,\Lambda,\,\infty})$ of $\mathrm{Spa}(R[\frac{1}{p}], R)$ contains $\mathrm{Spa}(R_{\lambda,\,\infty}[\frac{1}{p}], R_{\lambda,\,\infty})$ as a subcover, the claim follows from Lemma 3.7 and Remark 3.35. □

5.21. A functorial complex that computes $R\Gamma(\mathfrak{X}_{\mathrm{ét}}^{\mathrm{psh}}, A\Omega_{\mathfrak{X}}^{\mathrm{psh}})\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}$. For a fixed R, the isomorphisms of Proposition 5.20 are compatible with enlarging Σ and Λ. Therefore, by taking the filtered direct limit over all the closed immersions (5.17.1) for varying Σ and Λ (but fixed R), we may build the complex

$$\varinjlim_{\Sigma,\,\Lambda}\left(\left(\varinjlim_{m>0}\left(\eta_{(\mu)}\left(K_{\mathbb{A}_{\mathrm{cris}}^{(m)}(R_{\Sigma,\,\Lambda,\,\infty})}((\delta_\sigma-1)_{\sigma\in\Sigma},(\delta_{\lambda,\,i}-1)_{\lambda\in\Lambda,\,1\le i\le d})\right)\right)\right)^{\wedge}\right), \tag{5.21.1}$$

where the direct limits and the p-adic completion are formed termwise, that comes equipped with an A_{cris}-semilinear Frobenius endomorphism. By Proposition 5.20, in the derived category this complex is canonically and Frobenius-equivariantly isomorphic to

$$R\Gamma(\mathfrak{X}_{\mathrm{ét}}^{\mathrm{psh}}, A\Omega_{\mathfrak{X}}^{\mathrm{psh}})\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}.$$

Moreover, if R' is a formally étale R-algebra equipped with a closed immersion as in (5.17.1) for some sets Σ' and Λ', then the term indexed by Σ, Λ (and by a closed immersion (5.17.1)) of the direct limit (5.21.1) maps to the term indexed by $\Sigma\cup\Sigma'$, $\Lambda\cup\Lambda'$ (and by a closed immersion of $\mathrm{Spf}\,R'$) of the analogous direct limit for R', compatibly with the transition maps in (5.32.1) and the Frobenius. Consequently, the complex (5.21.1) equipped with its Frobenius is functorial in R (equipped with the closed immersion (5.17.1)).

Our next aim is to similarly reexpress the (presheaf version of the) right side of (5.4.1) in §5.32.

5.22. The completed log PD envelope $D_{\Sigma,\,\Lambda}$. By §5.9, the maps θ of (3.14.3) give a Frobenius-equivariant closed immersion

$$\mathrm{Spec}(R/p)\hookrightarrow \mathrm{Spf}(A(R_{\Sigma}^{\square}))\times \textstyle\prod_{\lambda\in\Lambda}\mathrm{Spf}(A(R_{\lambda}^{\square}))=:\mathrm{Spf}(A(R_{\Sigma,\,\Lambda}^{\square})) \tag{5.22.1}$$

of (p,μ)-adic formal log schemes, where the products are formed over the (p,μ)-adic formal log scheme $\mathrm{Spf}(A_{\mathrm{inf}})$. By [Kat89, 4.1 and 4.4], for each $n, n'\in\mathbb{Z}_{>0}$, the (quasi-coherent) log structure of $\mathrm{Spec}(A(R_{\Sigma,\,\Lambda}^{\square})/(p^n,\mu^{n'}))$ is integral and the map $\mathrm{Spec}(A(R_{\Sigma,\,\Lambda}^{\square})/(p^n,\mu^{n'}))\to\mathrm{Spec}(A_{\mathrm{inf}}/(p^n,\mu^{n'}))$ of log schemes is also integral.

For each $n, n'\in\mathbb{Z}_{>0}$, by [Bei13b, 1.3, Theorem], the $A_{\mathrm{inf}}/(p^n,\mu^{n'})$-base change of the closed immersion (5.22.1) has a log PD envelope $\mathrm{Spec}(D_{\Sigma,\,\Lambda,\,n,\,n'})$ over $(\mathbb{Z}/p^n\mathbb{Z}, p\mathbb{Z}/p^n\mathbb{Z})$, which, in particular, is a nil thickening of $\mathrm{Spec}(R/p)$, so is also affine as indicated. In fact, $D_{\Sigma,\,\Lambda,\,n,\,n'}$ is supplied already by [Kat89, 5.4] because the closed immersion (5.22.1) is a base change of a similar closed immersion of fine log schemes over A_{inf} along a "change of log structure" self-map of A_{inf} (compare with §5.9).[9]

If n' is large enough relative to n, so that $\mu^{n'}\in p^nA_{\mathrm{cris}}$, then, since $\mathrm{Spec}(A_{\mathrm{cris}}/p^n)$ is identified with the log PD envelope of the exact log closed immersion $\mathrm{Spec}(\mathcal{O}_C/p)\hookrightarrow\mathrm{Spec}(A_{\mathrm{inf}}/(p^n,\mu^{n'}))$ over $(\mathbb{Z}/p^n\mathbb{Z}, p\mathbb{Z}/p^n\mathbb{Z})$ (see §5.1), $\mathrm{Spec}(D_{\Sigma,\,\Lambda,\,n,\,n'})$ comes equipped with a canonical log PD morphism to $\mathrm{Spec}(A_{\mathrm{cris}}/p^n)$ that identifies it with the log PD envelope of

$$\mathrm{Spec}(R/p)\hookrightarrow\mathrm{Spec}(A(R_{\Sigma,\,\Lambda}^{\square})\otimes_{A_{\mathrm{inf}}}A_{\mathrm{cris}}/p^n)\qquad\text{over}\qquad\mathrm{Spec}(\mathcal{O}_C/p)\hookrightarrow\mathrm{Spec}(A_{\mathrm{cris}}/p^n).$$

[9] The two references characterize the log PD envelope differently, but this is not an issue for us essentially because the image of any monoid morphism $M\to M'$ with M finitely generated is finitely generated.

In particular, letting $D_{\Sigma,\Lambda,n}$ denote the common $D_{\Sigma,\Lambda,n,n'}$ for large enough n', so that we have $D_{\Sigma,\Lambda,n}/p^{n-1} \cong D_{\Sigma,\Lambda,n-1}$ for $n>1$, we obtain a formal log $\mathrm{Spf}(A_{\mathrm{cris}})$-scheme $\mathrm{Spf}(D_{\Sigma,\Lambda})$ that fits into a factorization

$$\mathrm{Spec}(R/p) \hookrightarrow \mathrm{Spf}(D_{\Sigma,\Lambda}) \to \mathrm{Spf}(A_{\mathrm{cris}}(R^{\square}_{\Sigma})) \times \textstyle\prod_{\lambda\in\Lambda} \mathrm{Spf}(A_{\mathrm{cris}}(R^{\square}_{\lambda})) =: \mathrm{Spf}(A_{\mathrm{cris}}(R^{\square}_{\Sigma,\Lambda})), \quad (5.22.2)$$

where the products are formed over the formal log scheme $\mathrm{Spf}(A_{\mathrm{cris}})$. By the functoriality of its construction, $\mathrm{Spf}(D_{\Sigma,\Lambda})$ comes equipped with an A_{cris}-semilinear Frobenius endomorphism. In addition, since, for each $n>0$, the ideal defining the exact closed immersion $\mathrm{Spec}(R/p) \hookrightarrow \mathrm{Spec}(R/p^n)$ inherits divided powers from $\mathbb{Z}/p^n$, the universal property of $D_{\Sigma,\Lambda}$ supplies the factorization

$$\mathrm{Spec}(R/p) \hookrightarrow \mathrm{Spf}(R) \hookrightarrow \mathrm{Spf}(D_{\Sigma,\Lambda}) \quad \text{over} \quad \mathrm{Spec}(\mathcal{O}_C/p) \hookrightarrow \mathrm{Spf}(\mathcal{O}_C) \hookrightarrow \mathrm{Spf}(A_{\mathrm{cris}}). \quad (5.22.3)$$

The profinite group $\Delta_{\Sigma,\Lambda}$ acts continuously and Frobenius-equivariantly on $A(R^{\square}_{\Sigma,\Lambda})$ over A_{inf} (see (5.18.2), §5.19, and §3.14), and, due to the last paragraph of §5.9, this action extends to a $\Delta_{\Sigma,\Lambda}$-action on the formal log scheme $\mathrm{Spf}(A(R^{\square}_{\Sigma,\Lambda}))$. Moreover, the closed immersion (5.22.1) is $\Delta_{\Sigma,\Lambda}$-equivariant, so $\Delta_{\Sigma,\Lambda}$ acts continuously and A_{cris}-linearly on each $D_{\Sigma,\Lambda,n}$ and also on $D_{\Sigma,\Lambda}$.

For our purposes, the utility of $D_{\Sigma,\Lambda}$ will manifest itself through the following proposition:

Proposition 5.23. *In the local setting of §5.17, the complex (where the inverse limit is termwise)*

$$\Omega^{\bullet}_{D_{\Sigma,\Lambda}/A_{\mathrm{cris}},\log,\mathrm{PD}} := \varprojlim_{n>0} \left(\Omega^{\bullet}_{(A_{\mathrm{cris}}(R^{\square}_{\Sigma,\Lambda})/p^n)/(A_{\mathrm{cris}}/p^n),\log} \otimes_{A_{\mathrm{cris}}(R^{\square}_{\Sigma,\Lambda})/p^n} D_{\Sigma,\Lambda,n}\right)$$

may be canonically and Frobenius-equivariantly identified (in the derived category) as follows:

$$R\Gamma_{\log\mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}) \cong \Omega^{\bullet}_{D_{\Sigma,\Lambda}/A_{\mathrm{cris}},\log,\mathrm{PD}}. \quad (5.23.1)$$

Under this identification, the natural map

$$R\Gamma_{\log\mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}) \to R\Gamma_{\log\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C) \quad \textit{is} \quad \Omega^{\bullet}_{D_{\Sigma,\Lambda}/A_{\mathrm{cris}},\log,\mathrm{PD}} \to \Omega^{\bullet}_{\mathrm{Spf}(R)/\mathcal{O}_C,\log} \quad (5.23.2)$$

induced by the factorization (5.22.3). In particular, we have a Frobenius-equivariant identification

$$R\Gamma_{\log\mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}) \cong K_{D_{\Sigma,\Lambda}}\left(\left(\tfrac{\partial}{\partial\log(X_\sigma)}\right)_{\sigma\in\Sigma}, \left(\tfrac{\partial}{\partial\log(X_{\lambda,i})}\right)_{\lambda\in\Lambda,\,1\le i\le d}\right) \quad (5.23.3)$$

where the $\frac{\partial}{\partial\log(X_\sigma)}$ *(resp.,* $\frac{\partial}{\partial\log(X_{\lambda,i})}$*) are defined as in (5.10.1) with* $R^{\square}_{\Sigma}$ *(resp.,* $R^{\square}_{\lambda}$*) in place of* R *and the Frobenius acts on the degree* j *term of the right side by* p^j *times the action induced from the Frobenius action on* $D_{\Sigma,\Lambda}$ *(compare with §5.10).*

Proof. By §5.11, each $A_{\mathrm{cris}}(R^{\square}_{\Sigma,\Lambda})/p^n$ is a log smooth thickening of R/p over A_{cris}/p^n. Therefore, by [Bei13b, 1.4, Remarks (ii)], the PD thickening $D_{\Sigma,\Lambda,n}$ of R/p is PD smooth over A_{cris}/p^n (see the proof of Lemma 5.12 for the definition). Consequently, by [Bei13b, (1.8.1)], the logarithmic PD de Rham complex $\Omega^{\bullet}_{D_{\Sigma,\Lambda,n}/(A_{\mathrm{cris}}/p^n),\log,\mathrm{PD}}$ computes $R\Gamma_{\log\mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/(A_{\mathrm{cris}}/p^n)})$; the Frobenius-equivariance aspect follows by functoriality. By [Bei13b, 1.7, Exercises, (i)],

$$\Omega^{\bullet}_{D_{\Sigma,\Lambda,n}/(A_{\mathrm{cris}}/p^n),\log,\mathrm{PD}} \cong \Omega^{\bullet}_{(A_{\mathrm{cris}}(R^{\square}_{\Sigma,\Lambda})/p^n)/(A_{\mathrm{cris}}/p^n),\log} \otimes_{A_{\mathrm{cris}}(R^{\square}_{\Sigma,\Lambda})/p^n} D_{\Sigma,\Lambda,n}, \quad (5.23.4)$$

so the identification (5.23.1) follows. Since each R/p^n is a log smooth thickening of R/p over $\mathcal{O}_C/p^n$, similar reasoning applies to $R\Gamma_{\log\mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/R}) \overset{\text{[Bei13b, (1.8.1)]}}{\cong} R\Gamma_{\log\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)$ and gives (5.23.2).

The identification (5.23.3) then results from the Frobenius-equivariant identifications

$$\Omega^{\bullet}_{(A_{\mathrm{cris}}(R^{\square}_{\Sigma,\Lambda})/p^n)/(A_{\mathrm{cris}}/p^n),\log} \cong K_{A_{\mathrm{cris}}(R^{\square}_{\Sigma,\Lambda})/p^n}\left(\left(\tfrac{\partial}{\partial\log(X_\sigma)}\right)_{\sigma\in\Sigma}, \left(\tfrac{\partial}{\partial\log(X_{\lambda,i})}\right)_{\lambda\in\Lambda,\,1\le i\le d}\right)$$

supplied by (5.10.3). ☐

Remark 5.24. In fact, by [Bei13b, (1.11.1)], the first map in (5.23.2) induces the identification

$$R\Gamma_{\log\mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}) \otimes^{\mathbb{L}}_{A_{\mathrm{cris}}} \mathcal{O}_C \cong R\Gamma_{\log\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C) \tag{5.24.1}$$

in the derived category, so the same holds for the second map:

$$\Omega^{\bullet}_{D_{\Sigma,\Lambda}/A_{\mathrm{cris}},\log,\mathrm{PD}} \otimes^{\mathbb{L}}_{A_{\mathrm{cris}}} \mathcal{O}_C \cong \Omega^{\bullet}_{\mathrm{Spf}(R)/\mathcal{O}_C,\log}. \tag{5.24.2}$$

To bring the identification (5.23.3) in a form that mimics the last display of the statement of Proposition 5.20, in §5.30 we will express $D_{\Sigma,\Lambda}$ as a completed direct limit of rings $D^{(m)}_{\Sigma,\Lambda}$ that, loosely speaking, are generated by divided powers of degree at most m, see (5.30.1). For this, it will be useful to exploit the ideas from the proof of [Kat89, (4.10) (1)] to identify $D_{\Sigma,\Lambda}$ with the p-adic completion of the (non log) divided power envelope of an *exact* closed immersion in Lemma 5.29.[10]

5.25. A chart for $A(R^{\square}_{\Sigma,\Lambda})$. To express $D_{\Sigma,\Lambda}$ as the p-adic completion of a usual (that is, non log) divided power envelope, we will build a chart for the (fine version) of the log closed immersion $\mathrm{Spec}(R/p) \hookrightarrow \mathrm{Spec}(A(R^{\square}_{\Sigma,\Lambda}))$ of (5.22.1). For this, we fix the unique $q \in \mathbb{Q}_{>0}$ for which

$$\mathbb{Z}\cdot q = \Sigma_{\lambda\in\Lambda}\,\mathbb{Z}\cdot q_\lambda \quad \text{inside} \quad \mathbb{Q},$$

so that $\frac{q_\lambda}{q} \in \mathbb{Z}_{>0}$ for every λ (and even $q = q_\lambda$ in the case when R/p is not $\mathcal{O}_C/p$-smooth, see §5.17). We endow $\mathcal{O}_C/p$ (resp., A_{inf}) with the (fine) log structure determined by

$$\mathbb{N}_{\geq 0} \to \mathcal{O}_C/p \quad \text{with} \quad 1 \mapsto p^q \qquad (\text{resp.,} \qquad \mathbb{N}_{\geq 0} \to A_{\mathrm{inf}} \quad \text{with} \quad 1 \mapsto [(p^q)^{1/p^\infty}]).$$

For each $\lambda \in \Lambda$, we let $Q_\lambda \subset \frac{q}{q_\lambda}\prod_{0\leq i\leq r_\lambda}\mathbb{N}_{\geq 0}$ be the submonoid generated by $\prod_{0\leq i\leq r_\lambda}\mathbb{N}_{\geq 0}$ and the diagonal $(\frac{q}{q_\lambda},\dots,\frac{q}{q_\lambda})$, so that the chart

$$Q_\lambda \to A(R^{\square}_\lambda) \qquad \text{given by} \qquad (\tfrac{q}{q_\lambda},\dots,\tfrac{q}{q_\lambda}) \mapsto [(p^{1/p^\infty})^q] \quad \text{and} \quad \textstyle\prod_{0\leq i\leq r_\lambda}\mathbb{N}_{\geq 0} \xrightarrow{(n_i)\mapsto\prod X_{\lambda,i}^{n_i}} A(R^{\square}_\lambda)$$

makes $A(R^{\square}_\lambda)$ a fine log A_{inf}-algebra. We let

$$Q := \left(\textstyle\prod_{\lambda\in\Lambda} Q_\lambda\right)/((\tfrac{q}{q_{\lambda_1}},\dots,\tfrac{q}{q_{\lambda_1}}) = (\tfrac{q}{q_{\lambda_2}},\dots,\tfrac{q}{q_{\lambda_2}}))_{\lambda_1\neq\lambda_2}$$

be the quotient monoid obtained by identifying the diagonal elements $(\frac{q}{q_\lambda},\dots,\frac{q}{q_\lambda})$, so that the map

$$Q \to A(R^{\square}_{\Sigma,\Lambda}) \qquad \text{that results from the charts} \qquad Q_\lambda \to A(R^{\square}_\lambda)$$

is a chart for the target $\mathrm{Spf}(A(R^{\square}_{\Sigma,\Lambda}))$ of the fine version of the log closed immersion (5.22.1). In terms of this chart, the Frobenius action multiplies each element of Q by p.

To prepare for building a convenient chart for R/p, for each $\lambda \in \Lambda$ we define an indexing set by

$$\mathcal{I}_\lambda := \{i \mid 0 \leq i \leq d,\ t_{\lambda,i} \notin R^\times\}.$$

5.26. A convenient chart in the smooth case. Assume that R/p is $\mathcal{O}_C/p$-smooth. Then for each λ, there is a unique $0 \leq i_\lambda \leq r_\lambda$ with $t_{\lambda,i_\lambda} \notin R^\times$. For every $\lambda_0 \in \Lambda$, we consider the monoid

$$P_{\lambda_0} := \left(\mathbb{N}_{\geq 0} \times \textstyle\prod_{0\leq i\leq r_{\lambda_0},\, i\neq i_{\lambda_0}}\mathbb{Z}\right) \times \prod_{\lambda\neq\lambda_0}\left(\left(\prod_{0\leq i\leq r_\lambda}\mathbb{Z}\right)/\mathbb{Z}\right), \tag{5.26.1}$$

[10] In fact, the arguments below would become more direct if we could "uncomplete" $D_{\Sigma,\Lambda}$ by constructing the log PD envelope of the (possibly nonexact) log closed immersion (5.22.1) itself. Neither [Kat89, 5.4] nor [Bei13b, 1.3, Theorem] gives this hypothetical envelope because p is not nilpotent in $A(R^{\square}_{\Sigma,\Lambda})$.

where each $\mathbb{Z}$ by which we quotient is embedded diagonally. For each (λ, i) with $0 \le i \le r_\lambda$, there is a unique $v_{\lambda,i} \in R^\times$ such that $t_{\lambda,i} = (p^q)^{n_{\lambda,i}} \cdot v_{\lambda,i}$ in R for a (unique) $n_{\lambda,i} \in \mathbb{Z}_{\ge 0}$: explicitly, $n_{\lambda,i} = \frac{q_\lambda}{q}$ if $i \in \mathcal{I}_\lambda$, and else $n_{\lambda,i} = 0$. In particular, $\prod_{0 \le i \le r_\lambda} v_{\lambda,i} = 1$ for each λ. The map

$$P_{\lambda_0} \to R/p \qquad \text{given by} \qquad \mathbb{N}_{\ge 0} \ni 1 \mapsto p^q, \quad \mathbb{Z}_{(\lambda,i)} \ni 1 \mapsto v_{\lambda,i},$$

where the subscript (λ, i) indicates index of the factor $\mathbb{Z}$ in (5.26.1), is a chart for the source $\mathrm{Spec}(R/p)$ of the fine version of the log closed immersion (5.22.1). In terms of this chart, the Frobenius action multiplies each element of P_{λ_0} by p.

Moreover, there is a natural Frobenius-equivariant chart

$$Q \to P_{\lambda_0} = \left(\mathbb{N}_{\ge 0} \times \textstyle\prod_{0 \le i \le r_{\lambda_0},\, i \ne i_{\lambda_0}} \mathbb{Z}\right) \times \textstyle\prod_{\lambda \ne \lambda_0} \left(\left(\prod_{0 \le i \le r_\lambda} \mathbb{Z}\right)/\mathbb{Z}\right) \tag{5.26.2}$$

for this fine version of (5.22.1): for instance, it maps $1 \in (\mathbb{N}_{\ge 0})_{(\lambda_0, i_{\lambda_0})}$ to the element $(\frac{q_{\lambda_0}}{q}, -1, \dots, -1)$ of $\mathbb{N}_{\ge 0} \times \prod_{0 \le i \le r_{\lambda_0},\, i \ne i_{\lambda_0}} \mathbb{Z}$, each $(\frac{q}{q_\lambda}, \dots, \frac{q}{q_\lambda})$ to $1 \in \mathbb{N}_{\ge 0}$, each $1 \in (\mathbb{N}_{\ge 0})_{(\lambda,i)}$ with $i \ne i_\lambda$ to $1 \in \mathbb{Z}_{(\lambda,i)}$, etc.—the key is that the image under $Q \to A(R^\square_{\Sigma,\Lambda}) \twoheadrightarrow R/p$ of every generator of Q is evidently expressible in terms of the images of the generators of P_{λ_0} (without knowing the "values" of these images).

The $A(R^\square_{\Sigma,\Lambda})$-algebra $A(R^\square_{\Sigma,\Lambda}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]$ comes equipped with an $A(R^\square_{\Sigma,\Lambda})$-semilinear Frobenius and is initial among the $A(R^\square_{\Sigma,\Lambda})$-algebras B equipped with a unit $V_{\lambda,i} \in B^\times$ for each (λ, i) with $0 \le i \le r_\lambda$ subject to the relations

$$X_{\lambda,i} = [((p^{1/p^\infty})^q)^{n_{\lambda,i}}] \cdot V_{\lambda,i}, \qquad \textstyle\prod_{0 \le i \le r_\lambda} V_{\lambda,i} = 1. \tag{5.26.3}$$

In particular,

$$R \quad \text{is naturally an} \quad (A(R^\square_{\Sigma,\Lambda}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}])\text{-algebra} \quad (\text{with } V_{\lambda,i} = v_{\lambda,i}), \tag{5.26.4}$$

so the scheme counterpart of the fine variant of the closed immersion (5.22.1) factors Frobenius-equivariantly as follows:

$$\mathrm{Spec}(R/p) \overset{j_{\lambda_0}}{\hookrightarrow} \mathrm{Spec}\left(A(R^\square_{\Sigma,\Lambda}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]\right) \xrightarrow{q_{\lambda_0}} \mathrm{Spec}(A(R^\square_{\Sigma,\Lambda})), \tag{5.26.5}$$

where $\mathrm{Spec}\left(A(R^\square_{\Sigma,\Lambda}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]\right)$ is equipped with the log structure pulled back from $\mathbb{Z}[P_{\lambda_0}]$. By construction, j_{λ_0} is an exact closed immersion and, by [Kat89, 3.5], the projection q_{λ_0} is log étale.

The relations (5.26.3) do not depend on the choice of λ_0, so neither does the factorization (5.26.5). More precisely, for any $\lambda_0' \in \Lambda$, we have the a natural isomorphism over Q of charts for R/p:

$$P_{\lambda_0} \xrightarrow{\sim} P_{\lambda_0'}, \tag{5.26.6}$$

and this isomorphism gives rise to the vertical Frobenius-equivariant isomorphism in the commutative diagram

$$\begin{tikzcd} & \mathrm{Spec}\left(A(R^\square_{\Sigma,\Lambda}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]\right) \arrow[dr, "q_{\lambda_0}"] & \\ \mathrm{Spec}(R/p) \arrow[ur, hook, "j_{\lambda_0}"] \arrow[dr, hook, "j_{\lambda_0'}"'] & & \mathrm{Spec}(A(R^\square_{\Sigma,\Lambda})). \\ & \mathrm{Spec}\left(A(R^\square_{\Sigma,\Lambda}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0'}]\right) \arrow[uu, "\wr"] \arrow[ur, "q_{\lambda_0'}"'] & \end{tikzcd} \tag{5.26.7}$$

5.27. A convenient chart in the nonsmooth case. Assume that R/p is not $\mathcal{O}_C/p$-smooth. We have $q_\lambda = q$ and $Q_\lambda \cong \prod_{0 \le i \le r_\lambda} \mathbb{N}_{\ge 0}$, so, letting $\Delta_\lambda \subset Q_\lambda$ denote the diagonal copy of $\mathbb{N}_{\ge 0}$, also

$$Q \cong \left(\textstyle\prod_{\lambda \in \Lambda} \left(\prod_{0 \le i \le r_\lambda} \mathbb{N}_{\ge 0}\right)\right) / \left(\Delta_{\lambda_1} = \Delta_{\lambda_2}\right)_{\lambda_1 \ne \lambda_2}.$$

By §5.17, each $t_{\lambda,\, i} \notin R^\times$ cuts out a unique irreducible component $\overline{\{y_{\lambda,\, i}\}}$ of $\mathrm{Spec}(R \otimes_{\mathcal{O}_C} k)$. Moreover, the generic point $y_{\lambda,\, i}$ of this component determines the ideal $(t_{\lambda,\, i}) \subset R$: indeed, $(p^q) \subset (t_{\lambda,\, i})$ in R and the ideal $(t_{\lambda,\, i})/(p^q) \subset R/(p^q)$ is the kernel of the localization map $R/(p^q) \to (R/(p^q))_{y_{\lambda,\, i}}$, as may be seen over $R^\square_\lambda$. Conversely, for each generic point y of $\mathrm{Spec}(R \otimes_{\mathcal{O}_C} k)$ and each $\lambda \in \Lambda$, a unique $t_{\lambda,\, i_\lambda(y)}$ with $0 \le i_\lambda(y) \le r_\lambda$ cuts out $\overline{\{y\}}$. Consequently, for each y and every $\lambda, \lambda_0 \in \Lambda$, there is a unique $u_{\lambda,\, \lambda_0,\, y} \in R^\times$ such that we have

$$t_{\lambda,\, i_\lambda(y)} = u_{\lambda,\, \lambda_0,\, y} \cdot t_{\lambda_0,\, i_{\lambda_0}(y)} \quad \text{in} \quad R. \tag{5.27.1}$$

Letting $\mathcal{Y}$ denote the set of generic points of $\mathrm{Spec}(R \otimes_{\mathcal{O}_C} k)$, for $\lambda_0 \in \Lambda$ we consider the monoid

$$P_{\lambda_0} := \left(\left(\prod_{\mathcal{Y}} \mathbb{N}_{\ge 0} \times \prod_{\{0 \le i \le r_{\lambda_0}\} \setminus i_{\lambda_0}(\mathcal{Y})} \mathbb{Z}\right) \times \prod_{\lambda \ne \lambda_0} \left(\prod_{0 \le i \le r_\lambda} \mathbb{Z}\right)\right) / (\Delta_\lambda = \Delta_{\lambda_0})_{\lambda \ne \lambda_0}, \tag{5.27.2}$$

where the quotient means that for every $\lambda \ne \lambda_0$ we are identifying every diagonal element of $\prod_{0 \le i \le r_\lambda} \mathbb{Z}$ with the corresponding diagonal element of $\prod_{\{0 \le i \le r_{\lambda_0}\} \setminus i_{\lambda_0}(\mathcal{Y})} \mathbb{Z}$ (interpreted to be 0 if the latter indexing set is empty). The map

$$P_{\lambda_0} \to R/p$$

given by (similarly to before, the subscript indicates the factor in (5.27.2))

$$(\mathbb{N}_{\ge 0})_y \ni 1 \mapsto t_{\lambda_0,\, i_{\lambda_0}(y)}, \quad \mathbb{Z}_{(\lambda,\, i)} \ni 1 \mapsto u_{\lambda,\, \lambda_0,\, i} \;\text{ for }\; i \in i_\lambda(\mathcal{Y}), \quad \mathbb{Z}_{(\lambda,\, i)} \ni 1 \mapsto t_{\lambda,\, i} \;\text{ for }\; i \notin i_\lambda(\mathcal{Y})$$

is a chart for the source $\mathrm{Spec}(R/p)$ of the fine version of the log closed immersion (5.22.1). In terms of this chart, the Frobenius action multiplies each element of P_{λ_0} by p.

Due to the relation (5.27.1), the images in R/p of the generators of Q are evidently expressible in terms of the images of the generators of P_{λ_0}, so, as in the smooth case, there is a natural Frobenius-equivariant chart

$$Q \to P_{\lambda_0}$$

for the fine version of (5.22.1): for instance, for $\lambda \ne \lambda_0$ and $i \in i_\lambda(y)$, it sends $1 \in (\mathbb{N}_{\ge 0})_{(\lambda,\, i)}$ to $(1, 1) \in (\mathbb{N}_{\ge 0})_{(\lambda_0,\, i_{\lambda_0}(y))} \times (\mathbb{Z})_{(\lambda,\, i)}$.

The $A(R^\square_{\Sigma,\, \Lambda})$-algebra $A(R^\square_{\Sigma,\, \Lambda}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]$ comes equipped with an $A(R^\square_{\Sigma,\, \Lambda})$-semilinear Frobenius endomorphism and is initial among the $A(R^\square_{\Sigma,\, \Lambda})$-algebras B for which $X_{\lambda,\, i} \in B^\times$ when $i \notin i_\lambda(\mathcal{Y})$ and that are equipped with, for each $y \in \mathcal{Y}$ and $\lambda \in \Lambda$, a unit $U_{\lambda,\, \lambda_0,\, y} \in B^\times$ subject to the relations

$$\begin{gathered} X_{\lambda,\, i_\lambda(y)} = U_{\lambda,\, \lambda_0,\, y} \cdot X_{\lambda_0,\, i_{\lambda_0}(y)}, \qquad U_{\lambda_0,\, \lambda_0,\, y} = 1, \qquad \text{and} \\ \prod_{y \in \mathcal{Y}} U_{\lambda,\, \lambda_0,\, y} = \left(\prod_{\{0 \le i \le r_{\lambda_0}\} \setminus i_{\lambda_0}(\mathcal{Y})} X_{\lambda_0,\, i}\right) / \left(\prod_{\{0 \le i \le r_\lambda\} \setminus i_\lambda(\mathcal{Y})} X_{\lambda,\, i}\right). \end{gathered} \tag{5.27.3}$$

In particular, up to a canonical isomorphism, $A(R^\square_{\Sigma,\, \Lambda}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]$ does not depend on λ_0: for a $\lambda_0' \in \Lambda$, we may set $U_{\lambda,\, \lambda_0',\, y} = U_{\lambda,\, \lambda_0,\, y} \cdot U^{-1}_{\lambda_0',\, \lambda_0,\, y}$ to express the $U_{\lambda,\, \lambda_0',\, y}$ in terms of the $U_{\lambda,\, \lambda_0,\, y}$.

Moreover,

$$R \quad \text{is naturally an} \quad (A(R^\square_{\Sigma,\, \Lambda}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}])\text{-algebra} \quad (\text{with } U_{\lambda,\, \lambda_0,\, y} = t_{\lambda,\, i_\lambda(y)}/t_{\lambda_0,\, i_{\lambda_0}(y)}) \tag{5.27.4}$$

and, as in the smooth case, the scheme version of the fine variant of the closed immersion (5.22.1) factors Frobenius-equivariantly as follows:

$$\mathrm{Spec}(R/p) \xhookrightarrow{j_{\lambda_0}} \mathrm{Spec}\left(A(R^\square_{\Sigma,\, \Lambda}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]\right) \xrightarrow{q_{\lambda_0}} \mathrm{Spec}(A(R^\square_{\Sigma,\, \Lambda})), \tag{5.27.5}$$

where j_{λ_0} is an exact closed immersion and, by [Kat89, 3.5], the projection q_{λ_0} is log étale. As in §5.26, we have natural isomorphisms $P_{\lambda_0} \simeq P_{\lambda_0'}$ over Q and the compatibility diagram (5.26.7).

5.28. The divided power envelope of j_{λ_0}. For each $\lambda_0 \in \Lambda$, we let $D_{j_{\lambda_0}}$ denote the divided power envelope over $(\mathbb{Z}_p, p\mathbb{Z}_p)$ of the closed immersion j_{λ_0} defined in (5.26.5) and (5.27.5). Similarly to §5.22, we may also regard $D_{j_{\lambda_0}}$ as the divided power envelope over $\mathrm{Spec}(\mathcal{O}_C/p) \hookrightarrow \mathrm{Spec}(A^0_{\mathrm{cris}})$ of the closed immersion

$$j_{\lambda_0,\,\mathrm{cris}} \colon \mathrm{Spec}(R/p) \hookrightarrow \mathrm{Spec}((A(R^{\square}_{\Sigma,\Lambda}) \otimes_{A_{\mathrm{inf}}} A^0_{\mathrm{cris}}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]).$$

Since j_{λ_0} underlies an *exact* closed immersion of log schemes (see §5.27), we may, in addition, regard $D_{j_{\lambda_0}}$ endowed with the log structure pulled back from $\mathbb{Z}[P_{\lambda_0}]$ as the log PD envelope of j_{λ_0} over $\mathbb{Z}_p$, or of $j_{\lambda_0,\,\mathrm{cris}}$ over A^0_{cris} (compare with [Kat89, 5.5.1]). For any $\lambda_0' \in \Lambda$, the isomorphism as in (5.26.7) induces an isomorphism

$$D_{j_{\lambda_0}} \cong D_{j_{\lambda_0'}}. \tag{5.28.1}$$

By functoriality, $D_{j_{\lambda_0}}$ comes equipped with an A^0_{cris}-semilinear Frobenius endomorphism, and the isomorphisms (5.28.1) are Frobenius equivariant. Due to (5.26.4) and (5.27.4), there is a map

$$D_{j_{\lambda_0}} \to R \qquad \text{that lifts} \qquad D_{j_{\lambda_0}} \twoheadrightarrow R/p \tag{5.28.2}$$

and whose formation is compatible with the isomorphisms (5.28.1).

Lemma 5.29. *For each $\lambda_0 \in \Lambda$, the map q_{λ_0} induces Frobenius-equivariant isomorphisms*

$$D_{\Sigma,\Lambda,n} \cong D_{j_{\lambda_0}}/p^n \quad \text{for} \quad n \in \mathbb{Z}_{\geq 1} \qquad (\text{resp.,} \qquad D_{\Sigma,\Lambda} \cong \widehat{D_{j_{\lambda_0}}}) \tag{5.29.1}$$

that are compatible with divided powers, maps to R/p^n (resp., R; see (5.22.3) and (5.28.2)), and the isomorphisms (5.28.1). In particular,

$$D_{\Sigma,\Lambda}/p^n \xrightarrow{\sim} D_{\Sigma,\Lambda,n} \qquad \text{for} \qquad n > 0.$$

Proof. Similarly to §5.22, we may regard $D_{\Sigma,\Lambda}$ and $D_{j_{\lambda_0}}$ as being defined using fine log structures and the trivial log structure on A_{inf}. In particular, $D_{j_{\lambda_0}}/p^n$ is identified with the (log) divided power envelope of

$$j_{\lambda_0,\,\mathrm{cris}} \otimes_{A^0_{\mathrm{cris}}} A^0_{\mathrm{cris}}/p^n \colon \mathrm{Spec}(R/p) \hookrightarrow \mathrm{Spec}((A_{\mathrm{cris}}(R^{\square}_{\Sigma,\Lambda})/p^n) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]) \qquad \text{over} \qquad A_{\mathrm{cris}}/p^n$$

(see [SP, 07HB]). Consider a commutative square

$$\begin{tikzcd} T_0 \arrow[r] \arrow[d, hook] & \mathrm{Spec}((A_{\mathrm{cris}}(R^{\square}_{\Sigma,\Lambda})/p^n) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]) \arrow[d, "q_{\lambda_0} \otimes_{A_{\mathrm{inf}}} A_{\mathrm{cris}}/p^n"] \\ T \arrow[r] \arrow[ur, dashed, "?"] & \mathrm{Spec}(A_{\mathrm{cris}}(R^{\square}_{\Sigma,\Lambda})/p^n) \end{tikzcd} \tag{5.29.2}$$

of log schemes over A_{cris}/p^n in which $T_0 \hookrightarrow T$ is a log PD thickening such that the log structure $\mathcal{N}_T$ of T (and hence also $\mathcal{N}_{T_0}$ of T_0) is integral and quasi-coherent. By [Bei13b, 1.1 Exercises (iii)], for any $t, t' \in \Gamma(T, \mathcal{N}_T)$ and $u_0 \in \mathcal{O}^{\times}_{T_0}$ with $t|_{T_0} = u_0 \cdot t'|_{T_0}$, there exists a unique lift $u \in \mathcal{O}_T^{\times}$ of u_0 such that $t = ut'$. Thus, by the construction of P_{λ_0} and the universal property described by the equations (5.26.3) and (5.27.3), there is a unique morphism indicated by a dashed arrow in (5.29.2) that makes the diagram commute. Consequently, q_{λ_0} induces an isomorphism between the log PD envelopes of $j_{\lambda_0,\,\mathrm{cris}} \otimes_{A^0_{\mathrm{cris}}} A^0_{\mathrm{cris}}/p^n$ and $(q_{\lambda_0} \circ j_{\lambda_0,\,\mathrm{cris}}) \otimes_{A_{\mathrm{inf}}} A_{\mathrm{cris}}/p^n$ over A_{cris}/p^n:

$$D_{j_{\lambda_0}}/p^n \cong D_{\Sigma,\Lambda,n}, \qquad \text{and, by letting } n \text{ vary, also} \qquad \widehat{D_{j_{\lambda_0}}} \cong D_{\Sigma,\Lambda}.$$

The Frobenius equivariance follows by functoriality. $\square$

5.30. The rings $D^{(m)}_{\Sigma,\Lambda}$. For each $\lambda_0 \in \Lambda$, the divided powers of the images in $D_{j_{\lambda_0}}$ of the elements of the ideal of $A(R^{\square}_{\Sigma,\Lambda}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]$ that cuts out R/p generate $D_{j_{\lambda_0}}$ as an $(A(R^{\square}_{\Sigma,\Lambda}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}])$-algebra. The divided powers of degree at most $m \in \mathbb{Z}_{\geq 1}$ of these images generate a Frobenius-stable subalgebra

$$D^{(m)}_{j_{\lambda_0}} \subset D_{j_{\lambda_0}}, \qquad \text{so that} \qquad D_{j_{\lambda_0}} = \textstyle\bigcup_{m\geq 1} D^{(m)}_{j_{\lambda_0}}.$$

Since $D_{j_{\lambda_0}}$ is naturally an algebra over A^0_{cris} (see §5.28), $D^{(m)}_{j_{\lambda_0}}$ is naturally and Frobenius-semilinearly an algebra over the subring $A^{0,(m)}_{\mathrm{cris}} \subset A^0_{\mathrm{cris}}$ defined in §3.26.

By Lemma 5.29, the image $D^0_{\Sigma,\Lambda}$ of $D_{j_{\lambda_0}}$ in $D_{\Sigma,\Lambda} \overset{(5.29.1)}{\cong} \widehat{D_{j_{\lambda_0}}}$ is Frobenius stable and does not depend on λ_0. Similarly, the image $D^{0,(m)}_{\Sigma,\Lambda}$ of $D^{(m)}_{j_{\lambda_0}}$ in $D_{\Sigma,\Lambda}$ is also Frobenius stable and does not depend on λ_0. For $m \in \mathbb{Z}_{\geq 1}$, we set

$$D^{(m)}_{\Sigma,\Lambda} := (D^{0,(m)}_{\Sigma,\Lambda})\widehat{\ }, \qquad \text{which is naturally an algebra over} \qquad A^{(m)}_{\mathrm{cris}}$$

and comes equipped with an $A^{(m)}_{\mathrm{cris}}$-semilinear Frobenius. In what follows, $D^0_{\Sigma,\Lambda}$ will play the role of the ring that underlies the hypothetical log PD envelope of the log closed immersion (5.22.1) that we started with (compare with footnote 10).

Both maps in the composition $D_{j_{\lambda_0}} \twoheadrightarrow D^0_{\Sigma,\Lambda} \hookrightarrow D_{\Sigma,\Lambda}$ become isomorphisms upon reduction modulo p^n (because so does their composition), so, since $D^0_{\Sigma,\Lambda} = \bigcup_{m\geq 1} D^{0,(m)}_{\Sigma,\Lambda}$, we obtain an Frobenius-equivariant identification

$$D_{\Sigma,\Lambda} \cong \left(\varinjlim D^{(m)}_{\Sigma,\Lambda}\right)\widehat{\ } \qquad \text{over} \qquad A_{\mathrm{cris}}. \tag{5.30.1}$$

The $\Delta_{\Sigma,\Lambda}$-action on $D_{\Sigma,\Lambda}$ discussed in §5.22 respects the subrings $D^{0,(m)}_{\Sigma,\Lambda} \subset D_{\Sigma,\Lambda}$ (compare with the last paragraph of §5.9). The induced continuous $\Delta_{\Sigma,\Lambda}$-action on the $A^{(m)}_{\mathrm{cris}}$-algebras $D^{(m)}_{\Sigma,\Lambda}$ is compatible as m varies, and the identification (5.30.1) is $\Delta_{\Sigma,\Lambda}$-equivariant.

5.31. The derivations $\frac{\partial}{\partial \log(X_\tau)}$. Similarly to Proposition 5.23, the log derivations defined in (5.10.1) with $R^{\square}_{\Sigma}$ (resp., $R^{\square}_{\lambda}$) in place of R give rise to the log A_{inf}-derivations

$$\tfrac{\partial}{\partial \log(X_\sigma)} \colon A(R^{\square}_{\Sigma,\Lambda}) \to A(R^{\square}_{\Sigma,\Lambda}) \qquad \text{and} \qquad \tfrac{\partial}{\partial \log(X_{\lambda,i})} \colon A(R^{\square}_{\Sigma,\Lambda}) \to A(R^{\square}_{\Sigma,\Lambda}) \tag{5.31.1}$$

for $\sigma \in \Sigma$ and $\lambda \in \Lambda$ with $i = 1, \dots, d$ (as in §5.10, we do not explicate the accompanying homomorphisms from the log structure). For brevity, let τ denote either the index "σ" for some $\sigma \in \Sigma$ or the index "λ, i" for some $\lambda \in \Lambda$ and $i = 1, \dots, d$. Then, since each q_{λ_0} is log étale (see §5.27), the log A_{inf}-derivation $\frac{\partial}{\partial \log(X_\tau)}$ from (5.31.1) extends uniquely to a log A_{inf}-derivation

$$\tfrac{\partial}{\partial \log(X_\tau)} \colon A(R^{\square}_{\Sigma,\Lambda}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}] \to A(R^{\square}_{\Sigma,\Lambda}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}] \qquad \text{for every} \qquad \lambda_0 \in \Lambda.$$

Consequently, by (5.23.4), that is, by [Bei13b, 1.7, Exercises, (i)], we obtain divided power A_{cris}-derivations

$$\tfrac{\partial}{\partial \log(X_\tau)} \colon D_{\Sigma,\Lambda} \to D_{\Sigma,\Lambda}, \tag{5.31.2}$$

where a divided power A_{cris}-derivation ∂ is, as usual, in addition to A_{cris}-linearity and the Leibniz rule, required to satisfy $\partial(x^{[m]}) = x^{[m-1]}\partial(x)$ for divided powers $x^{[m]}$ with $m \geq 1$. Likewise, we obtain divided power A^0_{cris}-derivations

$$\tfrac{\partial}{\partial \log(X_\tau)} \colon D_{j_{\lambda_0}} \to D_{j_{\lambda_0}} \quad \text{for} \quad \lambda_0 \in \Lambda$$

that are compatible with those in (5.31.2). Thus, $\frac{\partial}{\partial \log(X_\tau)}$ induces divided power $A_{\mathrm{cris}}^{(m)}$-derivations

$$\tfrac{\partial}{\partial \log(X_\tau)} \colon D_{\Sigma,\Lambda}^{(m)} \to D_{\Sigma,\Lambda}^{(m)} \qquad \text{for} \qquad m \in \mathbb{Z}_{\geq 1}$$

that are compatible as m varies and recover (5.31.2) under the identification $D_{\Sigma,\Lambda} \cong \left(\varinjlim D_{\Sigma,\Lambda}^{(m)}\right)^{\widehat{}}$. Consequently, we may reexpress (5.23.3) as the Frobenius-equivariant identification

$$R\Gamma_{\log \mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}) \cong \left(\varinjlim_{m>0} K_{D_{\Sigma,\Lambda}^{(m)}}\left(\left(\tfrac{\partial}{\partial \log(X_\sigma)}\right)_{\sigma\in\Sigma}, \left(\tfrac{\partial}{\partial \log(X_{\lambda,i})}\right)_{\lambda\in\Lambda,\, 1\leq i\leq d}\right)\right)^{\widehat{}} \quad (5.31.3)$$

(the Frobenius action on the right side is defined via the identification with the right side of (5.23.3)).

5.32. A functorial complex that computes $R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}})$. For a fixed R, the formation of the rings $D_{\Sigma,\Lambda}$, D_{λ_0}, $D_{\Sigma,\Lambda}^0$, and $D_{\Sigma,\Lambda}^{(m)}$, as well as the morphisms j_{λ_0} and q_{λ_0}, is compatible with enlarging Σ and Λ. Likewise, the formation of the identifications (5.23.3) and (5.31.3), is also compatible with such enlargement. Consequently, by taking the filtered direct limit over all the closed immersions (5.17.1) for varying Σ and Λ (but a fixed R), we may build the complex

$$\varinjlim_{\Sigma,\Lambda}\left(\left(\varinjlim_{m>0} K_{D_{\Sigma,\Lambda}^{(m)}}\left(\left(\tfrac{\partial}{\partial \log(X_\sigma)}\right)_{\sigma\in\Sigma}, \left(\tfrac{\partial}{\partial \log(X_{\lambda,i})}\right)_{\lambda\in\Lambda,\, 1\leq i\leq d}\right)\right)^{\widehat{}}\right), \quad (5.32.1)$$

where the direct limits and the p-adic completion are formed termwise, that, by the identification with the direct limit of the right sides of (5.23.3), comes equipped with an A_{cris}-semilinear Frobenius endomorphism. By (5.31.3), this complex in the derived category is canonically and Frobenius-equivariantly isomorphic to

$$R\Gamma_{\log \mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}).$$

If R' is a formally étale R-algebra equipped with a closed immersion as in (5.17.1) for some sets Σ' and Λ', then we may also equip it with the induced closed immersion as in (5.17.1) for the sets $\widetilde{\Sigma} := \Sigma \cup \Sigma'$ and $\widetilde{\Lambda} := \Lambda \cup \Lambda'$. The rings $D_{\Sigma,\Lambda}$, D_{λ_0} (with $\lambda_0 \in \Lambda$), $D_{\Sigma,\Lambda}^0$, and $D_{\Sigma,\Lambda}^{(m)}$ then map to their counterparts for R' constructed using $\widetilde{\Sigma}$ and $\widetilde{\Lambda}$: for this, the only slight subtlety is in the case when R/p is not $\mathcal{O}_C/p$-smooth but R'/p is $\mathcal{O}_C/p$-smooth, when one uses the relations (5.27.3) that describe the universal property of $A(R_{\Sigma,\Lambda}^{\square}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]$. Consequently, the term indexed by Σ, Λ (and by a closed immersion (5.17.1)) of the direct limit (5.32.1) maps to the term indexed by $\widetilde{\Sigma}$, $\widetilde{\Lambda}$ (and by a closed immersion of $\operatorname{Spf} R'$) of the analogous direct limit for R', compatibly with the transition maps in (5.32.1). In other words, the complex (5.32.1) is functorial in the ring R equipped with the closed immersion (5.17.1).

Since the formation of the maps (5.23.2) is compatible with enlarging Σ and Λ, and then also with replacing R by R', the map $R\Gamma_{\log \mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}) \to R\Gamma_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)$ is identified with a map

$$\varinjlim_{\Sigma,\Lambda}\left(\left(\varinjlim_{m>0} K_{D_{\Sigma,\Lambda}^{(m)}}\left(\left(\tfrac{\partial}{\partial \log(X_\sigma)}\right)_{\sigma}, \left(\tfrac{\partial}{\partial \log(X_{\lambda,i})}\right)_{\lambda\in\Lambda,\, 1\leq i\leq d}\right)\right)^{\widehat{}}\right) \to \Omega^\bullet_{\operatorname{Spf}(R)/\mathcal{O}_C,\log} \quad (5.32.2)$$

whose formation is compatible is compatible with replacing R by R'.

Having constructed the functorial complexes (5.21.1) and (5.32.1), we seek to exhibit a natural map between them and prove that this map is an isomorphism. These tasks, which will be completed in §5.38 and Proposition 5.39, are the last stepping stones to the proof of Theorem 5.4 given in §5.40.

Lemma 5.33. *For every $m \geq p^2$, the element $\delta_\tau \in \Delta$, where the index τ is either "σ" for some $\sigma \in \Sigma$ or "λ, i" for some $\lambda \in \Lambda$ and $i = 1, \ldots, d$ (see §5.19), acts on $D_{\Sigma,\Lambda}^{(m)}$ as the endomorphism*

$$\textstyle\sum_{n\geq 0} \frac{(\log([\epsilon]))^n}{n!}\left(\frac{\partial}{\partial \log(X_\tau)}\right)^n, \quad (5.33.1)$$

where $\frac{(\log([\epsilon]))^n}{n!}$ *lies in* $A_{\mathrm{cris}}^{(m)}$ *and* p*-adically converges to* 0 (*see* §5.14).

Proof. By Lemma 5.15, the action of δ_τ on $A_{\mathrm{cris}}(R^{\square}_{\Sigma,\Lambda})$ (defined in (5.22.2)) is given by the series (5.33.1). Moreover, analogously to the proof of Lemma 5.15, the series (5.33.1) *a priori* defines an A_{cris}-algebra endomorphism of $D_{\Sigma,\Lambda}$. Therefore, by the universal property of $D_{\Sigma,\Lambda}$ (see §5.22), the action of δ_τ on $D_{\Sigma,\Lambda}$, and hence also on $D^{0,(m)}_{\Sigma,\Lambda}$ and $D^{(m)}_{\Sigma,\Lambda}$ is given by the series (5.33.1). ☐

Proposition 5.34. *In the local setup of* §5.17, *for every* $m \geq p^2$, *the additive morphisms*

$$\left(D^{(m)}_{\Sigma,\Lambda} \xrightarrow{\frac{\partial}{\partial \log(X_\tau)}} D^{(m)}_{\Sigma,\Lambda}\right) \xrightarrow{\left(\mathrm{id},\, \sum_{n\geq 1} \frac{(\log([\epsilon]))^n}{n!} (\frac{\partial}{\partial \log(X_\tau)})^{n-1}\right)} \left(D^{(m)}_{\Sigma,\Lambda} \xrightarrow{\delta_\tau - 1} D^{(m)}_{\Sigma,\Lambda}\right) \tag{5.34.1}$$

of complexes concentrated in degree 0 *and* 1, *where* τ *ranges over "*σ*" for* $\sigma \in \Sigma$ *and "*λ, i*" for* $\lambda \in \Lambda$ *and* $i = 1, \dots, d$, *define a morphism* (*whose target is defined as in* (1.7.3))

$$K_{D^{(m)}_{\Sigma,\Lambda}}\left((\tfrac{\partial}{\partial \log(X_\sigma)})_{\sigma\in\Sigma}, (\tfrac{\partial}{\partial \log(X_{\lambda,i})})_{\lambda\in\Lambda,\, 1\leq i\leq d}\right) \to \eta_{(\mu)}\left(K_{D^{(m)}_{\Sigma,\Lambda}}((\delta_\sigma - 1)_{\sigma\in\Sigma}, (\delta_{\lambda,i} - 1)_{\lambda\in\Lambda,\, 1\leq i\leq d})\right).$$

Proof. The morphism (5.34.1) is well defined by Lemma 5.33. Moreover, the image of its degree 1 component lies in $\mu \cdot D^{(m)}_{\Sigma,\Lambda}$ because, by §5.14, $\frac{(\log([\epsilon]))^n}{\mu\cdot n!}$ lies in $A^{(m)}_{\mathrm{cris}}$ and p-adically tends to 0. The rest of the claim then follows from the definitions (1.7.2) and (1.7.3). ☐

Proposition 5.34 essentially reduces the task of exhibiting a natural map from the complex (5.32.1) to the complex (5.21.1) to that of exhibiting a natural $\Delta_{\Sigma,\Lambda}$-equivariant map $D^{(m)}_{\Sigma,\Lambda} \to \mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$. For this, in Proposition 5.36, we will realize $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$ inside the following ring $\mathbb{A}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$.

5.35. The ring $\mathbb{A}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$. Let

$$\mathbb{A}^0_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \subset \mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})[\tfrac{1}{p}]$$

be the $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$-subalgebra generated by the elements $\frac{\xi^n}{n!}$ for $n \in \mathbb{Z}_{\geq 1}$. Analogously to §5.8, by [Tsu99, proof of A2.8],

$$\mathbb{A}^0_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \cong (\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})[\tfrac{T^n}{n!}]_{n\geq 1})/(T-\xi), \quad \text{so} \quad \mathbb{A}^0_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \cong \mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty}) \otimes_{A_{\mathrm{inf}}} A^0_{\mathrm{cris}}$$

and $\mathbb{A}^0_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$ agrees with the divided power envelope of $(\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty}), (\xi, p)\cdot \mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty}))$ over $(\mathbb{Z}_p, p\mathbb{Z}_p)$. Thus, again as in §5.8,

$$\mathbb{A}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) := \mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty}) \widehat{\otimes}_{A_{\mathrm{inf}}} A_{\mathrm{cris}} \qquad \text{is identified with} \qquad (\mathbb{A}^0_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}))^{\wedge}.$$

Similarly to §3.26, for an $m \in \mathbb{Z}_{\geq 1}$, we let $\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \subset \mathbb{A}^0_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$ be the $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$-subalgebra generated by the divided powers of order at most m, that is, by the $\frac{\xi^n}{n!}$ with $n \leq m$. Since (p,ξ) is a regular sequence in $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$, the quotient of $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})[\frac{T^n}{n!}]_{n\geq 1}$ by the subalgebra $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})[\frac{T^n}{n!}]_{m\geq n\geq 1}$ is $(T-\xi)$-torsion free. Consequently,

$$\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \cong (\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})[\tfrac{T^n}{n!}]_{m\geq n\geq 1})/(T-\xi), \tag{5.35.1}$$

to the effect that

$$\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \cong \mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty}) \otimes_{A_{\mathrm{inf}}} A^{0,(m)}_{\mathrm{cris}}.$$

Thus, by letting the completion be p-adic if $m \geq p$ and (p,μ)-adic if $m < p$, we get the identification

$$(\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}))^{\wedge} \cong \mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}). \tag{5.35.2}$$

Proposition 5.36. *For any $m \in \mathbb{Z}_{\geq 1}$, the following natural maps are injective:*

$$\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \hookrightarrow \mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \hookrightarrow \mathbb{A}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \hookrightarrow \mathbb{B}^{+}_{\mathrm{dR}}(R_{\Sigma,\Lambda,\infty}) := (\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})[\tfrac{1}{p}])^{\wedge}$$

where the completion is ξ-adic and the definition of the last map will be explained in the proof.

Proof. Since (p, ξ) is a regular sequence in $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$ and $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$ is ξ-adically separated (see [SP, 090T]), the ring $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})[\frac{1}{p}]$ is also ξ-adically separated. Consequently, the map $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})[\frac{1}{p}] \to \mathbb{B}^{+}_{\mathrm{dR}}(R_{\Sigma,\Lambda,\infty})$ is injective, so $\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \to \mathbb{B}^{+}_{\mathrm{dR}}(R_{\Sigma,\Lambda,\infty})$ is also injective.

For varying $n \in \mathbb{Z}_{\geq 0}$, the $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$-submodules

$$\mathrm{Fil}^0_n \subset \mathbb{A}^0_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \qquad \text{generated by the} \quad \tfrac{\xi^{n'}}{n'!} \quad \text{for} \quad n' \geq n$$

form a decreasing filtration of $\mathbb{A}^0_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$ by ideals. By [Tsu99, A2.9 (2)],[11] each

$$\mathbb{A}^0_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})/\,\mathrm{Fil}^0_n \qquad \text{is } p\text{-torsion free and } p\text{-adically complete,} \tag{5.36.1}$$

so the p-adic completions $\mathrm{Fil}_n := (\mathrm{Fil}^0_n)^{\wedge}$ form a decreasing filtration of $\mathbb{A}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$ by ideals with

$$\mathbb{A}^0_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})/\,\mathrm{Fil}^0_n \cong \mathbb{A}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})/\,\mathrm{Fil}_n\,. \tag{5.36.2}$$

The p-torsion freeness also supplies a decreasing filtration modulo p:

$$\mathrm{Fil}^0_n/p\,\mathrm{Fil}^0_n \hookrightarrow \mathbb{A}^0_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})/p\mathbb{A}^0_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}). \tag{5.36.3}$$

The isomorphism $\mathbb{A}^0_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \cong (\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})[\frac{T^n}{n!}]_{n\geq 1})/(T-\xi)$ gives the explicit description

$$\mathbb{A}^0_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})/p\mathbb{A}^0_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \cong (R^{\flat}_{\Sigma,\Lambda,\infty}/\xi^p)[Y_1, Y_2, \ldots]/(Y_1^p, Y_2^p, \ldots) \tag{5.36.4}$$

where Y_j corresponds to $\frac{\xi^{p^j}}{(p^j)!}$ (compare with [BC09, 9.4.1 (3)]), so the filtration $\{\mathrm{Fil}^0_n/p\,\mathrm{Fil}^0_n\}_{n\geq 0}$ of (5.36.3) is separated. In particular, since

$$\mathrm{Fil}^0_n/p\,\mathrm{Fil}^0_n \cong \mathrm{Fil}_n/p\,\mathrm{Fil}_n \qquad \text{compatibly with} \qquad \mathbb{A}^0_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})/p \cong \mathbb{A}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})/p,$$

the p-adic separatedness of $\mathbb{A}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$ ensures that the filtration $\{\mathrm{Fil}_n\}_{n\geq 0}$ is also separated:

$$\mathbb{A}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \hookrightarrow \varprojlim\,(\mathbb{A}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})/\,\mathrm{Fil}_n) \overset{(5.36.2)}{\cong} \varprojlim\,(\mathbb{A}^0_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})/\,\mathrm{Fil}^0_n) \hookrightarrow \mathbb{B}^{+}_{\mathrm{dR}}(R_{\Sigma,\Lambda,\infty}),$$

where the last map is injective because so is each

$$\mathbb{A}^0_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})/\,\mathrm{Fil}^0_n \hookrightarrow (\mathbb{A}^0_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})/\,\mathrm{Fil}^0_n)[\tfrac{1}{p}] \cong (\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})[\tfrac{1}{p}])/\xi^n \cong \mathbb{B}^{+}_{\mathrm{dR}}(R_{\Sigma,\Lambda,\infty})/\xi^n.$$

This gives the desired natural injection $\mathbb{A}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \hookrightarrow \mathbb{B}^{+}_{\mathrm{dR}}(R_{\Sigma,\Lambda,\infty})$ of $\mathbb{A}^0_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$-algebras.

The filtration $\{\mathrm{Fil}^0_n\}_{n\geq 0}$ defines the decreasing filtration

$$\mathrm{Fil}^{0,(m)}_n := \mathrm{Fil}^0_n \bigcap \mathbb{A}^{0,(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \subset \mathbb{A}^{0,(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$$

of $\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$ by ideals. Explicitly, $\mathrm{Fil}^{0,(m)}_n$ is generated by the products $\frac{\xi^{n_1}}{n_1!}\cdots\frac{\xi^{n_s}}{n_s!}$ with $n_1 + \ldots + n_s \geq n$ and $0 \leq n_i \leq m$. By (5.36.1), the quotients

$$\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})/\,\mathrm{Fil}^{0,(m)}_n \qquad \text{are } p\text{-torsion free,} \tag{5.36.5}$$

so we again get the induced filtration modulo p:

$$\mathrm{Fil}^{0,(m)}_n/p\,\mathrm{Fil}^{0,(m)}_n \hookrightarrow \mathbb{A}^{0,(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})/p\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}).$$

[11] *Loc. cit.* is written in a different setting, but its proof continues to work if A there is replaced by our $R_{\Sigma,\Lambda,\infty}$.

Similarly to the case of the filtration $\{\mathrm{Fil}_n^0 / p \,\mathrm{Fil}_n^0\}_{n \geq 0}$, the analogous to (5.36.4) explicit description of $\mathbb{A}_{\mathrm{cris}}^{0,\,(m)}(R_{\Sigma,\,\Lambda,\,\infty})/p\mathbb{A}_{\mathrm{cris}}^{0,\,(m)}(R_{\Sigma,\,\Lambda,\,\infty})$ supplied by the isomorphism (5.35.1) shows that the filtration $\{\mathrm{Fil}_n^{0,\,(m)} / p \,\mathrm{Fil}_n^{0,\,(m)}\}_{n \geq 0}$ is separated.

For each $n > 0$, there is a $j_n > 0$ such that p^{j_n} kills

$$\mathbb{A}_{\mathrm{cris}}^0(R_{\Sigma,\,\Lambda,\,\infty})/(\mathbb{A}_{\mathrm{cris}}^{0,\,(m)}(R_{\Sigma,\,\Lambda,\,\infty}) + \mathrm{Fil}_n^0)$$

(for instance, $j_n := \mathrm{ord}_p(n!)$ has this property). Consequently, p^{j_n} kills the kernel of the map

$$\frac{\mathbb{A}_{\mathrm{cris}}^{0,\,(m)}(R_{\Sigma,\,\Lambda,\,\infty})/\,\mathrm{Fil}_n^{0,\,(m)}}{p^j \cdot (\mathbb{A}_{\mathrm{cris}}^{0,\,(m)}(R_{\Sigma,\,\Lambda,\,\infty})/\,\mathrm{Fil}_n^{0,\,(m)})} \to \frac{\mathbb{A}_{\mathrm{cris}}^{0}(R_{\Sigma,\,\Lambda,\,\infty})/\,\mathrm{Fil}_n^{0}}{p^j \cdot (\mathbb{A}_{\mathrm{cris}}^{0}(R_{\Sigma,\,\Lambda,\,\infty})/\,\mathrm{Fil}_n^{0})} \qquad \text{for each} \qquad j > 0,$$

so, for $j > j_n$, every element of this kernel is a multiple of $p^{j - j_n}$. The short exact sequences

$$0 \to \frac{\mathrm{Fil}_n^{0,\,(m)}}{p^j \cdot \mathrm{Fil}_n^{0,\,(m)}} \xrightarrow{(5.36.5)} \frac{\mathbb{A}_{\mathrm{cris}}^{0,\,(m)}(R_{\Sigma,\,\Lambda,\,\infty})}{p^j \cdot \mathbb{A}_{\mathrm{cris}}^{0,\,(m)}(R_{\Sigma,\,\Lambda,\,\infty})} \to \frac{\mathbb{A}_{\mathrm{cris}}^{0,\,(m)}(R_{\Sigma,\,\Lambda,\,\infty})/\,\mathrm{Fil}_n^{0,\,(m)}}{p^j \cdot (\mathbb{A}_{\mathrm{cris}}^{0,\,(m)}(R_{\Sigma,\,\Lambda,\,\infty})/\,\mathrm{Fil}_n^{0,\,(m)})} \to 0$$

then show that modulo p every element of $\mathrm{Ker}(\mathbb{A}_{\mathrm{cris}}^{(m)}(R_{\Sigma,\,\Lambda,\,\infty}) \to \mathbb{A}_{\mathrm{cris}}(R_{\Sigma,\,\Lambda,\,\infty}))$, that is, of

$$\mathrm{Ker}\left(\varprojlim_{j>0}(\mathbb{A}_{\mathrm{cris}}^{0,\,(m)}(R_{\Sigma,\,\Lambda,\,\infty})/p^j) \to \varprojlim_{j>0}(\mathbb{A}_{\mathrm{cris}}^{0}(R_{\Sigma,\,\Lambda,\,\infty})/p^j)\right) \tag{5.36.6}$$

(see §5.35), lies inside $\mathrm{Fil}_n^{0,\,(m)}/p \subset \mathbb{A}_{\mathrm{cris}}^{0,\,(m)}(R_{\Sigma,\,\Lambda,\,\infty})/p$ for each $n > 0$. However, by the previous paragraph, $\bigcap_{n>0}(\mathrm{Fil}_n^{0,\,(m)}/p) = 0$, so every element that lies in the kernel in (5.36.6) is divisible by p in $\mathbb{A}_{\mathrm{cris}}^{(m)}(R_{\Sigma,\,\Lambda,\,\infty})$. This implies that $\mathbb{A}_{\mathrm{cris}}^{(m)}(R_{\Sigma,\,\Lambda,\,\infty}) \to \mathbb{A}_{\mathrm{cris}}(R_{\Sigma,\,\Lambda,\,\infty})$ is injective, as desired. ☐

Lemma 5.37. *For each $\lambda_0 \in \Lambda$, there is a divided power morphism*

$$D_{j_{\lambda_0}} \to \mathbb{A}_{\mathrm{cris}}^0(R_{\Sigma,\,\Lambda,\,\infty}) \tag{5.37.1}$$

whose formation is compatible with the isomorphisms $D_{j_{\lambda_0}} \cong D_{j_{\lambda'_0}}$ discussed in (5.28.1).

Proof. By construction, $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\,\Lambda,\,\infty})$ is an $A(R_\Sigma^\square)$-algebra and an $A(R_\lambda^\square)$-algebra for every $\lambda \in \Lambda$ (compatibly with the maps θ of (3.14.3) and (5.19.1)), so it is also an $A(R_{\Sigma,\,\Lambda}^\square)$-algebra. Moreover, since $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\,\Lambda,\,\infty})$ is ξ-adically complete with $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\,\Lambda,\,\infty})/\xi \cong R_{\Sigma,\,\Lambda,\,\infty}$, if $t_{\lambda,\,i}$ is a unit in R, so also in $R_{\Sigma,\,\Lambda,\,\infty}$, then, since $X_{\lambda,\,i} \bmod \xi$ is $t_{\lambda,i}$ (see (3.14.3)), $X_{\lambda,\,i}$ is a unit in $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\,\Lambda,\,\infty})$. Thus, if R/p is $\mathcal{O}_C/p$-smooth, then the equations (5.26.3) have a unique solution in $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\,\Lambda,\,\infty})$, to the effect that, in this case, $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\,\Lambda,\,\infty})$ is naturally an $(A(R_{\Sigma,\,\Lambda}^\square) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}])$-algebra, compatibly with the "change of λ_0" isomorphisms exhibited in (5.26.7) and the maps (5.26.4) and (5.19.1) to R and $R_{\Sigma,\,\Lambda,\,\infty}$, respectively.

If R/p is not $\mathcal{O}_C/p$-smooth, then, in the notation of §5.27, for each $m \in \mathbb{Z}_{\geq 0}$ and $y \in \mathcal{Y}$, the element $t_{\lambda_0,\,i_{\lambda_0}(y)}^{1/p^m}$ is not a zero divisor in $R_{\Sigma,\,\Lambda,\,\infty}$ and is a unit in $R_{\Sigma,\,\Lambda,\,\infty}[\frac{1}{p}]$, so, since $R_{\Sigma,\,\Lambda,\,\infty}$ is integrally closed in $R_{\Sigma,\,\Lambda,\,\infty}[\frac{1}{p}]$, we conclude from (5.27.1) that

$$t_{\lambda,\,i_\lambda(y)}^{1/p^m}/t_{\lambda_0,\,i_{\lambda_0}(y)}^{1/p^m} \in R_{\Sigma,\,\Lambda,\,\infty}^\times \qquad \text{for every} \qquad \lambda \in \Lambda.$$

Thus, for such m, y, and λ, there is a unique $u_{\lambda,\,\lambda_0,\,y}^{(m)} \in R_{\Sigma,\,\Lambda,\,\infty}^\times$ such that

$$t_{\lambda,\,i_\lambda(y)}^{1/p^m} = u_{\lambda,\,\lambda_0,\,y}^{(m)} \cdot t_{\lambda_0,\,i_{\lambda_0}(y)}^{1/p^m} \qquad \text{in} \qquad R_{\Sigma,\,\Lambda,\,\infty}.$$

The uniqueness ensures that $(u^{(m)}_{\lambda,\,\lambda_0,\,y})^p = u^{(m-1)}_{\lambda,\,\lambda_0,\,y}$, so the system $u^\flat_{\lambda,\,\lambda_0,\,y} := (u^{(m)}_{\lambda,\,\lambda_0,\,y})_{m\geq 0}$ is an element of $(R^\flat_{\Sigma,\,\Lambda,\,\infty})^\times$ for which (see §3.11 and §3.14)

$$X_{\lambda,\,i_\lambda(y)} = [u^\flat_{\lambda,\,\lambda_0,\,y}]\cdot X_{\lambda_0,\,i_{\lambda_0}(y)} \qquad \text{in} \qquad \mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\,\Lambda,\,\infty}). \tag{5.37.2}$$

Since $[(p^{1/p^\infty})]$, and hence also each $X_{\lambda,\,i}$, is a nonzero divisor in $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$, the equalities (5.37.2) provide a solution in $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$ to the equations (5.27.3) that lifts the solution in $R \subset R_{\Sigma,\Lambda,\infty}$ provided by (5.27.4). Thus, also in the nonsmooth case, $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$ is naturally an $(A(R^\square_{\Sigma,\,\Lambda})\otimes_{\mathbb{Z}[Q]}\mathbb{Z}[P_{\lambda_0}])$-algebra, compatibly with the "change of λ_0" isomorphisms and the maps (5.27.4) and (5.19.1) to R and $R_{\Sigma,\,\Lambda,\,\infty}$, respectively.

In conclusion, in all cases we get the compatible with change of λ_0 commutative square

$$\begin{tikzcd} A(R^\square_{\Sigma,\,\Lambda})\otimes_{\mathbb{Z}[Q]}\mathbb{Z}[P_{\lambda_0}] \arrow[r, two heads, "j_{\lambda_0}"] \arrow[d] & R \arrow[d] \\ \mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\,\Lambda,\,\infty}) \arrow[r, two heads, "\theta"] & R_{\Sigma,\,\Lambda,\,\infty}, \end{tikzcd} \qquad \text{so also} \qquad \begin{tikzcd} A(R^\square_{\Sigma,\,\Lambda})\otimes_{\mathbb{Z}[Q]}\mathbb{Z}[P_{\lambda_0}] \arrow[r, two heads, "j_{\lambda_0}"] \arrow[d] & R \arrow[d] \\ \mathbb{A}^0_{\mathrm{cris}}(R_{\Sigma,\,\Lambda,\,\infty}) \arrow[r, two heads, "\theta"] & R_{\Sigma,\,\Lambda,\,\infty}. \end{tikzcd}$$

The universal property of D_{λ_0} now supplies the desired divided power morphism (5.37.1). $\square$

5.38. The comparison map. Upon p-adic completion, the map (5.37.1) induces a map

$$D_{\Sigma,\,\Lambda} \to \mathbb{A}_{\mathrm{cris}}(R_{\Sigma,\,\Lambda,\,\infty}), \tag{5.38.1}$$

which, by Lemma 5.37, does not depend on the choice of λ_0. By its construction, the map (5.38.1) is compatible with the maps

$$D_{\Sigma,\,\Lambda} \xrightarrow{(5.22.3)}\!\!\!\!\rightarrow R \qquad \text{and} \qquad \mathbb{A}_{\mathrm{cris}}(R_{\Sigma,\,\Lambda,\,\infty}) \overset{\theta}{\to} R_{\Sigma,\,\Lambda,\,\infty}. \tag{5.38.2}$$

The restriction of this map to $D^{0,\,(m)}_{\Sigma,\,\Lambda}$ factors through the subring $\mathbb{A}^{0,\,(m)}_{\mathrm{cris}}(R_{\Sigma,\,\Lambda,\,\infty}) \overset{5.36}{\subset} \mathbb{A}_{\mathrm{cris}}(R_{\Sigma,\,\Lambda,\,\infty})$, so, by passing to p-adic completions and using (5.35.2), we obtain compatible maps

$$D^{(m)}_{\Sigma,\,\Lambda} \to \mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\,\Lambda,\,\infty}) \qquad \text{for} \qquad m \geq p. \tag{5.38.3}$$

By construction, the maps (5.38.3) are $\Delta_{\Sigma,\,\Lambda}$-equivariant, so they give to the morphisms

$$K_{D^{(m)}_{\Sigma,\,\Lambda}}((\delta_\sigma - 1)_{\sigma\in\Sigma}, (\delta_{\lambda,\,i}-1)_{\lambda\in\Lambda,\,1\leq i\leq d}) \to K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\,\Lambda,\,\infty})}((\delta_\sigma - 1)_{\sigma\in\Sigma}, (\delta_{\lambda,\,i}-1)_{\lambda\in\Lambda,\,1\leq i\leq d}).$$

After applying the functor $\eta_{(\mu)}$ (see (1.7.3)), these morphisms compose with the ones constructed in Proposition 5.34 and give rise to the desired comparison map of complexes:

$$\left(\varinjlim_{m>0} K_{D^{(m)}_{\Sigma,\,\Lambda}}\left(\left(\tfrac{\partial}{\partial\log(X_\tau)}\right)_\tau\right)\right)^\wedge \to \left(\varinjlim_{m>0}\left(\eta_{(\mu)}\left(K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\,\Lambda,\,\infty})}((\delta_\tau - 1)_\tau)\right)\right)\right)^\wedge, \tag{5.38.4}$$

where the direct limits and the p-adic completions formed termwise and, for brevity, we let the label τ range over "σ" for $\sigma\in\Sigma$ and "(λ, i)" for $\lambda\in\Lambda$ and $i = 1,\dots,d$. The source (resp., target) of this map is a term of the direct limit (5.32.1) (resp., (5.21.1)) and its formation is compatible with the transition maps of the direct limits (5.32.1) and (5.21.1) (in other words, with enlarging Σ and Λ). Moreover, if R' is a formally étale R-algebra equipped with a closed immersion as in (5.17.1) for some sets Σ' and Λ', then the map (5.38.4) and its analogue for R' and the sets $\Sigma\cup\Sigma'$, $\Lambda\cup\Lambda'$ (and the induced closed immersion) are compatible with the maps between their sources (resp., targets) discussed in §5.21 and §5.32.

In conclusion, by taking the filtered direct limit of the maps (5.38.4) over all the closed immersions (5.17.1) for varying Σ and Λ (but a fixed R), we obtain a comparison map from the complex (5.32.1) to the complex (5.21.1), and the formation of this map is compatible with replacing R by a formally étale R-algebra R'. It follows from the following proposition that this map is a quasi-isomorphism.

Proposition 5.39. *The comparison map* (5.38.4) *is a Frobenius-equivariant quasi-isomorphism.*

Proof. The proof is similar to that of [BMS16, 12.8], and the key idea is to reduce to the case of a single coordinate morphism settled in Proposition 5.16. More precisely, for $m \geq p$, let

$$\mathrm{Spec}(R/p) \hookrightarrow \mathrm{Spf}(D^{(m)}_{\Sigma,\,\Lambda}) \tag{5.39.1}$$

be the closed immersion induced by its analogue for $D_{\Sigma,\,\Lambda}$, that is, by the first map in (5.22.2). For each $\lambda_0 \in \Lambda$, the ideal of $A(R^{\square}_{\Sigma,\,\Lambda}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]$ that cuts out R/p (see (5.27.5)) is finitely generated. Consequently, for each $m \geq p$, the ideal of $D^{(m)}_{\Sigma,\,\Lambda}$ that cuts out R/p is finitely generated, too, and hence, due to divided powers, it is also topologically nilpotent. Thus, if we fix a $\lambda \in \Lambda$ and for $m \geq p$ let $A^{(m)}_{\mathrm{cris}}(R)_\lambda$ be the ring $A^{(m)}_{\mathrm{cris}}(R)$ of §3.27 constructed using the semistable coordinate map $R^{\square}_\lambda \to R$, then the formal étaleness of $A^{(m)}_{\mathrm{cris}}(R^{\square}_\lambda) \to A^{(m)}_{\mathrm{cris}}(R)_\lambda$ (see §3.14) ensures the existence of the unique indicated lifts in the commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(R_{\Sigma,\,\Lambda,\,\infty}/p) \arrow[r] \arrow[d, hook, "\theta"] & \mathrm{Spec}(R/p) \arrow[r] \arrow[d, hook] & \mathrm{Spf}(A^{(m)}_{\mathrm{cris}}(R)_\lambda) \arrow[d] \\
\mathrm{Spf}(\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\,\Lambda,\,\infty})) \arrow[r, "(5.38.3)"'] \arrow[urr, dashed] & \mathrm{Spf}(D^{(m)}_{\Sigma,\,\Lambda}) \arrow[r] \arrow[ur, dashed] & \mathrm{Spf}(A^{(m)}_{\mathrm{cris}}(R^{\square}_\lambda))
\end{tikzcd}$$

in which the bottom horizontal map results from the fact that, by construction, each $D^{(m)}_{\Sigma,\,\Lambda}$ is an $A(R^{\square}_\lambda)$-algebra and an $A^{(m)}_{\mathrm{cris}}$-algebra. The uniqueness ensures that the resulting maps

$$A^{(m)}_{\mathrm{cris}}(R)_\lambda \to D^{(m)}_{\Sigma,\,\Lambda} \tag{5.39.2}$$

are compatible as m varies, Frobenius-equivariant, $\Delta_{\Sigma,\,\Lambda}$-equivariant, where $\Delta_{\Sigma,\,\Lambda}$ acts on $A^{(m)}_{\mathrm{cris}}(R)_\lambda$ through the projection $\Delta_{\Sigma,\,\Lambda} \twoheadrightarrow \Delta_\lambda$, and are compatible with the maps from its source and target to $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\,\Lambda,\,\infty})$. By construction, these maps are also compatible with the derivations $\frac{\partial}{\partial \log(X_{\lambda,\,i})}$ for $i = 1, \dots, d$ discussed in §5.10 and §5.31. Consequently, we get a commutative diagram

$$\begin{tikzcd}
K_{A^{(m)}_{\mathrm{cris}}(R)_\lambda}\left(\frac{\partial}{\partial \log(X_{\lambda,\,1})}, \dots, \frac{\partial}{\partial \log(X_{\lambda,\,d})}\right) \arrow[r, "(5.16.2)"] \arrow[d, "(5.39.2)"] & \eta_{(\mu)}\left(K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\lambda,\infty})}(\delta_{\lambda,\,1} - 1, \dots, \delta_{\lambda,\,d} - 1)\right) \arrow[d] \\
K_{D^{(m)}_{\Sigma,\,\Lambda}}\left(\left(\frac{\partial}{\partial \log(X_\tau)}\right)_\tau\right) \arrow[r, "5.34 \text{ and } (5.38.3)"] & \eta_{(\mu)}\left(K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\,\Lambda,\,\infty})}((\delta_\tau - 1)_\tau)\right)
\end{tikzcd} \tag{5.39.3}$$

where we again let the label τ range over "σ" for $\sigma \in \Sigma$ and "(λ', i)" for $\lambda' \in \Lambda$ and $i = 1, \dots, d$. By Proposition 5.16, the top horizontal map in (5.39.3) is a Frobenius-equivariant quasi-isomorphism and, by Lemma 3.7, Remark 3.35, and the Frobenius-equivariance of the homomorphism $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\lambda,\infty}) \to \mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\,\Lambda,\,\infty})$, so is the right vertical map. By Proposition 5.13 and (5.31.3), the left vertical map in (5.39.3) becomes a Frobenius-equivariant quasi-isomorphism after applying $\varinjlim_{m>0}$ and forming the termwise p-adic completion. These operations turn bottom horizontal map in (5.39.3) into the comparison map (5.38.4), so we conclude that the latter is also a Frobenius-equivariant quasi-isomorphism, as desired. $\square$

5.40. *Proof of Theorem 5.4.* By §5.38 and Proposition 5.39, the functorial in R complexes (5.21.1) and (5.32.1) define canonically and Frobenius-equivariantly quasi-isomorphic complexes of presheaves on a basis for the topology of $\mathfrak{X}_{\text{ét}}$. Their associated complexes of sheaves on $\mathfrak{X}_{\text{ét}}$ are then also canonically and Frobenius-equivariantly quasi-isomorphic. By §5.21 and §5.32, these complexes of sheaves represent $A\Omega_{\mathfrak{X}}\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}A_{\mathrm{cris}}$ and $Ru_*(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}})$, respectively, so that, in conclusion, Proposition 5.39 supplies a Frobenius-equivariant isomorphism

$$Ru_*(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}) \xrightarrow{\sim} A\Omega_{\mathfrak{X}}\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}A_{\mathrm{cris}}, \tag{5.40.1}$$

which gives the desired identification (5.4.1). $\square$

We now have two ways to identify the de Rham specialization of $A\Omega_{\mathfrak{X}}$: we could either use (4.16.1) or combine (5.4.1) with the fact that the logarithmic crystalline cohomology of $\mathfrak{X}_{\mathcal{O}_C/p}$ over $\mathcal{O}_C$ is computed by $\Omega^{\bullet}_{\mathfrak{X}/\mathcal{O}_C,\,\log}$. We now check that the two identifications agree (this will be used in §8).

Proposition 5.41. *The following diagram commutes*:

$$\begin{tikzcd} Ru_*(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}) \arrow[rr, "(5.40.1)", "\sim"'] \arrow[dr] & & A\Omega_{\mathfrak{X}}\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}A_{\mathrm{cris}} \arrow[dl, "(4.16.1)"] \\ & \Omega^{\bullet}_{\mathfrak{X}/\mathcal{O}_C,\,\log} & \end{tikzcd} \tag{5.41.1}$$

where the left diagonal map is induced by the identification $Ru_*(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}})\otimes^{\mathbb{L}}_{A_{\mathrm{cris}},\,\theta}\mathcal{O}_C \cong \Omega^{\bullet}_{\mathfrak{X}/\mathcal{O}_C,\,\log}$ *supplied by* [Bei13b, (1.8.1)]. *In particular, the two ways to identify* $A\Omega_{\mathfrak{X}}\otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\,\theta}\mathcal{O}_C$ *with* $\Omega^{\bullet}_{\mathfrak{X}/\mathcal{O}_C,\,\log}$ *mentioned in the preceding paragraph agree.*

Proof. We build on the corresponding proof given in the smooth case in [BMS16, proof of 14.1].

The claim is local, so we place ourselves in the setup of §5.17. Then, since the terms of $\Omega^{\bullet}_{\mathrm{Spf}(R)/\mathcal{O}_C,\,\log}$ are p-torsion free, each $t_{\lambda,\,i}$ is a unit in $R[\frac{1}{p}]$, and the elements $d\log(X_\sigma)$ and $d\log(X_{\lambda,\,i})$ generate the commutative differential graded algebra $\Omega^{\bullet}_{D_{\Sigma,\,\Lambda}/A_{\mathrm{cris}},\,\log,\,\mathrm{PD}}$ from Proposition 5.23 over $D_{\Sigma,\,\Lambda}$, there is a unique map of commutative differential graded algebras

$$\Omega^{\bullet}_{D_{\Sigma,\,\Lambda}/A_{\mathrm{cris}},\,\log,\,\mathrm{PD}} \to \Omega^{\bullet}_{\mathrm{Spf}(R)/\mathcal{O}_C,\,\log} \tag{5.41.2}$$

that in degree 0 is given by the map $D_{\Sigma,\,\Lambda} \twoheadrightarrow R$ from (5.22.3). By Proposition 5.23, the left diagonal map of (5.41.1) is described by this unique map (5.41.2). Thus, it remains to show that so is the composition in (5.41.1).

We recall from the proof of Theorem 4.16 that the right diagonal map in (5.41.2) is defined by using the Frobenius endomorphism of $A\Omega_{\mathfrak{X}}$ and the canonical identification (supplied by [BMS16, 6.11]) of $(L\eta_{(\varphi(\xi))}(A\Omega_{\mathfrak{X}}))/\varphi(\xi)$ with the complex[12] $H^{\bullet}(A\Omega_{\mathfrak{X}}/\varphi(\xi))$ whose differentials are given by Bockstein homomorphisms (defined in *loc. cit.* using $A\Omega_{\mathfrak{X}}/\varphi(\xi)^2$; *a posteriori*, $H^{\bullet}(A\Omega_{\mathfrak{X}}/\varphi(\xi))$ is canonically identified with $\Omega^{\bullet}_{\mathrm{Spf}(R)/\mathcal{O}_C,\,\log}$). Letting τ range over the same indexing set as in the proof of Proposition 5.39, this construction also applies to the complex $\eta_{(\mu)}\left(K_{\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\,\Lambda,\,\infty})}((\delta_\tau - 1)_\tau)\right)$: Frobenius maps it isomorphically to $\eta_{(\varphi(\mu))}\left(K_{\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\,\Lambda,\,\infty})}((\delta_\tau - 1)_\tau)\right)$, for which the reduction modulo $\varphi(\xi)$ map is

$$\eta_{(\varphi(\mu))}\left(K_{\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\,\Lambda,\,\infty})}((\delta_\tau - 1)_\tau)\right) \to H^{\bullet}\left(\left(\eta_{(\mu)}\left(K_{\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\,\Lambda,\,\infty})}((\delta_\tau - 1)_\tau)\right)\right)/\varphi(\xi)\right). \tag{5.41.3}$$

[12]For the sake of simplicity, we notationally suppress the twists inherent in the construction $H^{\bullet}(-)$ of *loc. cit.*

Since, by Theorem 4.2 and Remarks 3.10 and 3.35,

$$\left(\eta_{(\mu)}\left(K_{\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})}((\delta_\tau-1)_\tau)\right)\right)/\varphi(\xi)\cong\eta_{(\zeta_p-1)}\left(K_{R_{\Sigma,\Lambda,\infty}}((\delta_\tau-1)_\tau)\right),\tag{5.41.4}$$

the cited remarks imply that the composition of Frobenius and (5.41.3)–(5.41.4) gives the de Rham specialization map $A\Omega_{\mathfrak{X}}\to\Omega^\bullet_{\mathfrak{X}/\mathcal{O}_C,\log}$ in terms of the complex $\eta_{(\mu)}\left(K_{\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})}((\delta_\tau-1)_\tau)\right)$.

We can now describe the right diagonal map of (5.41.1) in terms of $\eta_{(\mu)}\left(K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})}((\delta_\tau-1)_\tau)\right)$, which is a variable term that comprises the target of the comparison map (5.38.4). Namely, we first let $\varphi(\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}))$ be the analogue of the ring $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$ built using the element $\varphi(\xi)$ instead of ξ, so that the Frobenius gives the isomorphism $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})\xrightarrow{\sim}\varphi(\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}))$.[13] Then Frobenius maps the complex $\eta_{(\mu)}\left(K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})}((\delta_\tau-1)_\tau)\right)$ isomorphically to the complex $\eta_{(\varphi(\mu))}\left(K_{\varphi(\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}))}((\delta_\tau-1)_\tau)\right)$, for which the reduction modulo $\varphi(\xi)$ map is

$$\eta_{(\varphi(\mu))}\left(K_{\varphi(\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}))}((\delta_\tau-1)_\tau)\right)\to H^\bullet\left(\eta_{(\mu)}\left(K_{\varphi(\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}))}((\delta_\tau-1)_\tau)\right)/\varphi(\xi)\right).\tag{5.41.5}$$

The target of the map (5.41.5) maps to

$$H^\bullet\left(\eta_{(\zeta_p-1)}\left(K_{R_{\Sigma,\Lambda,\infty}}((\delta_\tau-1)_\tau)\right)\right)\overset{3.10\text{ and }4.11}{\cong}\Omega^\bullet_{\mathrm{Spf}(R)/\mathcal{O}_C,\log}\tag{5.41.6}$$

via a morphism induced by the map $\theta\circ\varphi^{-1}\colon\varphi(\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}))\twoheadrightarrow R_{\Sigma,\Lambda,\infty}$; indeed, since each $H^i\left(\eta_{(\zeta_p-1)}\left(K_{R_{\Sigma,\Lambda,\infty}}((\delta_\tau-1)_\tau)\right)\right)$ is p-torsion free, the agreement of the Bockstein differentials may be checked after inverting p by using the fact that $(\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})/\varphi(\xi)^2)[\frac{1}{p}]$ is an algebra over $\varphi(\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}))$ via a map that lifts $\theta\circ\varphi^{-1}$. The resulting composition

$$\eta_{(\mu)}\left(K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})}((\delta_\tau-1)_\tau)\right)\to H^\bullet\left(\eta_{(\zeta_p-1)}\left(K_{R_{\Sigma,\Lambda,\infty}}((\delta_\tau-1)_\tau)\right)\right)\overset{3.10\text{ and }4.11}{\cong}\Omega^\bullet_{\mathrm{Spf}(R)/\mathcal{O}_C,\log}$$

gives the promised description of the right diagonal map of (5.41.1) and, by construction and [BMS16, 6.13], is a morphism of commutative differential graded algebras[14] that in degree 0 is given by the map θ of (5.19.3). On the other hand, the comparison map

$$\Omega^\bullet_{D_{\Sigma,\Lambda}/A_{\mathrm{cris}},\log,\mathrm{PD}}\cong K_{D_{\Sigma,\Lambda}}\left(\left(\tfrac{\partial}{\partial\log(X_\tau)}\right)_\tau\right)\to\left(\varinjlim_{m>0}\left(\eta_{(\mu)}\left(K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})}((\delta_\tau-1)_\tau)\right)\right)\right)^{\wedge}$$

from (5.38.4) would only become a morphism of commutative differential graded algebras if in the formula $\log([\epsilon])\cdot\sum_{n\geq 0}\frac{(\log([\epsilon]))^n}{(n+1)!}(\frac{\partial}{\partial\log(X_\tau)})^n$ that describes the morphism (5.34.1) in degree 1 we could ignore the terms with $n\geq 1$. However, $\log([\epsilon])$ and μ are unit multiples of each other and $\theta(\frac{\mu^n}{(n+1)!})=0$ in $\mathcal{O}_C$ for $n\geq 1$ (see §5.14), so we can indeed ignore these terms if we are only interested in the composition

$$\Omega^\bullet_{D_{\Sigma,\Lambda}/A_{\mathrm{cris}},\log,\mathrm{PD}}\to\left(\varinjlim_{m>0}\left(\eta_{(\mu)}\left(K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})}((\delta_\tau-1)_\tau)\right)\right)\right)^{\wedge}\to\Omega^\bullet_{\mathrm{Spf}(R)/\mathcal{O}_C,\log}.$$

that describes the composition in (5.41.1). In conclusion, this composition is a morphism of commutative differential graded algebras that, due to (5.38.2), is given in degree 0 by the map $D_{\Sigma,\Lambda}\twoheadrightarrow R$ from (5.22.3), so, as desired, it is indeed the unique morphism (5.41.2). $\square$

We now use Theorem 5.4 to analyze the crystalline specialization of $R\Gamma(\mathfrak{X},A\Omega_{\mathfrak{X}})$ in Corollary 5.43.

[13] Composition with the map $\varphi(\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}))\to\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$ recovers the Frobenius of $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$.

[14] See [BMS16, 7.5] and its proof for the description of the commutative differential graded algebra structure on the Koszul complex $K_*((\delta_\tau-1)_\tau)$ that computes continuous group cohomology.

5.42. The crystalline specialization map. The Witt vector functoriality gives the surjection

$$A_{\mathrm{inf}} \twoheadrightarrow W(k), \qquad \text{the so-called } \textit{crystalline specialization} \text{ map of } A_{\mathrm{inf}}.$$

Since ξ maps to p in $W(k)$, this surjection factors through A_{cris} as follows: $A_{\mathrm{inf}} \hookrightarrow A_{\mathrm{cris}} \twoheadrightarrow W(k)$. We equip $W(k)$ with the pullback of the log structure on A_{cris} defined in §5.2. Explicitly, the resulting log structure on $W(k)$ is associated to the prelog structure $\mathbb{Q}_{\geq 0} \xrightarrow{0} W(k)$.

Corollary 5.43. *If $\mathfrak{X}$ is quasi-compact and quasi-separated, then we have the Frobenius-equivariant identifications*

$$\begin{aligned} R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}} &\cong R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}), \\ R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} W(k) &\cong R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k)). \end{aligned} \tag{5.43.1}$$

If $\mathfrak{X}$ is even proper over $\mathcal{O}_C$, then we have the Frobenius-equivariant identifications

$$\begin{aligned} R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}} &\cong R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}), \\ R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} W(k) &\cong R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k)), \end{aligned} \tag{5.43.2}$$

and the cohomology groups of $R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}[\frac{1}{p}]$ are finite free as $A_{\mathrm{cris}}[\frac{1}{p}]$-modules.

Proof. By [BMS16, 4.9], any finitely presented A_{inf}/p^n-module is perfect as an A_{inf}-module. Consequently, any A_{inf}/p^n-module M is a filtered direct limit of perfect A_{inf}-modules, so, by [SP, 0739],

$$R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} M) \cong R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} M.$$

In particular, this applies to $M = A_{\mathrm{cris}}/p^n$, so, since $R\Gamma(\mathfrak{X}_{\text{ét}}, -)$ commutes with derived limits (see [SP, 0A07]), the first identification in (5.43.1) follows from Theorem 5.4.

For each finite subextension of $C/\operatorname{Frac}(W(k))$, consider its ring of integers $\mathcal{O} \subset \mathcal{O}_C$ equipped with the (fine) log structure associated to the prelog structure $\mathcal{O} \cap (\mathcal{O}[\frac{1}{p}])^\times \hookrightarrow \mathcal{O}$. By using étale local semistable coordinates (1.5.1) and Claim 1.6.1, we may employ limit arguments to find such an $\mathcal{O}$ together with a quasi-compact and quasi-separated log smooth log scheme $\mathcal{X}$ over $\mathcal{O}/p$ that descends $\mathfrak{X}_{\mathcal{O}_C/p}$ and is of Cartier type (see [Kat89, 4.8]). Then the base change theorem [Bei13b, (1.11.1)] applies[15] and shows that

$$R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) \widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{cris}}} W(k) \cong R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k)), \tag{5.43.3}$$

so that the second identification in (5.43.1) follows from the first.

If $\mathfrak{X}$ is $\mathcal{O}_C$-proper, then, by Corollary 4.19, the object $R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}})$ is quasi-isomorphic to a bounded complex of finite free A_{inf}-modules, so the identifications in (5.43.2) follows from those in (5.43.1). Moreover, then $\mathcal{X}$ is $\mathcal{O}$-proper and [Bei13b, 1.18, Theorem] proves that the cohomology groups of

$$R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}[\tfrac{1}{p}], \qquad \text{and hence also of} \qquad R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}[\tfrac{1}{p}],$$

are finite free $A_{\mathrm{cris}}[\frac{1}{p}]$-modules. □

Remarks.

[15] In loc. cit., the map f of fine log schemes is quasi-compact and separated. One may relax this to quasi-compact and quasi-separated: once Y there is affine, the iterated intersections of opens in an affine cover of Z are quasi-compact and separated over Y, so the Čech technique (compare with [SP, 08BN]) reduces to the original assumptions.

5.44. In the notation of the preceding proof, the special fiber $\mathcal{X}_k$ of $\mathcal{X}$ is a descent of $\mathfrak{X}_k$ to a fine log scheme over the "standard log point" k whose log structure is associated to $\mathbb{N}_{\geq 0} \xrightarrow{0} k$ (the base change map is a "change of log structure" self-map of k determined by the map $\mathbb{N}_{\geq 0} \to \mathbb{Q}_{\geq 0}$ that sends 1 to the valuation of a uniformizer of $\mathcal{O}$). Given any such descent, the base change theorem [Bei13b, (1.11.1)] also gives the further Frobenius-equivariant identification

$$R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k)) \cong R\Gamma_{\log \mathrm{cris}}(\mathcal{X}_k/W(k)), \tag{5.44.4}$$

where $W(k)$ on the right side is equipped with the log structure associated to $\mathbb{N}_{\geq 0} \xrightarrow{0} W(k)$. Likewise, if $\mathfrak{X}_k$ is k-smooth, then *loc. cit.* gives the Frobenius-equivariant identification

$$R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k)) \cong R\Gamma_{\mathrm{cris}}(\mathfrak{X}_k/W(k)). \tag{5.44.5}$$

5.45. The identification (5.43.3) expresses $R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k))$ in terms of $R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}})$. Further results from [Bei13b] imply that for proper $\mathfrak{X}$ a converse holds after base change to B_{st}^+: see (9.2.1) below (when $\mathfrak{X}_k$ is smooth, $A_{\mathrm{cris}}[\frac{1}{p}]$ in place of B_{st}^+ suffices, see [BMS16, 13.9]).

The results of [Bei13b] also give a Hyodo–Kato type isomorphism in our context:

Proposition 5.46. *If $\mathfrak{X}$ is proper over $\mathcal{O}_C$, then there is an isomorphism*

$$R\Gamma_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C) \otimes^{\mathbb{L}}_{\mathcal{O}_C} C \simeq R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k)) \otimes^{\mathbb{L}}_{W(k)} C. \tag{5.46.1}$$

Proof. By [Bei13b, (1.8.1)], letting $p\mathcal{O}_C \subset \mathcal{O}_C$ be endowed with its standard divided powers, we have

$$R\Gamma_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C) \cong R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/\mathcal{O}_C). \tag{5.46.2}$$

Moreover, letting $\mathcal{X}$ be a descent of $\mathfrak{X}_{\mathcal{O}_C/p}$ as in the proof of Corollary 5.43, by [Bei13b, (1.16.2) and §1.15, Remarks, (iv)],[16] we have a (noncanonical) isomorphism

$$R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/\mathcal{O}_C) \otimes^{\mathbb{L}}_{\mathcal{O}_C} C \simeq R\Gamma_{\log \mathrm{cris}}(\mathcal{X}_k/W(k)) \otimes^{\mathbb{L}}_{W(k)} C, \tag{5.46.3}$$

where $W(k)$ is equipped with the log structure associated to $\mathbb{N}_{\geq 0} \xrightarrow{0} W(k)$. It remains to combine the isomorphisms (5.44.4), (5.46.2), and (5.46.3). $\square$

6. The comparison to the B_{dR}^+-cohomology

The main goal of this section is Theorem 6.6, which, for quasi-compact and quasi-separated $\mathfrak{X}$, identifies the B_{dR}^+-base change of the absolute crystalline cohomology $R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}})$ with the "crystalline cohomology of $\mathfrak{X}_C^{\mathrm{ad}}$ over B_{dR}^+," denoted by $R\Gamma_{\mathrm{cris}}(\mathfrak{X}_C^{\mathrm{ad}}/B_{\mathrm{dR}}^+)$, that was defined in [BMS16, §13] (see §6.2 for a brief review). The definition of $R\Gamma_{\mathrm{cris}}(\mathfrak{X}_C^{\mathrm{ad}}/B_{\mathrm{dR}}^+)$ is purely in terms of $\mathfrak{X}_C^{\mathrm{ad}}$ and was engineered in *op. cit.* to be compatible with $R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}})$ in the case when $\mathfrak{X}$ is smooth. Therefore, for the desired base change, we only need to check that a slightly more general definition of $R\Gamma_{\mathrm{cris}}(\mathfrak{X}_C^{\mathrm{ad}}/B_{\mathrm{dR}}^+)$ that uses the étale topology instead of Zariski and more general embeddings than those furnished by annuli leads to the same cohomology (see §§6.2–6.3). For this, we adapt the arguments of *op. cit.*; in fact, our C is given as $(\overline{\mathrm{Frac}(W(k))})^\wedge$ (see §1.5), so we may simplify the "descent to a discretely valued base" aspects of these arguments by taking advantage of a result of Huber on the local structure of étale maps of adic spaces (see §6.3).

[16] We are citing the post-publication arXiv version of the article, which slightly differs from the published version.

6.1. The ring B_{dR}^+. Since ξ is not a zero divisor in $A_{\mathrm{inf}}[\frac{1}{p}]$ and generates $\mathrm{Ker}(\theta[\frac{1}{p}])$, the $\mathrm{Ker}(\theta[\frac{1}{p}])$-adic completion of $A_{\mathrm{inf}}[\frac{1}{p}]$ is a complete discrete valuation ring B_{dR}^+ with ξ as a uniformizer and C as the residue field. By Proposition 5.36, both A_{inf} and A_{cris} are subalgebras of B_{dR}^+. By the "glueing of flatness" [RG71, II.1.4.2.1], the ring B_{dR}^+ is flat as an A_{inf}-algebra. We set

$$B_{\mathrm{dR}} := \mathrm{Frac}(B_{\mathrm{dR}}^+).$$

Our A_{inf} is a $W(k)$-algebra (see §2.1), so, by Hensel's lemma, B_{dR}^+ is naturally a $\overline{W(k)}[\frac{1}{p}]$-algebra.

6.2. The B_{dR}^+-cohomology using the étale topology. In [BMS16, §13], Bhatt–Morrow–Scholze used the Zariski site of a smooth adic C-space X to define the "B_{dR}^+-cohomology" of X, denoted by

$$R\Gamma_{\mathrm{cris}}(X/B_{\mathrm{dR}}^+) \in D^{\geq 0}(B_{\mathrm{dR}}^+).$$

We will now review their construction to show that it may also be carried out in the étale topology.

By [Hub94, 1.6.10, 2.2.8], the Zariski (resp., étale) topology of X has a basis of affinoid opens $\mathrm{Spa}(A, A^\circ)$ each of which admits a map

$$\mathrm{Spa}(A, A^\circ) \to \mathbb{T}_C^d := \mathrm{Spa}(C\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle, \mathcal{O}_C\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle) \quad \text{for some} \quad d \in \mathbb{Z}_{\geq 0} \tag{6.2.1}$$

that is a composition of a rational embedding, a finite étale map, and a rational embedding. Since A is topologically of finite type over C and for any $a \in A$ there exists an $n > 0$ with $1 + p^n a \in (A^\circ)^\times$, there is a finite subset $\Psi \subset (A^\circ)^\times$ such that the following map is surjective:

$$C\left\langle (X_u^{\pm 1})_{u \in \Psi} \right\rangle \xrightarrow{X_u \mapsto u} A. \tag{6.2.2}$$

Thus, endowing each A_{inf}/ξ^n with the p-adic topology, each $(A_{\mathrm{inf}}/\xi^n)[\frac{1}{p}]$ with the unique ring topology for which A_{inf}/ξ^n is an open subring, setting

$$B_{\mathrm{dR}}^+\left\langle (X_u^{\pm 1})_{u \in \Psi} \right\rangle := \varprojlim_{n>0}((B_{\mathrm{dR}}^+/\xi^n)\left\langle (X_u^{\pm 1})_{u \in \Psi} \right\rangle), \tag{6.2.3}$$

and composing the projection onto the $n = 1$ term with (6.2.2), one obtains the surjection

$$s\colon B_{\mathrm{dR}}^+\left\langle (X_u^{\pm 1})_{u \in \Psi} \right\rangle \twoheadrightarrow A \qquad \text{and sets} \qquad D_\Psi(A) := \varprojlim_{n>0}((B_{\mathrm{dR}}^+\left\langle (X_u^{\pm 1})_{u \in \Psi} \right\rangle)/(\mathrm{Ker}\, s)^n). \tag{6.2.4}$$

By the Leibniz rule, any derivation of $B_{\mathrm{dR}}^+\left\langle (X_u^{\pm 1})_{u \in \Psi} \right\rangle$ extends to $D_\Psi(A)$. In particular, the derivations $\frac{\partial}{\partial \log(X_u)} := X_u \cdot \frac{\partial}{\partial X_u}$ allow one to define the Koszul complex

$$\Omega^\bullet_{D_\Psi(A)/B_{\mathrm{dR}}^+} := K_{D_\Psi(A)}\left((\tfrac{\partial}{\partial \log(X_u)})_{u \in \Psi} \right)$$

that is functorial in enlarging Ψ. The resulting complex

$$\Omega^\bullet_{A/B_{\mathrm{dR}}^+} := \varinjlim_\Psi \left(\Omega^\bullet_{D_\Psi(A)/B_{\mathrm{dR}}^+} \right) \tag{6.2.5}$$

is functorial in A. Consequently, by varying $\mathrm{Spa}(A, A^\circ)$, we obtain a complex of presheaves on a basis for the Zariski (resp., étale) topology of X. The cohomology of the associated complex of sheaves is, by definition, the B_{dR}^+-cohomology of X:

$$R\Gamma_{\mathrm{cris}}(X/B_{\mathrm{dR}}^+) \qquad \text{(resp., its variant for the étale topology} \qquad R\Gamma_{\mathrm{cris}}(X_{\mathrm{\acute{e}t}}/B_{\mathrm{dR}}^+)) \tag{6.2.6}$$

By [BMS16, 13.5 (ii)], if A is fixed and Ψ is sufficiently large, then $D_\Psi(A)$ is ξ-torsion free and ξ-adically complete. Consequently, the B_{dR}^+-cohomology objects

$$R\Gamma_{\mathrm{cris}}(X/B_{\mathrm{dR}}^+) \qquad \text{and} \qquad R\Gamma_{\mathrm{cris}}(X_{\mathrm{\acute{e}t}}/B_{\mathrm{dR}}^+) \qquad \text{are derived } \xi\text{-adically complete.} \tag{6.2.7}$$

By [BMS16, 13.6], their (derived) reductions modulo ξ are canonically and compatibly identified with the de Rham cohomology objects $R\Gamma(X, \Omega^{\bullet,\,\mathrm{cont}}_{X/C})$ and $R\Gamma(X_{\text{ét}}, \Omega^{\bullet,\,\mathrm{cont}}_{X/C})$, respectively, for instance:

$$R\Gamma_{\mathrm{cris}}(X/B^{+}_{\mathrm{cris}}) \otimes^{\mathbb{L}}_{B^{+}_{\mathrm{dR}}} C \cong R\Gamma(X, \Omega^{\bullet,\,\mathrm{cont}}_{X/C}) =: R\Gamma_{\mathrm{dR}}(X/C). \tag{6.2.8}$$

Thus, since, by the Hodge-to-de Rham spectral sequence and [Sch13, 9.2 (ii)], the formation of de Rham cohomology is insensitive to passage to the étale topology, we have the pullback isomorphism:

$$R\Gamma_{\mathrm{cris}}(X/B^{+}_{\mathrm{dR}}) \xrightarrow{\sim} R\Gamma_{\mathrm{cris}}(X_{\text{ét}}/B^{+}_{\mathrm{dR}}). \tag{6.2.9}$$

In addition, if (for simplicity) X is proper over C and there is a complete discretely valued subfield $K \subset C$ with a perfect residue field and a proper, smooth adic space X_0 over K equipped with an isomorphism $X \cong X_0 \widehat{\otimes}_K C$, then, by [BMS16, 13.7], there is a canonical identification

$$R\Gamma_{\mathrm{cris}}(X/B^{+}_{\mathrm{dR}}) \cong R\Gamma_{\mathrm{dR}}(X_0/K) \otimes_K B^{+}_{\mathrm{dR}}, \quad \text{where} \quad R\Gamma_{\mathrm{dR}}(X_0/K) := R\Gamma(X_0, \Omega^{\bullet,\,\mathrm{cont}}_{X_0/K}). \tag{6.2.10}$$

In this situation, by the proof of *loc. cit.*, the reduction modulo ξ of the identification (6.2.10) recovers the identification (6.2.8) under the canonical identification $R\Gamma_{\mathrm{dR}}(X/C) \cong R\Gamma_{\mathrm{dR}}(X_0/K) \otimes^{\mathbb{L}}_K C$.

6.3. The B^{+}_{dR}-cohomology using more general embeddings. To relate the B^{+}_{dR}-cohomology and the logarithmic crystalline cohomology studied in §5, we wish to mildly generalize the construction of $R\Gamma_{\mathrm{cris}}(X_{\text{ét}}/B^{+}_{\mathrm{dR}})$. Namely, we consider the (larger) basis of $X_{\text{ét}}$ that consists of those affinoids $\mathrm{Spa}(A, A^{\circ})$ that have an étale morphism

$$\mathrm{Spa}(A, A^{\circ}) \to \mathrm{Spa}(C\langle T_1, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_d^{\pm 1}\rangle, \mathcal{O}_C\langle T_1, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_d^{\pm 1}\rangle) \tag{6.3.1}$$

for some $r, d \in \mathbb{Z}_{\geq 0}$ with $r \leq d$ such that $T_i \in A^{\times}$ for each i (even when $1 \leq i \leq r$). By [Hub96, 1.7.3 iii)][17] and limit arguments, there is a complete discretely valued subfield $K \subset C$ with the ring of integers $\mathcal{O}$ and a perfect residue field together with a finite type $\mathcal{O}[T_1, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_d^{\pm 1}]$-algebra A_0 that is étale after inverting p, flat over $\mathcal{O}$, and normal such that the morphism (6.3.1) is the C-base change of an étale $\mathrm{Spa}(K, \mathcal{O})$-morphism

$$\mathrm{Spa}((\widehat{A_0})[\tfrac{1}{p}], \widehat{A_0}) \to \mathrm{Spa}(K\langle T_1, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_d^{\pm 1}\rangle, \mathcal{O}\langle T_1, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_d^{\pm 1}\rangle) \tag{6.3.2}$$

such that $T_i \in ((\widehat{A_0})[\frac{1}{p}])^{\times}$. For each element $\mathrm{Spa}(A, A^{\circ})$ of this basis, we consider variable finite subsets $\Psi \subset (A^{\circ})^{\times}$ and $\Xi \subset A^{\circ} \cap A^{\times}$ such that the map

$$C\langle (X_u^{\pm 1})_{u \in \Psi}, (X_a)_{a \in \Xi}\rangle \xrightarrow{X_u \mapsto u,\, X_a \mapsto a} A \tag{6.3.3}$$

is surjective and Ψ (resp., Ξ) contains the images of the T_i with $r+1 \leq i \leq d$ (resp., $1 \leq i \leq r$) for some coordinate map as in (6.3.1) whose choice, together with a choice of its descent (6.3.2), we fix when discussing fixed Ψ and Ξ. Defining the ring $B^{+}_{\mathrm{dR}}\left\langle (X_u^{\pm 1})_{u \in \Psi},\, (X_a)_{a \in \Xi}\right\rangle$ analogously to (6.2.3), so that the map (6.3.3) gives rise to the surjection

$$s\colon B^{+}_{\mathrm{dR}}\left\langle (X_u^{\pm 1})_{u \in \Psi},\, (X_a)_{a \in \Xi}\right\rangle \twoheadrightarrow A,$$

for $n \in \mathbb{Z}_{>0}$ we set

$$D_{\Psi,\,\Xi,\,n}(A) := (B^{+}_{\mathrm{dR}}\left\langle (X_u^{\pm 1})_{u \in \Psi},\, (X_a)_{a \in \Xi}\right\rangle)/(\mathrm{Ker}\, s)^n \qquad \text{and} \qquad D_{\Psi,\,\Xi}(A) := \varprojlim_{n>0} D_{\Psi,\,\Xi,\,n}(A).$$

[17] Noncomplete A are allowed in loc. cit., so we choose $A^{+} := \overline{W(k)}[T_1, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_d^{\pm 1}]$ and $A^{\rhd} := A^{+}[\frac{1}{p}]$.

The ring B_{dR}^{+} is naturally a K-algebra and, for each $n > 0$, we let $(B_{\mathrm{dR}}^{+}/\xi^n)_0 \subset B_{\mathrm{dR}}^{+}/\xi^n$ be the A_{inf}/ξ^n-subalgebra generated by the image of $\mathcal{O}$. The proof of [BMS16, 13.4] shows[18] (with R_A there replaced by our $(\widehat{A_0})[\frac{1}{p}]$) that the B_{dR}^{+}-algebra

$$B_{\mathrm{dR}}^{+}\widehat{\otimes}_K(A_0[\tfrac{1}{p}]) := \varprojlim_{n>0} \left(((B_{\mathrm{dR}}^{+}/\xi^n)_0 \otimes_{\mathcal{O}} A_0)\widehat{\ }[\tfrac{1}{p}]\right) \tag{6.3.4}$$

is ξ-adically complete and ξ-torsion free with

$$(B_{\mathrm{dR}}^{+}\widehat{\otimes}_K(A_0[\tfrac{1}{p}]))/\xi \cong A \quad \text{and, more generally,} \quad (B_{\mathrm{dR}}^{+}\widehat{\otimes}_K(A_0[\tfrac{1}{p}]))/\xi^n \cong ((B_{\mathrm{dR}}^{+}/\xi^n)_0 \otimes_{\mathcal{O}} A_0)\widehat{\ }[\tfrac{1}{p}].$$

Moreover, we have the following analogue of [BMS16, 13.5 (ii)] whose proof will be given in §6.4:

Lemma 6.3.5. *If Ξ contains the images of the T_i with $1 \le i \le r$ under some coordinate morphism as in (6.3.1) and Ψ is large enough, then we have the isomorphism*

$$D_{\Psi,\Xi}(A) \cong (B_{\mathrm{dR}}^{+}\widehat{\otimes}_K(A_0[\tfrac{1}{p}]))[\![(X_a - \widetilde{a})_{a\in(\Psi\cup\Xi)\setminus\{T_1,\dots,T_d\}}]\!] \tag{6.3.6}$$

where $\widetilde{a} \in B_{\mathrm{dR}}^{+}\widehat{\otimes}_K(A_0[\frac{1}{p}])$ denotes a fixed lift of a. In particular, for large Ψ and Ξ, the B_{dR}^{+}-algebra $D_{\Psi,\Xi}(A)$ is ξ-adically complete and ξ-torsion free.

Similarly to §6.2, for any Ψ and Ξ the derivations $\frac{\partial}{\partial\log(X_a)} := X_a \cdot \frac{\partial}{\partial X_a}$ with $a \in \Psi \cup \Xi$ extend to $D_{\Psi,\Xi}(A)$ and we may define the Koszul complex

$$\Omega^{\bullet}_{D_{\Psi,\Xi}(A)/B_{\mathrm{dR}}^{+}} := K_{D_{\Psi,\Xi}(A)}\left((\tfrac{\partial}{\partial\log(X_u)})_{u\in\Psi}, (\tfrac{\partial}{\partial\log(X_a)})_{a\in\Xi}\right)$$

that is functorial in replacing Ψ and Ξ by larger Ψ' and Ξ'. Since $a \in A^{\times}$ for every $a \in \Psi \cup \Xi$, Lemma 6.3.5 and the proof of [BMS16, 13.6] show that

$$\Omega^{\bullet}_{D_{\Psi,\Xi}(A)/B_{\mathrm{dR}}^{+}}/\xi \cong \Omega^{\bullet,\,\mathrm{cont}}_{A/C} \qquad \text{in the derived category,} \tag{6.3.7}$$

compatibly with enlarging Ψ and Ξ. In particular, due to the derived ξ-adic completeness supplied by Lemma 6.3.5, if Ψ is large enough, then the map

$$\Omega^{\bullet}_{D_{\Psi,\Xi}(A)/B_{\mathrm{dR}}^{+}} \to \Omega^{\bullet}_{D_{\Psi',\Xi'}(A)/B_{\mathrm{dR}}^{+}} \qquad \text{is a quasi-isomorphism.}$$

Thus, if $\mathrm{Spa}(A, A^{\circ})$ even has a coordinate map as in (6.2.1), then we obtain the functorial in $\mathrm{Spa}(A, A^{\circ})$ quasi-isomorphism with the complex $\Omega^{\bullet}_{A/B_{\mathrm{dR}}^{+}}$ of (6.2.5):

$$\Omega^{\bullet}_{A/B_{\mathrm{dR}}^{+}} \xrightarrow{\sim} \varinjlim_{\Psi,\Xi}\left(\Omega^{\bullet}_{D_{\Psi,\Xi}(A)/B_{\mathrm{dR}}^{+}}\right). \tag{6.3.8}$$

Since those $\mathrm{Spa}(A, A^{\circ})$ for which the coordinate map as in (6.2.1) exists also form a basis for $X_{\mathrm{\acute{e}t}}$, we conclude that the cohomology of the sheafification of the complex of presheaves furnished by the target of (6.3.8) is identified with $R\Gamma_{\mathrm{cris}}(X_{\mathrm{\acute{e}t}}/B_{\mathrm{cris}}^{+})$. In conclusion, we may summarize informally:

$$\text{the complexes} \quad \Omega^{\bullet}_{D_{\Psi,\Xi}(A)/B_{\mathrm{dR}}^{+}} \quad \text{also compute the } B_{\mathrm{dR}}^{+}\text{-cohomology} \quad R\Gamma_{\mathrm{cris}}(X/B_{\mathrm{cris}}^{+}) \tag{6.3.9}$$

and the maps (6.3.7) recover the following identification (6.2.8).

[18] In fact, in our case the argument is simpler than in *loc. cit.* and we sketch it here. We may assume that $\mathrm{Spec}(A_0)$ has no connected components on which p is a unit (such components do not contribute to (6.3.2)), so, by [RG71, 3.3.5] and [SP, 0593], the ring A_0 is free as an $\mathcal{O}$-module. Consequently, the n^{th} term of the inverse limit in (6.3.4) is a p-adically completed direct sum of copies of $(A_{\mathrm{inf}}/\xi^n)[\frac{1}{p}]$. This makes the multiplication by ξ^m map on this n^{th} term explicit and the desired claims follow by passing to the inverse limit over n.

6.4. *Proof of Lemma 6.3.5.* We adapt the proof of [BMS16, 13.5 (ii)] as follows.

In addition to the fixed coordinate morphism (6.3.1) used in the statement and its descent (6.3.2), we set $\Xi_0 := \{T_1, \dots, T_r\} \overset{(6.3.2)}{\subset} \widehat{A_0} \cap (\widehat{A_0}[\frac{1}{p}])^\times$ and fix a subset $\Psi_0 \subset (\widehat{A_0})^\times$ such that the map

$$s_0 \colon K\left\langle (x_u^{\pm 1})_{u \in \Psi_0}, (x_a)_{a \in \Xi_0} \right\rangle \xrightarrow{x_u \mapsto u,\, x_a \mapsto a} \widehat{A_0}[\tfrac{1}{p}]$$

is surjective and Ψ_0 contains the images under the map (6.3.2) of the variables T_i with $r+1 \le i \le d$. We require that Ψ contains the image of Ψ_0 in R° (this is the meaning of "large enough" in the statement). We set

$$D_{0,\,n} := K\left\langle (x_u^{\pm 1})_{u \in \Psi_0}, (x_a)_{a \in \Xi_0} \right\rangle / (\operatorname{Ker} s_0)^n \quad \text{for} \quad n > 0 \qquad \text{and} \qquad D_0 := \varprojlim_{n>0} D_{0,\,n},$$

so that, by the $K[T_1, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_d^{\pm 1}]$-étaleness of $A_0[\frac{1}{p}]$, the map

$$A_0[\tfrac{1}{p}] \to \widehat{A_0}[\tfrac{1}{p}] \qquad \text{lifts to a map} \qquad A_0[\tfrac{1}{p}] \to D_0 \quad \text{with} \quad T_i \mapsto x_{T_i}. \tag{6.4.1}$$

By [GR03, 7.3.15] (alternatively, by [Hub93, 3.3] and the fact that $D_{0,\,n}$ is a Tate ring with a Noetherian ring of definition), for each $n > 0$, the subring $D_{0,\,n}^\circ \subset D_{0,\,n}$ of powerbounded elements is the preimage of its counterpart $(\widehat{A_0}[\frac{1}{p}])^\circ \subset \widehat{A_0}[\frac{1}{p}]$. Thus, the lift (6.4.1) maps A_0 to $D_{0,\,n}^\circ$ and hence also to some subring of definition of $D_{0,\,n}$, to the effect that we obtain a continuous section

$$\widehat{A_0}[\tfrac{1}{p}] \hookrightarrow D_0 \qquad \text{of the surjection} \qquad D_0 \twoheadrightarrow \widehat{A_0}[\tfrac{1}{p}]. \tag{6.4.2}$$

The continuous map $K \to B_{\mathrm{dR}}^+$ mentioned in §6.3 gives a compatible with s_0 and s continuous map

$$K\left\langle (x_u^{\pm 1})_{u \in \Psi_0}, (x_a)_{a \in \Xi_0} \right\rangle \xrightarrow{x_u \mapsto X_u,\, x_a \mapsto X_a} B_{\mathrm{dR}}^+\left\langle (X_u^{\pm 1})_{u \in \Psi_0}, (X_a)_{a \in \Xi_0} \right\rangle, \quad \text{so also} \quad D_0 \to D_{\Psi,\,\Xi}(A).$$

Thus, the section (6.4.2) gives the continuous map y in the commutative diagram

$$\begin{tikzcd}
& (B_{\mathrm{dR}}^+ \widehat{\otimes}_K (A_0[\frac{1}{p}]))[\![(X_a - \widetilde{a})_{a \in (\Psi \cup \Xi) \setminus \{T_1, \dots, T_d\}}]\!] \arrow[dr, two heads] \arrow[d, bend right, "y"'] & \\
B_{\mathrm{dR}}^+\left\langle (X_u^{\pm 1})_{u \in \Psi}, (X_a)_{a \in \Xi} \right\rangle \arrow[ur, "X_{T_i} \mapsto T_i"] \arrow[dr] & & A \\
& D_{\Psi,\,\Xi}(A) \arrow[ur, two heads] \arrow[u, bend right, "z"'] &
\end{tikzcd}$$

in which the continuous map z is defined by combining the top part of the diagram, the ξ-adic completeness of $B_{\mathrm{dR}}^+ \widehat{\otimes}_K (A_0[\frac{1}{p}])$ (see §6.3), and the definition of $D_{\Psi,\,\Xi}(A)$. By construction, $y \circ z = \mathrm{id}$. By the $\mathcal{O}[T_1, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_d^{\pm 1}]$-étaleness of A_0, the B_{dR}^+-algebra endomorphism $z \circ y$ of $(B_{\mathrm{dR}}^+ \widehat{\otimes}_K (A_0[\frac{1}{p}]))[\![(X_a - \widetilde{a})_{a \in (\Psi \cup \Xi) \setminus \{T_1, \dots, T_d\}}]\!]$ is the identity on A_0, so also on $(B_{\mathrm{dR}}^+ \widehat{\otimes}_K (A_0[\frac{1}{p}]))$. Since, in addition, it fixes every X_a, it must be the identity. Thus, z is the desired isomorphism (6.3.6). ☐

6.5. The map from the absolute crystalline cohomology. Returning to the $\mathfrak{X}$ of §1.5, our next goal is to use the discussion of §§6.2–6.3 to exhibit a map

$$R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) \to R\Gamma_{\mathrm{cris}}(\mathfrak{X}_C^{\mathrm{ad}}/B_{\mathrm{dR}}^+). \tag{6.5.1}$$

For this, we work on the basis for $\mathfrak{X}_{\text{ét}}$ consisting of affine opens $\operatorname{Spf} R$ as in the "all possible coordinates" setup of §5.17 and use the notation introduced in §§5.17–5.40. To relate to §6.3, we set

$$A := R[\tfrac{1}{p}], \quad \Psi := \{t_\sigma\}_{\sigma \in \Sigma} \cup \textstyle\bigcup_{\lambda \in \Lambda} \{t_{\lambda,\, r_\lambda + 1}, \dots, t_{\lambda,\, d}\}, \quad \text{and} \quad \Xi := \textstyle\bigcup_{\lambda \in \Lambda} \{t_{\lambda,\, 1}, \dots, t_{\lambda,\, r_\lambda}\} \tag{6.5.2}$$

(so that $A^\circ \cong R$ and $t_{\lambda,\,0}$ is omitted). For each $\lambda \in \Lambda$, the adic generic fiber of

$$\operatorname{Spf}(\mathcal{O}_C\{t_{\lambda,\,0}, \dots, t_{\lambda,\,r_\lambda}, t_{\lambda,\,r_\lambda+1}^{\pm 1}, \dots, t_{\lambda,\,d}^{\pm 1}\}/(t_{\lambda,\,0} \cdots t_{\lambda,\,r_\lambda} - p^{q_\lambda}))$$

is the rational subset of

$$\mathrm{Spa}(C\langle T_{\lambda,1},\dots,T_{\lambda,r_\lambda},T_{\lambda,r_\lambda+1}^{\pm1},\dots,T_{\lambda,d}^{\pm1}\rangle,\mathcal{O}_C\langle T_{\lambda,1},\dots,T_{\lambda,r_\lambda},T_{\lambda,r_\lambda+1}^{\pm1},\dots,T_{\lambda,d}^{\pm1}\rangle)$$

cut out by the condition "$|p^q|\le|T_{\lambda,1}\cdots T_{\lambda,r_\lambda}|$," so our $\mathrm{Spa}(A,A^\circ)$ is an element of the basis considered in §6.3. Moreover, for each $\lambda\in\Lambda$, we may descend the étale map (5.17.3) to a discrete valuation subring $\mathcal{O}\subset\mathcal{O}_C$ as in (1.5.2) and then obtain the descended coordinate map (6.3.2) on the generic fiber. In conclusion, the above choices of A, Ψ, and Ξ satisfy the assumptions of §6.3: specifically, due to (5.17.2), the resulting map (6.3.3) is surjective and, by construction, Ξ contains $\{t_{\lambda,1},\dots,t_{\lambda,r_\lambda}\}$. We assume that Σ is large enough, so that so is Ψ and the entire §6.3 applies.

By [BMS16, 13.3 (ii) (b)], each $D_{\Psi,\Xi,n}(A)$ is a complete Tate ring (in the sense of [Hub93, §1]), whose ring of definition may be taken to be the image of $(A_{\mathrm{inf}}/\xi^n)\left\langle(X_u^{\pm1})_{u\in\Sigma},(X_a)_{a\in\Xi}\right\rangle$ endowed with its p-adic topology, and, by construction, $D_{\Psi,\Xi,n}(A)$ is a nilpotent thickening of $D_{\Psi,\Xi,1}(A)\cong A$. For each $\lambda\in\Lambda$, the relation "$|[(p^{1/p^\infty})^{q_\lambda}]|\le|X_{t_{\lambda,1}}\cdots X_{t_{\lambda,r_\lambda}}|$" holds in A, and hence also in every $D_{\Psi,\Xi,n}(A)$, so $D_{\Psi,\Xi}(A)$ is naturally an algebra over the ring $A(R^\square_{\Sigma,\Lambda})$ defined in (5.22.1). In fact, since each $D_{\Psi,\Xi,n}(A)$ is a $\mathbb{Q}$-algebra in which ξ is nilpotent and each X_a is a unit in $D_{\Psi,\Xi}(A)$, the universal relations (5.26.3) and (5.27.3) imply that $D_{\Psi,\Xi}(A)$ is naturally an algebra even over every

$$(A(R^\square_{\Sigma,\Lambda})\otimes_{A_{\mathrm{inf}}}A^0_{\mathrm{cris}})\otimes_{\mathbb{Z}[Q]}\mathbb{Z}[P_{\lambda_0}]\qquad\text{for }\lambda_0\in\Lambda,\tag{6.5.3}$$

compatibly with the isomorphisms (5.26.7). Moreover, the elements $\frac{\xi^m}{m!}$ with $m\ge n$ vanish in $D_{\Psi,\Xi,n}(A)$, so the algebra structure map factors through some (necessarily p-adically complete) ring of definition $(D_{\Psi,\Xi,n}(A))_0$ (see [Hub93, 1.3 and 1.5]):

$$(A(R^\square_{\Sigma,\Lambda})\otimes_{A_{\mathrm{inf}}}A^0_{\mathrm{cris}})\otimes_{\mathbb{Z}[Q]}\mathbb{Z}[P_{\lambda_0}]\to(D_{\Psi,\Xi,n}(A))_0\hookrightarrow(D_{\Psi,\Xi,n}(A))^\circ\hookrightarrow D_{\Psi,\Xi,n}(A).$$

The maps (the first of which was described in (5.26.4) and (5.27.4))

$$(A(R^\square_{\Sigma,\Lambda})\otimes_{A_{\mathrm{inf}}}A^0_{\mathrm{cris}})\otimes_{\mathbb{Z}[Q]}\mathbb{Z}[P_{\lambda_0}]\twoheadrightarrow R\qquad\text{and}\qquad(D_{\Psi,\Xi,n}(A))^\circ\to A^\circ\cong R\tag{6.5.4}$$

are compatible, so the map $(D_{\Psi,\Xi,n}(A))_0\to R$ is surjective. In addition, by [SP, 07GM], the kernel of the map $(D_{\Psi,\Xi,n}(A))^\circ\twoheadrightarrow R/p$ has a unique divided power structure, so we obtain a map

$$D_{j_{\lambda_0}}\to(D_{\Psi,\Xi,n}(A))^\circ\tag{6.5.5}$$

from the divided power envelope $D_{j_{\lambda_0}}$ defined in §5.28. Modulo the ideal generated by the $\frac{\xi^m}{m!}$ with $m\ge n$ for a fixed n, the kernel of the first surjection in (6.5.4) is finitely generated, so, since the $(A(R^\square_{\Sigma,\Lambda})\otimes_{A_{\mathrm{inf}}}A^0_{\mathrm{cris}})\otimes_{\mathbb{Z}[Q]}\mathbb{Z}[P_{\lambda_0}])$-algebra $D_{j_{\lambda_0}}$ is generated by the divided powers of the elements in this kernel, after enlarging $(D_{\Psi,\Xi,n}(A))_0$ we may assume that the map (6.5.5) factors as follows:

$$D_{j_{\lambda_0}}\to(D_{\Psi,\Xi,n}(A))_0\hookrightarrow(D_{\Psi,\Xi,n}(A))^\circ\hookrightarrow D_{\Psi,\Xi,n}(A),\tag{6.5.6}$$

and hence induces a continuous map

$$\widehat{D_{j_{\lambda_0}}}\cong D_{\Sigma,\Lambda}\to(D_{\Psi,\Xi,n}(A))_0\hookrightarrow D_{\Psi,\Xi,n}(A),\tag{6.5.7}$$

which, by construction, does not depend on the choice of λ_0. These maps are compatible as n varies, so by passing to the limit in n we get compatible continuous maps (where $D_{j_{\lambda_0}}$ is discrete)

$$D_{j_{\lambda_0}}\to D_{\Psi,\Xi}(A)\qquad\text{and}\qquad D_{\Sigma,\Lambda}\to D_{\Psi,\Xi}(A).\tag{6.5.8}$$

By construction, the derivations $\frac{\partial}{\partial\log(X_\sigma)}$ for $\sigma\in\Sigma$ and $\frac{\partial}{\partial\log(X_{\lambda,i})}$ for $\lambda\in\Lambda$ and $1\le i\le d$ of $D_{j_{\lambda_0}}$ are compatible with their corresponding derivations of $D_{\Psi,\Xi}(A)$ (see (6.5.2) and §6.3). Therefore, since all the derivations in question are continuous and $D_{j_{\lambda_0}}$ is dense in $D_{\Sigma,\Lambda}$, the map $D_{\Sigma,\Lambda} \to D_{\Psi,\Xi}(A)$ is also compatible with derivations, to the effect that we obtain a map of complexes

$$K_{D_{\Sigma,\Lambda}}\left(\left(\tfrac{\partial}{\partial\log(X_\sigma)}\right)_{\sigma\in\Sigma}, \left(\tfrac{\partial}{\partial\log(X_{\lambda,i})}\right)_{\lambda\in\Lambda,\,1\le i\le d}\right) \to K_{D_{\Psi,\Xi}(A)}\left((\tfrac{\partial}{\partial\log(X_a)})_{a\in\Psi\cup\Xi}\right). \tag{6.5.9}$$

These maps are compatible with enlarging Σ and Λ (and correspondingly enlarging Ψ and Ξ), so we obtain a map of complexes

$$\varinjlim_{\Sigma,\Lambda}\left(K_{D_{\Sigma,\Lambda}}\left(\left(\tfrac{\partial}{\partial\log(X_\sigma)}\right)_{\sigma\in\Sigma}, \left(\tfrac{\partial}{\partial\log(X_{\lambda,i})}\right)_{\lambda\in\Lambda,\,1\le i\le d}\right)\right) \to \varinjlim_{\Psi,\Xi}\left(\Omega^\bullet_{D_{\Psi,\Xi}(R[\frac{1}{p}])/B^+_{\mathrm{dR}}}\right) \tag{6.5.10}$$

whose formation is compatible with varying R. Due to (5.23.3) and (6.3.8), after applying $R\Gamma(\mathfrak{X}_{\text{ét}}, -)$, the sheafification of the resulting map of complexes of presheaves gives the desired map (6.5.1).

In addition, since the map $D_{\Sigma,\Lambda} \to D_{\Psi,\Xi}(R[\frac{1}{p}])$ is compatible with the maps of both sides to $R[\frac{1}{p}]$ (see Lemma 5.29), the proof of [BMS16, 13.6] implies that the map (6.5.10) is compatible with the maps (in the derived category) of both sides of (6.5.10) to $\Omega^{\bullet,\,\mathrm{cont}}_{R[\frac{1}{p}]/C}$ (see (5.32.2) and (6.3.7)). In conclusion, the map (6.5.1) fits into the commutative diagram:

$$\begin{tikzcd} R\Gamma_{\log\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) \arrow[r, "(6.5.1)"] \arrow[d, "(5.23.2)"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_C^{\mathrm{ad}}/B^+_{\mathrm{dR}}) \arrow[d, "(6.2.8)"] \\ R\Gamma_{\log\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C) \arrow[r] & R\Gamma_{\mathrm{dR}}(\mathfrak{X}_C^{\mathrm{ad}}/C). \end{tikzcd} \tag{6.5.11}$$

Having constructed the map (6.5.1), we are ready for the following generalization of [BMS16, 13.11].

Theorem 6.6. *If $\mathfrak{X}$ is quasi-compact and quasi-separated, then (6.5.1) induces an identification*

$$R\Gamma_{\log\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}})\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{cris}}}B^+_{\mathrm{dR}} \cong R\Gamma_{\mathrm{cris}}(\mathfrak{X}_C^{\mathrm{ad}}/B^+_{\mathrm{dR}}), \tag{6.6.1}$$

where $-\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{cris}}}B^+_{\mathrm{dR}} := R\lim_n(-\otimes^{\mathbb{L}}_{A_{\mathrm{cris}}}(B^+_{\mathrm{dR}}/\xi^n))$.

Proof. Since both sides of (6.6) are derived ξ-adically complete (see (6.2.7)) and (6.5.1) induces a map between them, it suffices to show that this map is an isomorphism modulo ξ. However, modulo ξ both sides of (6.6.1) are identified with $R\Gamma_{\mathrm{dR}}(\mathfrak{X}_C^{\mathrm{ad}}/C)$ (see (5.24.1) and (6.2.8)), so the claim follows from the commutativity of the diagram (6.5.11). ☐

Corollary 6.7. *If $\mathfrak{X}$ is $\mathcal{O}_C$-proper, then we have the identification*

$$R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}})\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}B^+_{\mathrm{dR}} \cong R\Gamma_{\mathrm{cris}}(\mathfrak{X}_C^{\mathrm{ad}}/B^+_{\mathrm{dR}}) \tag{6.7.1}$$

that is compatible with the identifications given by (4.17.1) and (6.2.8) of the reductions modulo ξ of both sides with $R\Gamma_{\mathrm{dR}}(\mathfrak{X}_C^{\mathrm{ad}}/C)$; in particular, then the cohomology groups of $R\Gamma_{\mathrm{cris}}(\mathfrak{X}_C^{\mathrm{ad}}/B^+_{\mathrm{dR}})$ are finite free B^+_{dR}-modules.

Proof. A combination of (5.43.2) and (6.6.1) gives the identification. The asserted compatibility of the reductions modulo ξ follows from Proposition 5.41 and the commutativity of the diagram (6.5.11). By Corollary 5.43, each $H^j(R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}})\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}A_{\mathrm{cris}}[\frac{1}{p}])$ is a finite free $A_{\mathrm{cris}}[\frac{1}{p}]$-module, so

$$H^j(R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}})\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}B^+_{\mathrm{dR}}) \cong H^j(R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}})\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}A_{\mathrm{cris}}[\tfrac{1}{p}])\otimes_{A_{\mathrm{cris}}[\frac{1}{p}]}B^+_{\mathrm{dR}},$$

to the effect that also each $H^j(R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}})\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}B^+_{\mathrm{dR}})$ is a finite free B^+_{dR}-module. ☐

6.8. The B_{dR}^+-cohomology and the étale cohomology. For any proper and smooth adic space X over C, in [BMS16, 13.1] Bhatt–Morrow–Scholze proved the following identification:

$$R\Gamma_{\mathrm{cris}}(X/B_{\mathrm{dR}}^+) \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}} \cong R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}. \tag{6.8.1}$$

Due to (6.2.10), when $X \cong X_0 \widehat{\otimes}_K C$ for a proper, smooth adic space X_0 defined over a complete discretely valued subfield $K \subset C$ that has a perfect residue field, (6.8.1) supplies the "de Rham comparison isomorphism"

$$R\Gamma_{\text{ét}}(X_0 \widehat{\otimes}_K C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}} \cong R\Gamma_{\mathrm{dR}}(X_0/K) \otimes_K B_{\mathrm{dR}}. \tag{6.8.2}$$

If $C \cong \widehat{\overline{K}}$, then, by transport of structure, the identification (6.8.2) is $\mathrm{Gal}(\overline{K}/K)$-equivariant (by functoriality, $\mathrm{Gal}(\overline{K}/K)$ acts nontrivially on B_{dR} and $R\Gamma_{\text{ét}}(X_0 \widehat{\otimes}_K C, \mathbb{Z}_p)$) and, by *loc. cit.*, it recovers the de Rham comparison isomorphism constructed in [Sch13, 8.4]. In particular, in this case (6.8.2) is compatible with filtrations, where B_{dR} is filtered by its discrete valuation and $R\Gamma_{\mathrm{dR}}(X_0/K)$ (resp., $R\Gamma_{\text{ét}}(X_0 \widehat{\otimes}_K C, \mathbb{Z}_p)$) is equipped with the the Hodge (resp., trivial) filtration.

For proper $\mathfrak{X}$, we now have two ways to identify $R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} B_{\mathrm{dR}}$ with $R\Gamma_{\text{ét}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} B_{\mathrm{dR}}$: we can either base change (2.3.1) to B_{dR} or combine (6.7.1) and (6.8.1). We now prove that the two ways give the same identification; this will be important in the proof of Theorem 8.7 below.

Proposition 6.9. *If $\mathfrak{X}$ is $\mathcal{O}_C$-proper, then the map $R\Gamma_{\mathrm{cris}}(\mathfrak{X}_C^{\mathrm{ad}}/B_{\mathrm{dR}}^+) \to R\Gamma(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}^+$ of* [BMS16, proof of 13.1] *that underlies the identification* (6.8.1) *for $X = \mathfrak{X}_C^{\mathrm{ad}}$ makes the diagram*

$$\begin{tikzcd}
R\Gamma_{\log\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) \arrow[rr, "(6.5.1)"] \arrow[d, "(5.40.1)"] & & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_C^{\mathrm{ad}}/B_{\mathrm{dR}}^+) \arrow[d] \\
R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}} \arrow[r, "{[BMS16,\ 6.10]}"] & R\Gamma_{\text{ét}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{A}_{\mathrm{inf},\, \mathfrak{X}_C^{\mathrm{ad}}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} B_{\mathrm{dR}}^+ & R\Gamma(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} B_{\mathrm{dR}}^+ . \arrow[l, "(2.3.2)"', "\sim"]
\end{tikzcd}$$

commute; in particular, the identification of $R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} B_{\mathrm{dR}}$ with $R\Gamma_{\text{ét}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} B_{\mathrm{dR}}$ that results from (2.3.1) (*and is encoded by the bottom part of the above diagram*) *agrees with the identification that results from* (6.7.1) *and* (6.8.1) (*and is encoded by the top part of the diagram*).

Proof. Since $\varphi^{-1}(\mu)$ lies in $W(\mathfrak{m}^\flat)$ and is a unit in B_{dR}^+, the discussion after Theorem 2.3 implies that the map labeled "(2.3.2)" in the diagram is an isomorphism. We will now review the definition given in [BMS16, proof of 13.1] of the composition f of the right vertical map with this map "(2.3.2)."

Let $\mathrm{Spa}(A, A^\circ)$ be an element of the basis for the Zariski topology of $\mathfrak{X}_C^{\mathrm{ad}}$ discussed in §6.2. For a large enough set Ψ as in §6.2, consider the surjection $C\left\langle (X_u^{\pm 1})_{u\in\Psi}\right\rangle \xrightarrow{X_u \mapsto u} A$ from (6.2.2), as well as the perfectoid $(\prod_\Psi \mathbb{Z}_p(1))$-cover $C\left\langle (X_u^{\pm 1/p^\infty})_{u\in\Psi}\right\rangle$ of $C\langle (X_u^{\pm 1})_{u\in\Psi}\rangle$. The base change of this cover to $\mathrm{Spa}(A, A^\circ)$ is a perfectoid $(\prod_\Psi \mathbb{Z}_p(1))$-cover

$$\mathrm{Spa}(A_{\Psi,\infty}, A_{\Psi,\infty}^+) \to \mathrm{Spa}(A, A^\circ). \tag{6.9.1}$$

By applying the definition given in Proposition 5.36 to the perfectoid ring $A_{\Psi,\infty}^+$, we obtain the B_{dR}^+-algebra $\mathbb{B}_{\mathrm{dR}}^+(A_{\Psi,\infty}^+)$ that may be viewed as a pro-(infinitesimal thickening) of $A_{\Psi,\infty}$. By construction, each $u \in \Psi$ has a canonical system u^{1/p^∞} of p-power roots in $A_{\Psi,\infty}^+$, which gives rise to the element $[u^{1/p^\infty}] \in \mathbb{B}_{\mathrm{dR}}^+(A_{\Psi,\infty}^+)$. The assignment $X_u \mapsto [u^{1/p^\infty}]$ extends to a B_{dR}^+-algebra morphism

$$D_\Psi(A) \to \mathbb{B}_{\mathrm{dR}}^+(A_{\Psi,\infty}^+) \tag{6.9.2}$$

that is compatible with the map $A \to A_{\Psi,\infty}$ and, for each $u \in \Psi$, intertwines $\exp(\log([\epsilon]) \cdot \frac{\partial}{\partial \log(X_u)})$ defined by the formula (5.15.1) and viewed as an endomorphism of $D_\Psi(A)$ with the action of the generator $[\epsilon]$ of the u^{th} copy of $\mathbb{Z}_p(1)$ on $\mathbb{B}^+_{\mathrm{dR}}(A^+_{\Psi,\infty})$. In particular, letting γ_u denote this generator, one may use the same formula as in (5.16.1) to define a morphism of complexes

$$\Omega^\bullet_{D_\Psi(A)/B^+_{\mathrm{dR}}} = K_{D_\Psi(A)}\left((\tfrac{\partial}{\partial \log(X_u)})_{u\in\Psi}\right) \to K_{\mathbb{B}^+_{\mathrm{dR}}(A^+_{\Psi,\infty})}((\gamma_u - 1)_{u\in\Psi}), \tag{6.9.3}$$

whose formation is functorial in $\mathrm{Spa}(A, A^\circ)$. The almost purity theorem identifies the cohomology of the sheaf of complexes determined by the target of (6.9.3) with $R\Gamma_{\text{ét}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{A}_{\mathrm{inf},\mathfrak{X}^{\mathrm{ad}}_C}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} B^+_{\mathrm{dR}}$. On the other hand, by definition, the cohomology of the sheaf of complexes determined by the source of (6.9.3) is $R\Gamma_{\mathrm{cris}}(\mathfrak{X}^{\mathrm{ad}}_C/B^+_{\mathrm{dR}})$. Thus, by sheafifying and forming cohomology, the maps (6.9.3) produce the aforementioned composition f defined in *op. cit.*

We may carry out the construction of the morphisms (6.9.3) using the étale topology of $\mathfrak{X}^{\mathrm{ad}}_C$ instead of Zariski. Due to (6.2.9), this leads to the same map f. We may also generalize the construction of (6.9.3) further by using both the étale topology of $\mathfrak{X}^{\mathrm{ad}}_C$ and the more general embeddings (6.3.3) described in §6.3: in this case, the cover (6.9.1) is replaced by the cover

$$\mathrm{Spa}(A_{\Psi,\Xi,\infty}, A^+_{\Psi,\Xi,\infty}) \to \mathrm{Spa}(A, A^\circ). \tag{6.9.4}$$

that is the base change of the perfectoid $(\prod_\Psi \mathbb{Z}_p(1) \times \prod_\Xi \mathbb{Z}_p(1))$-cover $C\langle (X_u^{\pm 1/p^\infty})_{u\in\Psi}, (X_a^{1/p^\infty})_{a\in\Xi}\rangle$ of $C\langle (X_u^{\pm 1})_{u\in\Psi}, (X_a)_{a\in\Xi}\rangle$, and the rest of the construction remains the same. Due to (6.3.8) and (6.3.9), this again gives the same map f.

In conclusion, since the construction of f may be carried out using the more general embeddings described in §6.3 and follows the same pattern as the construction of the map (5.40.1) (namely, is based on the map as in (5.16.1)), all we need to check is that, in the notation of §6.5, the following diagram commutes:

$$\begin{tikzcd} D_{\Sigma,\Lambda} \arrow[r, "(6.5.8)"] \arrow[d, "(5.38.1)"'] & D_{\Psi,\Xi}(A) \arrow[d, "(6.9.2)"] \\ \mathbb{A}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \arrow[r, "5.36"] & \mathbb{B}^+_{\mathrm{dR}}(R_{\Sigma,\Lambda,\infty}), \end{tikzcd} \tag{6.9.5}$$

where we have used the agreement $R_{\Sigma,\Lambda,\infty} \cong A_{\Psi,\Xi,\infty}$ that results from the choices in (6.5.2). For this desired commutativity, we may first replace $\mathbb{B}^+_{\mathrm{dR}}(R_{\Sigma,\Lambda,\infty})$ by $\mathbb{B}^+_{\mathrm{dR}}(R_{\Sigma,\Lambda,\infty})/\xi^n$ for a variable $n > 0$, then replace $D_{\Sigma,\Lambda}$ by $D_{j_{\lambda_0}}$ for some $\lambda_0 \in \Lambda$, and, finally, since $\mathbb{B}^+_{\mathrm{dR}}(R_{\Sigma,\Lambda,\infty})/\xi^n$ is a $\mathbb{Q}$-algebra and $D_{j_{\lambda_0}}$ is generated by divided powers, replace $D_{j_{\lambda_0}}$ by $(A(R^\square_{\Sigma,\Lambda}) \otimes_{A_{\mathrm{inf}}} A^0_{\mathrm{cris}}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]$. However, each X_τ from (5.19.2) with either $\tau = \sigma$ for some $\sigma \in \Sigma$ or $\tau = (\lambda, i)$ for some $\lambda \in \Lambda$ and $1 \le i \le d$ maps to the (necessarily invertible) Teichmüller element $[X_\tau^{1/p^\infty}]$ in $\mathbb{B}^+_{\mathrm{dR}}(R_{\Sigma,\Lambda,\infty})$ under either of the two maps from $(A(R^\square_{\Sigma,\Lambda}) \otimes_{A_{\mathrm{inf}}} A^0_{\mathrm{cris}}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]$ to $\mathbb{B}^+_{\mathrm{dR}}(R_{\Sigma,\Lambda,\infty})/\xi^n$ supplied by the diagram (6.9.5), so these two maps indeed agree, as desired. $\square$

7. The A_{inf}-cohomology modules $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ and their specializations

We are ready to define the A_{inf}-cohomology groups $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ for a proper $\mathfrak{X}$ and to detail some of their properties. We prove that each $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ is a Breuil–Kisin–Fargues module (see Theorem 7.4) and use this to deduce that, loosely speaking, the p-adic étale cohomology of $\mathfrak{X}^{\mathrm{ad}}_C$ has at most the amount of torsion that is contained in the logarithmic crystalline cohomology of $\mathfrak{X}_k$ or the logarithmic de Rham cohomology of $\mathfrak{X}$ (see Theorems 7.10 and 7.13 for precise statements). Most of these results are variants of their analogues established in the smooth case in [BMS16]. Their proofs, granted the inputs from §§3–6, are generally similar to those of *op. cit.* and in large part rely on commutative algebra results over A_{inf}.

7.1. Properness of $\mathfrak{X}$. In §7 we assume that $\mathfrak{X}$ is proper and $\mathfrak{X}_k$ is purely d-dimensional.

7.2. The A_{inf}-cohomology $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$. We use the object $A\Omega_{\mathfrak{X}} \in D^{\geq 0}(\mathfrak{X}_{\text{ét}}, A_{\mathrm{inf}})$ of §2.2 to set

$$R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) := R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \in D^{\geq 0}(A_{\mathrm{inf}}) \quad \text{and} \quad H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) := H^i(R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}})) \quad \text{for } i \in \mathbb{Z}.$$

Since the functor $L\eta$ commutes with pullback along a flat morphism of topoi (see [BMS16, 6.14]), the object $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ and its cohomology groups $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ are contravariantly functorial in $\mathfrak{X}$: more precisely, any $\mathcal{O}_C$-morphism $\mathfrak{X}' \to \mathfrak{X}$ induces a morphism $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \to R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}')$ in $D^{\geq 0}(A_{\mathrm{inf}})$, and hence, for $i \in \mathbb{Z}$, also the morphism $H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) \to H^i_{A_{\mathrm{inf}}}(\mathfrak{X}')$ of A_{inf}-modules.

By Corollary 4.19, the object $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ is perfect, that is, isomorphic to a bounded complex of finite free A_{inf}-modules. Moreover, by (2.3), (4.17.1), and (5.43.2), we have the following identifications:

$$\begin{aligned} R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}[\tfrac{1}{\mu}] &\cong R\Gamma_{\text{ét}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} A_{\mathrm{inf}}[\tfrac{1}{\mu}]; \\ R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}, \theta} \mathcal{O}_C &\cong R\Gamma_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C); \\ R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} W(k) &\cong R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k)). \end{aligned} \tag{7.2.1}$$

In the case when $\mathfrak{X}$ is $\mathcal{O}_C$-smooth, one may drop "log" from the subscripts (compare with (5.44.5)).

The morphism (2.2.6) gives rise to the morphism

$$R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}, \varphi} A_{\mathrm{inf}} \to R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \qquad \text{in} \qquad D^{\geq 0}(A_{\mathrm{inf}}) \tag{7.2.2}$$

that becomes an isomorphism after inverting $\varphi(\xi)$ (see (2.2.7)), and the last identification in (7.2.1) is in fact Frobenius-equivariant (see (5.43.2)). Consequently the cohomology modules $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ come equipped with the A_{inf}-module morphisms

$$\varphi \colon H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}, \varphi} A_{\mathrm{inf}} \to H^i_{A_{\mathrm{inf}}}(\mathfrak{X}). \tag{7.2.3}$$

that become isomorphisms after inverting $\varphi(\xi)$. We will prove in Theorem 7.4 that these morphisms make each $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ a Breuil–Kisin–Fargues module in the sense of [BMS16, Def. 4.22].

7.3. Breuil–Kisin–Fargues modules. A *Breuil–Kisin–Fargues module* is a finitely presented A_{inf}-module M equipped with an isomorphism

$$\varphi_M \colon (M \otimes_{A_{\mathrm{inf}}, \varphi} A_{\mathrm{inf}})[\tfrac{1}{\varphi(\xi)}] \xrightarrow{\sim} M[\tfrac{1}{\varphi(\xi)}] \tag{7.3.1}$$

of $A_{\mathrm{inf}}[\frac{1}{\varphi(\xi)}]$-modules such that $M[\frac{1}{p}]$ is $A_{\mathrm{inf}}[\frac{1}{p}]$-free. By [BMS16, 4.9 (i)], such an M is perfect as an A_{inf}-module, that is, M has a finite resolution by finite free A_{inf}-modules. A morphism of Breuil–Kisin–Fargues modules is an A_{inf}-module morphism that commutes with the isomorphisms φ_M.

Theorem 7.4. *Each $(H^i_{A_{\mathrm{inf}}}(\mathfrak{X}), \varphi)$ is a Breuil–Kisin–Fargues module and vanishes unless $i \in [0, 2d]$. In particular, each $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ is perfect as an A_{inf}-module and each $(H^i_{A_{\mathrm{inf}}}(\mathfrak{X}))[\frac{1}{p}]$ is $A_{\mathrm{inf}}[\frac{1}{p}]$-free.*

Proof. Due to the relation with $R\Gamma_{\text{ét}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{Z}_p)$, each $(H^i_{A_{\mathrm{inf}}}(\mathfrak{X}))[\frac{1}{p\mu}]$ is a free $A_{\mathrm{inf}}[\frac{1}{p\mu}]$-module. Moreover, by Corollary 5.43, the cohomology groups of $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}[\frac{1}{p}]$ are finite free $A_{\mathrm{cris}}[\frac{1}{p}]$-modules. Therefore, [BMS16, 4.20] applies and proves that each $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ is a finitely presented A_{inf}-module that becomes free upon inverting p, so $(H^i_{A_{\mathrm{inf}}}(\mathfrak{X}), \varphi)$ is a Breuil–Kisin–Fargues module.

Since $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ is perfect, its top degree cohomology of is finitely presented and of formation compatible with base change. Thus, by the de Rham specialization of (7.2.1) and the Nakayama lemma, $H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) = 0$ for $i > 2d$. The same holds for $i < 0$ because $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \in D^{\geq 0}(A_{\mathrm{inf}})$. $\square$

Corollary 7.5. *For each $i \in \mathbb{Z}$, the rank of the finitely presented $\mathbb{Z}_p$-module $H^i_{\text{ét}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{Z}_p)$ is equal to the rank of the finitely presented $W(k)$-module $H^i_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k))$, and is also equal to the rank of the finitely presented $\mathcal{O}_C$-module $H^i_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C) := R^i\Gamma(\mathfrak{X}, \Omega^\bullet_{\mathfrak{X}/\mathcal{O}_C, \log})$ (see also (7.11.1) below).*

Proof. The finite presentation assertions follow, for instance, from the perfectness of $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$, the comparisons (7.2.1), and the coherence of the ring $\mathcal{O}_C$. Due to Theorem 7.4 and the comparisons (7.2.1), all the ranks in question are equal to the rank of the free $A_{\mathrm{inf}}[\frac{1}{p}]$-module $(H^i_{A_{\mathrm{inf}}}(\mathfrak{X}))[\frac{1}{p}]$. $\square$

7.6. Base change for individual $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$. Since $A_{\mathrm{inf}}[\frac{1}{\mu}]$ is A_{inf}-flat, for each $i \in \mathbb{Z}$, (7.2.1) gives:

$$(H^i_{A_{\mathrm{inf}}}(\mathfrak{X}))[\tfrac{1}{\mu}] \cong H^i_{\text{ét}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}[\tfrac{1}{\mu}]. \tag{7.6.1}$$

In particular, since μ is a unit in $W(C^\flat)$ and $W(C^\flat)$ is A_{inf}-flat (the localization of A_{inf} at pA_{inf} is a discrete valuation ring whose completion is $W(C^\flat)$, see [BMS16, proof of Lem. 4.10]),

$$H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}} W(C^\flat) \cong H^i_{\text{ét}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(C^\flat). \tag{7.6.2}$$

Analogous comparison to the logarithmic de Rham cohomology groups is more complex: by (7.2.1) and [SP, 0662], for each $i \in \mathbb{Z}$ we have a short exact sequence

$$0 \to H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}, \theta} \mathcal{O}_C \to H^i_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C) \to (H^{i+1}_{A_{\mathrm{inf}}}(\mathfrak{X}))[\xi] \to 0. \tag{7.6.3}$$

Similarly, by Theorem 7.4 and [BMS16, 4.9], for each $i \in \mathbb{Z}$ we have a Frobenius-equivariant short exact sequence

$$0 \to H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}} W(k) \to H^i_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k)) \to \mathrm{Tor}^1_{A_{\mathrm{inf}}}(H^{i+1}_{A_{\mathrm{inf}}}(\mathfrak{X}), W(k)) \to 0. \tag{7.6.4}$$

In particular, by Theorem 7.4, in addition to (7.6.1), we have

$$H^{2d}_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}, \theta} \mathcal{O}_C \cong H^{2d}_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C) \qquad \text{and} \qquad H^{2d}_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}} W(k) \cong H^{2d}_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k)).$$

For general i, it is most pleasant to deal with such base changes when $H^{i+1}_{A_{\mathrm{inf}}}(\mathfrak{X})$ is A_{inf}-free. For such freeness, we have the following consequence of Theorem 7.4 and [BMS16, §4].

Proposition 7.7. *For each $i \in \mathbb{Z}$, the $\mathcal{O}_C$-module $H^i_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)$ is p-torsion free (equivalently, free) if and only if the $W(k)$-module $H^i_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k))$ is p-torsion free (equivalently, free), in which case $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ is free as an A_{inf}-module and $H^i_{\text{ét}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{Z}_p)$ is free as a $\mathbb{Z}_p$-module.*

Proof. Due to Theorem 7.4, we may apply [BMS16, 4.18] and combine it with (7.2.1) to conclude that $H^i_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)$ is p-torsion free if and only if so is $H^i_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k))$. When these conditions hold, the freeness of $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ and $H^i_{\text{ét}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{Z}_p)$ follows from [BMS16, 4.17] and (7.6.1). The parenthetical assertions follow, for instance, from the following elementary lemma.

Lemma 7.8. *For a local domain $(R, \mathfrak{m})$, a finite R-module M is free if and only if*

$$\dim_{R/\mathfrak{m}R}(M/\mathfrak{m}M) = \dim_{\mathrm{Frac}(R)}(M_{\mathrm{Frac}(R)}). \tag{7.8.1}$$

Proof. By the Nakayama lemma, a lift $m_1, \dots, m_d \in M$ of an $R/\mathfrak{m}R$-basis of $M/\mathfrak{m}M$ generates M, and hence also contains a basis of $M_{\mathrm{Frac}(R)}$. Thus, if (7.8.1) holds, then the m_i can have no R-relation, and hence must define an isomorphism $R^d \simeq M$. The converse is clear. $\square$ $\square$

Remark 7.9. As was observed by Jesse Silliman and Ravi Fernando during the Arizona Winter School 2017, the first assertion of Proposition 7.7 may be strengthened as follows: for each $i \in \mathbb{Z}$,

$$\dim_k \left(H^i_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)_{\mathrm{tors}} \otimes_{\mathcal{O}_C} k\right) = \dim_k \left(H^i_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k))_{\mathrm{tors}} \otimes_{W(k)} k\right), \tag{7.9.1}$$

that is, $H^i_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)$ and $H^i_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k))$ have the same number of cyclic summands (in the sense of (7.11.1) below). Indeed, by Corollary 7.5, the ranks of $H^i_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)$ and $H^i_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k))$ agree and, by [Bei13b, (1.8.1)], so do the k-fibers of $R\Gamma_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)$ and $R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k))$, so the claim follows by descending induction on i from the following exact sequences supplied by [SP, 0662]:

$$0 \to H^i_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C) \otimes_{\mathcal{O}_C} k \to H^i(R\Gamma_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C) \otimes^{\mathbb{L}}_{\mathcal{O}_C} k) \to \mathrm{Tor}_1^{\mathcal{O}_C}(H^{i+1}_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C), k) \to 0,$$
$$0 \to H^i_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k)) \otimes_{W(k)} k \to H^i(R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k)) \otimes^{\mathbb{L}}_{W(k)} k) \to H^{i+1}_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k))[p] \to 0.$$

The following variant of [BMS16, 14.5 (ii)] strengthens the part of Proposition 7.7 that deduces the freeness of $H^i_{\mathrm{\acute{e}t}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{Z}_p)$ from the freeness of $H^i_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k))$.

Theorem 7.10. *For every $i \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$, we have*

$$\begin{aligned} \mathrm{length}_{\mathbb{Z}_p}((H^i_{\mathrm{\acute{e}t}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{Z}_p)_{\mathrm{tors}})/p^n) &\leq \mathrm{length}_{W(k)}((H^i_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k))_{\mathrm{tors}})/p^n), \\ \mathrm{length}_{\mathbb{Z}_p}(H^i_{\mathrm{\acute{e}t}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{Z}/p^n\mathbb{Z})) &\leq \mathrm{length}_{W(k)}(H^i_{\log \mathrm{cris}}(\mathfrak{X}_k/W_n(k))). \end{aligned} \tag{7.10.1}$$

Proof. The proof of the first inequality analogous to the proof of *loc. cit.* Namely, by Corollary 7.5, we may drop the subscripts "tors" and, by Theorem 7.4, [BMS16, 4.15 (ii)], and (7.6.2), we have

$$\mathrm{length}_{\mathbb{Z}_p}(H^i_{\mathrm{\acute{e}t}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{Z}_p)/p^n) \leq \mathrm{length}_{W(k)}(H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}} W(k)/p^n). \tag{7.10.2}$$

Moreover, since $\mathrm{length}_{W(k)}(Q/p^n) = \mathrm{length}_{W(k)}(\mathrm{Tor}_1^{W(k)}(Q, W(k)/p^n))$ for every finite torsion $W(k)$-module Q, the short exact sequence (7.6.4) yields the inequality

$$\mathrm{length}_{W(k)}(H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}} W(k)/p^n) \leq \mathrm{length}_{W(k)}(H^i_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k))/p^n),$$

and the first inequality in (7.10.1) follows. Due to the short exact sequences

$$0 \to H^i_{\mathrm{\acute{e}t}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{Z}_p)/p^n \to H^i_{\mathrm{\acute{e}t}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{Z}/p^n\mathbb{Z}) \to \mathrm{Tor}_1^{\mathbb{Z}_p}(H^{i+1}_{\mathrm{\acute{e}t}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{Z}_p), \mathbb{Z}/p^n\mathbb{Z}) \to 0,$$
$$0 \to H^i_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k))/p^n \to H^i_{\log \mathrm{cris}}(\mathfrak{X}_k/W_n(k)) \to \mathrm{Tor}_1^{W(k)}(H^{i+1}_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k)), W(k)/p^n) \to 0$$

that result from [SP, 0662] and [Bei13b, §1.16, Theorem, (i)], the second inequality in (7.10.1) follows from the first. $\square$

The de Rham analogue of Theorem 7.10 (see Theorem 7.13) uses the following formalism.

7.11. The normalized length. Let $\mathfrak{o}$ be a rank 1 valuation ring of mixed characteristic $(0, p)$ and normalize its valuation $\mathrm{val}_{\mathfrak{o}}$ by requiring that $\mathrm{val}_{\mathfrak{o}}(p) = 1$. By the structure theorem [SP, 0ASP] (see also [GR03, 6.1.14]), every finitely presented $\mathfrak{o}$-module M is of the form

$$M \cong \textstyle\bigoplus_{i=1}^n \mathfrak{o}/(a_i) \qquad \text{for some} \qquad a_i \in \mathfrak{o}. \tag{7.11.1}$$

If M is, in addition, torsion, to the effect that the a_i are nonzero, then we set

$$\mathrm{val}_{\mathfrak{o}}(M) := \textstyle\sum_{i=1}^n \mathrm{val}(a_i).$$

More intrinsically, $\mathrm{val}_{\mathfrak{o}}(M)$ is the valuation of any generator of the 0^{th} Fitting ideal $\mathrm{Fitt}_0(M) \subset \mathfrak{o}$ of M, so it depends only on M. If $\mathfrak{o}$ is a discrete valuation ring for which p is a uniformizer, then

$\mathrm{val}_{\mathfrak{o}}(M) = \mathrm{length}_{\mathfrak{o}}(M)$. In general, $\mathrm{val}_{\mathfrak{o}}$ has the advantage of being invariant under extension of scalars to a larger $\mathfrak{o}$. Any short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

of finitely presented torsion $\mathfrak{o}$-modules gives rise to the equality $\mathrm{Fitt}_0(M_2) = \mathrm{Fitt}_0(M_1)\,\mathrm{Fitt}_0(M_3)$ (see [GR03, 6.3.1 and 6.3.5 (i)]), so the assignment $\mathrm{val}_{\mathfrak{o}}(-)$ satisfies

$$\mathrm{val}_{\mathfrak{o}}(M_2) = \mathrm{val}_{\mathfrak{o}}(M_1) + \mathrm{val}_{\mathfrak{o}}(M_3). \tag{7.11.2}$$

The following lemma is the de Rham version of [BMS16, 4.14], which gave the inequality (7.10.2).

Lemma 7.12. *For an $n \in \mathbb{Z}_{\geq 1}$ and a finitely presented $W_n(\mathcal{O}_C^\flat)$-module M, we have*

$$\mathrm{val}_{W(C^\flat)}(M \otimes_{A_{\mathrm{inf}}} W(C^\flat)) = \mathrm{val}_{\mathcal{O}_C}(M/\xi M) - \mathrm{val}_{\mathcal{O}_C}(M[\xi]). \tag{7.12.1}$$

Proof. Since $W_n(\mathcal{O}_C^\flat)$ is a coherent ring (see [BMS16, 3.24]), the $W_n(\mathcal{O}_C^\flat)$-module $M[\xi]$ is finitely presented. Moreover, due to (7.11.2), the flatness of $A_{\mathrm{inf}} \to W(C^\flat)$ (see §7.6), and the snake lemma, both sides of (7.12.1) are additive in short exact sequences. Therefore, we may assume that $n = 1$ and, due to the structure theorem [SP, 0ASP], that $M = \mathcal{O}_C^\flat/(x)$ for some $x \in \mathcal{O}_C^\flat$.

If $x = 0$, then both sides of (7.12.1) are equal to 1. If $x \neq 0$, then the left side vanishes, and so does the right side because $M[\xi] \cong \mathrm{Tor}^1_{\mathcal{O}_C^\flat}(M, \mathcal{O}_C/p)$ and the following sequence is exact:

$$0 \to \mathrm{Tor}^1_{\mathcal{O}_C^\flat}(\mathcal{O}_C^\flat/(x), \mathcal{O}_C/p) \to \mathcal{O}_C/p \xrightarrow{\theta([x])} \mathcal{O}_C/p \to M/\xi M \to 0. \qquad \square$$

Theorem 7.13. *For every $i \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$, we have (recall from §7.11 that $\mathrm{val}_{\mathbb{Z}_p} = \mathrm{length}_{\mathbb{Z}_p}$)*

$$\begin{aligned} \mathrm{val}_{\mathbb{Z}_p}((H^i_{\mathrm{\acute{e}t}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{Z}_p)_{\mathrm{tors}})/p^n) &\leq \mathrm{val}_{\mathcal{O}_C}((H^i_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)_{\mathrm{tors}})/p^n), \\ \mathrm{val}_{\mathbb{Z}_p}(H^i_{\mathrm{\acute{e}t}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{Z}/p^n\mathbb{Z})) &\leq \mathrm{val}_{\mathcal{O}_C}(R^i\Gamma_{\log \mathrm{dR}}(\mathfrak{X}_{\mathcal{O}_C/p^n}, \Omega^\bullet_{\mathfrak{X}_{\mathcal{O}_C/p^n}/(\mathcal{O}_C/p^n), \log})). \end{aligned} \tag{7.13.1}$$

Proof. The proof is analogous to that of Theorem 7.10. Namely, we may drop the subscripts "tors" and, by Theorem 7.4, (7.6.2), and Lemma 7.12, have

$$\mathrm{val}_{\mathbb{Z}_p}(H^i_{\mathrm{\acute{e}t}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{Z}_p)/p^n) \leq \mathrm{val}_{\mathcal{O}_C}(H^i_{A_{\mathrm{inf}}}(\mathfrak{X})/(\xi, p^n)).$$

The presentation (7.11.1) implies that $\mathrm{val}_{\mathcal{O}_C}(Q/p^n) = \mathrm{val}_{\mathcal{O}_C}(\mathrm{Tor}_1^{\mathcal{O}_C}(Q, \mathcal{O}_C/p^n))$ for every finitely presented torsion $\mathcal{O}_C$-module Q, so the short exact sequence (7.6.3) gives the inequality

$$\mathrm{val}_{\mathcal{O}_C}(H^i_{A_{\mathrm{inf}}}(\mathfrak{X})/(\xi, p^n)) \leq \mathrm{val}_{\mathcal{O}_C}(H^i_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)/p^n).$$

This proves the first inequality in (7.13.1) and, analogously to the proof of Theorem 7.10, the second inequality in (7.13.1) follows from the first. $\square$

The results above, specifically, (7.9.1) and Theorems 7.10 and 7.13 prompt the following question.

Question 7.14. *Are there examples of $\mathfrak{X}$ as above for which*

$$\mathrm{val}_{\mathcal{O}_C}(H^i_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)) \neq \mathrm{val}_{W(k)}(H^i_{\log \mathrm{cris}}(\mathfrak{X}/W(k)))?$$

For a proper smooth scheme X over a complete discretely valued extension K of $\mathbb{Q}_p$ with a perfect residue field, we explain in Example 8.6 how to functorially associate an $\mathcal{O}_K$-lattice

$$L^i_{\mathrm{dR}}(X) \subset H^i_{\mathrm{dR}}(X/K) \qquad \text{for every} \qquad i \in \mathbb{Z}.$$

In fact, $L^i_{\mathrm{dR}}(X)$ depends (functorially) only on $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)$ and its construction, which in the form given below is based on the theory of Breuil–Kisin–Fargues modules, proceeds along familiar lines of integral p-adic Hodge theory (compare, for instance, with [Liu17, §4]). Using the work of the preceding sections, for suitable X we interpret $L^i_{\mathrm{dR}}(X)$ geometrically: we prove in Theorem 8.7 that if X has a proper, flat, semistable $\mathcal{O}_K$-model $\mathcal{X}$ for which $H^i_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ and $H^{i+1}_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ are p-torsion free, then

$$L^i_{\mathrm{dR}}(X) = H^i_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}_K) \qquad \text{inside} \qquad H^i_{\mathrm{dR}}(X/K).$$

We do not know whether this equality continues to hold "modulo torsion" if $H^i_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ and $H^{i+1}_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ are not assumed to be torsion free.

8.1. The base field K. Throughout §8, we assume that $C \cong \widehat{\overline{K}}$ for some fixed complete discretely valued field K that is of mixed characteristic $(0, p)$ and has a perfect residue field k_0. We set

$$G := \mathrm{Gal}(\overline{K}/K),$$

so that G acts continuously on C and hence also on A_{inf}. The continuous maps φ and θ are G-equivariant, and the ideals (ξ), $(\varphi(\xi))$, and (μ) of A_{inf} are G-stable (see §2.1).

Consequently, if $\mathcal{X}$ is a proper p-adic formal $\mathcal{O}_K$-scheme for which $\mathfrak{X} := \mathcal{X} \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_C$ satisfies the assumptions of §1.5, then the functoriality of $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ (see §7.2) induces a semilinear G-action on the finite A_{inf}-modules $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$.

8.2. The Fargues equivalence. By [BMS16, 4.26], for any Breuil–Kisin–Fargues module (M, φ_M) (see §7.3), its *étale realization*, namely,

$$M_{\text{ét}} := (M \otimes_{A_{\mathrm{inf}}} W(C^\flat))^{\varphi_M \otimes \varphi = 1}, \tag{8.2.1}$$

is a finitely generated $\mathbb{Z}_p$-module and comes equipped with an identification

$$M \otimes_{A_{\mathrm{inf}}} W(C^\flat) \cong M_{\text{ét}} \otimes_{\mathbb{Z}_p} W(C^\flat) \qquad \text{under which} \qquad M \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}[\tfrac{1}{\mu}] \cong M_{\text{ét}} \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}[\tfrac{1}{\mu}].$$

In particular, $M_{\text{ét}}$ is $\mathbb{Z}_p$-free if M is A_{inf}-free, and, for any M, we also have

$$M \otimes_{A_{\mathrm{inf}}} B_{\mathrm{dR}} \cong M_{\text{ét}} \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}, \tag{8.2.2}$$

so that $M_{\text{ét}}$ comes equipped with a B^+_{dR}-sublattice (recall that $M[\frac{1}{p}]$ is $A_{\mathrm{inf}}[\frac{1}{p}]$-free, see §7.3)

$$M \otimes_{A_{\mathrm{inf}}} B^+_{\mathrm{dR}} \subset M_{\text{ét}} \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}.$$

By a theorem of Fargues [BMS16, 4.28, 4.29], the functor

$$(M, \varphi_M) \mapsto (M_{\text{ét}}, M \otimes_{A_{\mathrm{inf}}} B^+_{\mathrm{dR}}) \tag{8.2.3}$$

from the category of Breuil–Kisin–Fargues modules for which M is A_{inf}-free to that of pairs (T, Ξ) consisting of a finite free $\mathbb{Z}_p$-module T and a B^+_{dR}-lattice $\Xi \subset T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$ is an equivalence.

8.3. Breuil–Kisin–Fargues G-modules. Granted the origin of our C, it is natural to consider *Breuil–Kisin–Fargues G-modules*, that is, Breuil–Kisin–Fargues modules (M, φ_M) equipped with an A_{inf}-semilinear G-action on M for which φ_M is G-equivariant. For example, for $\mathcal{X}$ as in §8.1, each

$H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ is naturally a Breuil–Kisin–Fargues G-module (see Theorem 7.4). The étale realization $M_{\text{ét}}$ of a Breuil–Kisin–Fargues G-module (M, φ_M) carries an induced $\mathbb{Z}_p$-linear G-action.

Proposition 8.4. *The category of those Breuil–Kisin–Fargues G-modules (M, φ_M) for which M is A_{inf}-free is equivalent via the functor*

$$(M, \varphi_M) \mapsto (M_{\text{ét}}, M \otimes_{A_{\mathrm{inf}}} B^+_{\mathrm{dR}})$$

to the category of pairs (T, Ξ) consisting of a finite free $\mathbb{Z}_p$-module T equipped with a G-action and a G-stable B^+_{dR}-lattice $\Xi \subset T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$.

Proof. The claim follows immediately from the Fargues equivalence reviewed in §8.2. □

8.5. An étale lattice determines a de Rham lattice. Let T be a finite free $\mathbb{Z}_p$-module endowed with a continuous action of G for which the G-representation $T[\frac{1}{p}]$ is de Rham, so that there is a G-equivariant identification

$$T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}} \cong D_{\mathrm{dR}}(T) \otimes_K B_{\mathrm{dR}} \qquad \text{where} \qquad D_{\mathrm{dR}}(T) := (T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}})^G.$$

For such T, the B^+_{dR}-lattice $D_{\mathrm{dR}}(T) \otimes_K B^+_{\mathrm{dR}}$ is evidently G-stable in $T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$. Therefore, by Proposition 8.4, the pair $(T, D_{\mathrm{dR}}(T) \otimes_K B^+_{\mathrm{dR}})$, so, effectively, T, determines an A_{inf}-free Breuil–Kisin–Fargues G-module

$$(M(T), \varphi_{M(T)})$$

that depends functorially on T and is determined up to a unique isomorphism by the G-equivariant identification $M(T)_{\text{ét}} \cong T$. The de Rham realization

$$M(T)_{\mathrm{dR}} := M(T) \otimes_{A_{\mathrm{inf}}, \theta} \mathcal{O}_C \qquad \text{of} \qquad (M(T), \varphi_{M(T)})$$

is an $\mathcal{O}_C$-lattice in

$$(M(T) \otimes_{A_{\mathrm{inf}}} B^+_{\mathrm{dR}})/\xi \cong (D_{\mathrm{dR}}(T) \otimes_K B^+_{\mathrm{dR}})/\xi \cong D_{\mathrm{dR}}(T) \otimes_K C.$$

Therefore, we obtain the following $\mathcal{O}_K$-lattice that is functorial in T:

$$(M(T)_{\mathrm{dR}})^G \qquad \text{inside the } K\text{-vector space} \qquad D_{\mathrm{dR}}(T).$$

Example 8.6. We fix a K-scheme (or even a K-rigid space[19]) X that is proper and smooth, and set

$$L^i_{\text{ét}}(X) := H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)/H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)_{\mathrm{tors}} \cong H^i_{\text{ét}}(X_C, \mathbb{Z}_p)/H^i_{\text{ét}}(X_C, \mathbb{Z}_p)_{\mathrm{tors}} \qquad \text{for} \qquad i \geq 0.$$

As is well known and follows from (6.8.2), the G-representation $L^i_{\text{ét}}(X)[\frac{1}{p}]$ is de Rham and

$$D_{\mathrm{dR}}(L^i_{\text{ét}}(X)) \cong (L^i_{\text{ét}}(X) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}})^G \overset{(6.8.2)}{\cong} (H^i_{\mathrm{dR}}(X/K) \otimes_K B_{\mathrm{dR}})^G \cong H^i_{\mathrm{dR}}(X/K). \tag{8.6.1}$$

Thus,

$$L^i_{\mathrm{dR}}(X) := (M(L^i_{\text{ét}}(X))_{\mathrm{dR}})^G \subset H^i_{\mathrm{dR}}(X/K)$$

is an $\mathcal{O}_K$-lattice that is functorial in X. Its definition implies that for a finite Galois extension K'/K,

$$L^i_{\mathrm{dR}}(X) = (L^i_{\mathrm{dR}}(X_{K'}))^{\mathrm{Gal}(K'/K)} \qquad \text{inside} \qquad H^i_{\mathrm{dR}}(X/K) = (H^i_{\mathrm{dR}}(X_{K'}/K'))^{\mathrm{Gal}(K'/K)}.$$

Due to GAGA techniques (see Remark 4.18 and the proof of Claim 4.14.2) and the discussion in §§1.5–1.6, the following result implies that if X extends to a proper, flat, semistable $\mathcal{O}_K$-scheme $\mathcal{X}$ such that $H^i_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ and $H^{i+1}_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ have no nonzero p-torsion (where we endow $\mathcal{X}$ with the log structure $\mathcal{O}_{\mathcal{X}} \cap (\mathcal{O}_{\mathcal{X}}[\frac{1}{p}])^\times$), then

$$L^i_{\mathrm{dR}}(X) = H^i_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}_K) \qquad \text{inside} \qquad H^i_{\mathrm{dR}}(X/K); \tag{8.6.2}$$

[19] Which we view as an adic space, see [Hub96, 1.1.11 (d)].

in particular, if $\mathcal{X}'$ is another such model of X, then

$$H^i_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}_K) = H^i_{\log \mathrm{dR}}(\mathcal{X}'/\mathcal{O}_K) \qquad \text{inside} \qquad H^i_{\mathrm{dR}}(X/K). \tag{8.6.3}$$

Theorem 8.7. *Let $\mathcal{X}$ be a proper, flat p-adic formal $\mathcal{O}_K$-scheme endowed with the log structure $\mathcal{O}_\mathcal{X} \cap (\mathcal{O}_\mathcal{X}[\frac{1}{p}])^\times$ such that $\mathcal{X}$ has an étale cover by affines $\mathcal{U}$ each of which has an étale morphism*

$$\mathcal{U} \to \mathrm{Spf}(\mathcal{O}_K\{t_0, \dots, t_r, t_{r+1}, \dots, t_d\}/(t_0 \cdots t_r - \pi)) \quad \textit{for some nonunit} \quad \pi \in \mathcal{O}_K \setminus \{0\} \tag{8.7.1}$$

(where π, r, and d may depend on $\mathcal{U}$). If $H^i_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ and $H^{i+1}_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ are p-torsion free, then

$$L^i_{\mathrm{dR}}(\mathcal{X}^{\mathrm{ad}}_K) = H^i_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}_K) \qquad \textit{inside} \qquad H^i_{\mathrm{dR}}(\mathcal{X}^{\mathrm{ad}}_K/K); \tag{8.7.2}$$

in fact, then, setting $\mathfrak{X} := \mathcal{X}\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_C$, we have the identification

$$M(L^i_{\acute{e}t}(\mathcal{X}^{\mathrm{ad}}_K)) \cong H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) \tag{8.7.3}$$

of Breuil–Kisin–Fargues G-modules.

Proof. By working locally on $\mathcal{U}$, we may replace each t_i with $r+1 \le i \le d$ in the target of (8.7.1) by $t_i^{\pm 1}$, so $\mathfrak{X}$ meets the requirements of §1.5. Moreover, by the Grothendieck comparison theorem and flat base change (compare with Remark 4.18), for $j = i$ and $j = i+1$, we have

$$H^j_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C) \cong H^j_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}_K) \otimes_{\mathcal{O}_K} \mathcal{O}_C, \quad \text{so} \quad H^j_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}_K) \cong (H^j_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C))^G. \tag{8.7.4}$$

Consequently, by Proposition 7.7, the Breuil–Kisin–Fargues G-modules $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ and $H^{i+1}_{A_{\mathrm{inf}}}(\mathfrak{X})$ (see §8.3) are A_{inf}-free. By Theorem 2.3, we have the G-equivariant identification of the étale realization:

$$(H^i_{A_{\mathrm{inf}}}(\mathfrak{X}))_{\acute{e}t} \cong H^i_{\acute{e}t}(\mathcal{X}^{\mathrm{ad}}_C, \mathbb{Z}_p)$$

(which is then torsion free). By Proposition 6.9, the B_{dR}-base change of this identification agrees with the identification $H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}} B_{\mathrm{dR}} \cong H^i_{\acute{e}t}(\mathcal{X}^{\mathrm{ad}}_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$ that results by combining

$$H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}} B^+_{\mathrm{dR}} \overset{(6.7.1)}{\cong} H^i_{\mathrm{cris}}(\mathcal{X}^{\mathrm{ad}}_C/B^+_{\mathrm{dR}}) \overset{(6.2.10)}{\cong} H^i_{\mathrm{dR}}(\mathcal{X}^{\mathrm{ad}}_K/K) \otimes_K B^+_{\mathrm{dR}}$$

and

$$H^i_{\mathrm{dR}}(\mathcal{X}^{\mathrm{ad}}_K/K) \otimes_K B_{\mathrm{dR}} \overset{(8.6.1)}{\cong} H^i_{\acute{e}t}(\mathcal{X}^{\mathrm{ad}}_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}.$$

In particular, we obtain the desired G-equivariant identification

$$M(L^i_{\acute{e}t}(\mathcal{X}^{\mathrm{ad}}_K)) \cong H^i_{A_{\mathrm{inf}}}(\mathfrak{X}),$$

under which, by Corollary 6.7 and the sentence after (6.2.10), the identifications

$$M(L^i_{\acute{e}t}(\mathcal{X}^{\mathrm{ad}}_K)) \otimes_{A_{\mathrm{inf}}, \theta} C \overset{(8.6.1)}{\cong} H^i_{\mathrm{dR}}(\mathcal{X}^{\mathrm{ad}}_C/C) \qquad \text{and} \qquad H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}, \theta} C \overset{(4.17.1)}{\cong} H^i_{\mathrm{dR}}(\mathcal{X}^{\mathrm{ad}}_C/C)$$

agree. In particular, by (7.6.3), we obtain the following equality inside $H^i_{\mathrm{dR}}(\mathcal{X}^{\mathrm{ad}}_C/C)$:

$$M(L^i_{\acute{e}t}(\mathcal{X}^{\mathrm{ad}}_K))_{\mathrm{dR}} = M(L^i_{\acute{e}t}(\mathcal{X}^{\mathrm{ad}}_K)) \otimes_{A_{\mathrm{inf}}, \theta} \mathcal{O}_C = H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}, \theta} \mathcal{O}_C = H^i_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C),$$

which, together with the second identification in (8.7.4), gives the desired (8.7.2). ☐

Remark 8.8. In the proof above we have seen that both $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ and $H^{i+1}_{A_{\mathrm{inf}}}(\mathfrak{X})$ are A_{inf}-free, so, by (7.6.4), we have the G-equivariant and Frobenius-equivariant identifications

$$H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}} W(k) \cong H^i_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k)) \overset{(5.44.4)}{\cong} H^i_{\log \mathrm{cris}}(\mathcal{X}_k/W(k)),$$

and hence also the Frobenius-equivariant identification

$$(H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}} W(k))^G \cong H^i_{\log \mathrm{cris}}(\mathcal{X}_{k_0}/W(k_0)). \tag{8.8.1}$$

In particular, (8.7.3) and (8.8.1) show that, under the assumptions of Theorem 8.7, the integral p-adic étale cohomology $H^i_{\text{ét}}(\mathcal{X}^{\text{ad}}_K, \mathbb{Z}_p)$ endowed with its Galois action functorially determines the integral logarithmic crystalline cohomology $H^i_{\log \text{cris}}(\mathcal{X}_{k_0}/W(k_0))$ endowed with its Frobenius.

9. The semistable comparison isomorphism

Our final goal is to use the preceding results to deduce the semistable comparison isomorphism for suitable "semistable" formal schemes (see Theorem 9.5). This extends [BMS16, 1.1 (i)], which treated the good reduction case (see also [TT15, 1.2] for a result "with coefficients" over an absolutely unramified base), and is similar to the semistable comparison [CN17, 5.26]. More precisely, *loc. cit.* also includes cases in which the log structures are not "vertical."

9.1. The ring B_{st}. We consider the log PD thickenings A_{cris}/p^n of $\mathcal{O}_C/p$ of §5.2 and set

$$J_n := \text{Ker}(A_{\text{cris}}/p^n \twoheadrightarrow \mathcal{O}_C/p) \qquad \text{and} \qquad J := \varprojlim_{n \geq 1} J_n \cong \text{Ker}(A_{\text{cris}} \twoheadrightarrow \mathcal{O}_C/p).$$

The element $p \in \mathcal{O}_C \setminus \{0\}$ belongs to the log structure of $\mathcal{O}_C/p$ (see §1.6 (1)), so its preimage in the log structure of A_{cris}/p^n is a $(1 + J_n, \times)$-torsor, which is trivial because $(1 + J_n, \times)$ is a successive extension of A_{cris}/p^n-modules (compare with [Bei13b, §1.15, p. 23]). Consequently, as n varies, these torsors comprise a trivial $(1 + J, \times)$-torsor τ_0, whose base change along the logarithm map $(1+J, \times) \to (J, +) \subset (A_{\text{cris}}, +)$ furnished by the divided power structure on J is a trivial $(A_{\text{cris}}, +)$-torsor τ, the so-called *Fontaine–Hyodo–Kato torsor*. The functor which to an A_{cris}-algebra A assigns the underlying set of the $(A, +)$-torsor $\tau \times_{(A_{\text{cris}},+)} (A, +)$ is represented by the A_{cris}-algebra A_{st}, so A_{st} is the initial A_{cris}-algebra over which the Fontaine–Hyodo–Kato torsor is canonically trivialized.

Concretely, we may noncanonically trivialize τ_0 (for instance, $[p^{1/p^\infty}]$ is a trivialization, see (5.2.1)) to obtain an isomorphism $A_{\text{st}} \simeq A_{\text{cris}}[T]$, which, upon adjusting the trivialization by an $a \in 1 + J$, gets postcomposed with the A_{cris}-automorphism $A_{\text{cris}}[T] \xrightarrow{T \mapsto T + \log(a)} A_{\text{cris}}[T]$. The A_{cris}-derivation $-\frac{d}{dT}$ of $A_{\text{cris}}[T]$ commutes with these automorphisms, so it induces a canonical A_{cris}-derivation, the *monodromy operator*,

$$N\colon A_{\text{st}} \to A_{\text{st}} \qquad \text{for which} \qquad (A_{\text{st}})^{N=0} = A_{\text{cris}}$$

(our N agrees with that of *op. cit.*, see [Bei13b, §1.15, Remarks (i)]; compare also with [Tsu99, 4.1.1]).

By [Bei13b, (1.15.2)], the Frobenius pullback of τ_0 is isomorphic to the p-fold self-product of τ_0, and hence likewise for the base change of τ_0 (that is, of τ) to any $(A, +)$. Consequently, Frobenius base change of torsors determines an A_{cris}-semilinear "Frobenius" morphism

$$\varphi\colon A_{\text{st}} \to A_{\text{st}} \tag{9.1.1}$$

which in terms of an isomorphism $A_{\text{st}} \simeq A_{\text{cris}}[T]$ obtained by trivializing τ_0 is described by $T \mapsto pT$. The interaction of φ and N is described by the formula $N\varphi = p\varphi N$.

Since μ and $\log([\epsilon])$ are unit multiples of each other in A_{cris} (see §5.14) and $\varphi(\log([\epsilon])) = p\log([\epsilon])$, the Frobenius (9.1.1) and, evidently, also the derivation N induce their counterparts on the localizations

$$B^+_{\text{st}} := A_{\text{st}}[\tfrac{1}{p}] \qquad \text{and} \qquad B_{\text{st}} := A_{\text{st}}[\tfrac{1}{p\mu}].$$

The relation $N\varphi = p\varphi N$ continues to hold for B^+_{st} and B_{st}. As is explained in [Bei13b, §1.17], the A_{cris}-algebras B^+_{st} and B_{st} reviewed above agree with the ones constructed in [Fon94, §3].

Proposition 9.2. *Assume that $\mathfrak{X}$ is $\mathcal{O}_C$-proper and let $\mathcal{Y}$ be a descent of $\mathfrak{X}_{\mathcal{O}_C/p}$ to a proper log smooth fine log $\mathcal{O}/p$-scheme of Cartier type, where $\mathcal{O} \subset \mathcal{O}_C$ is a discrete valuation subring with a* *perfect residue field k_0 such that $C \cong (\overline{\mathrm{Frac}(\mathcal{O})})^\wedge$. Then we have the following identification that is compatible with the actions of N and φ (which are described in the proof):*

$$R\Gamma_{\log \mathrm{cris}}(\mathcal{Y}_{k_0}/W(k_0)) \otimes^{\mathbb{L}}_{W(k_0)} B^+_{\mathrm{st}} \cong R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) \otimes^{\mathbb{L}}_{A_{\mathrm{cris}}} B^+_{\mathrm{st}}, \tag{9.2.1}$$

where $W(k_0)$ is endowed with the log structure associated to $\mathbb{N}_{\geq 0} \xrightarrow{0} W(k_0)$.

Proof. A descent $\mathcal{Y}$ always exists by the proof of Corollary 5.43 and the claim is a direct consequence of [Bei13b, (1.18.5)]. On the left side of (9.2.1), the operator N combines the monodromy operators of $R\Gamma_{\log \mathrm{cris}}(\mathcal{X}_{k_0}/W(k_0))$ and B^+_{st}, so is of the form $N \otimes 1 + 1 \otimes N$; on the right side, N is induced by the monodromy operator of B^+_{st}. On either side of (9.2.1), the Frobenius φ acts on both factors. $\square$

Remark 9.3. One may eliminate the dependence of (9.2.1) on the choice of $\mathcal{Y}$ by forming a direct limit over all the possible $\mathcal{Y}$, see *loc. cit.*

9.4. The base field K. For the rest of §9, we assume that $C = \widehat{\overline{K}}$ for a fixed complete discretely valued subfield $K \subset C$ with a perfect residue field k_0 and set $G := \mathrm{Gal}(\overline{K}/K)$ (compare with §8.1). By functoriality, G acts continuously on A_{cris}, A_{st}, B^+_{st}, and, since the ideal (μ) does not depend on the choice of ϵ (see §2.1), also on B_{st}. These G-actions commute with the operators φ and N.

In the case when $\mathcal{O}$ from Proposition 9.2 is our $\mathcal{O}_K$, the identification (9.2.1) is G-equivariant granted that we let G act on both sides by functoriality.

Theorem 9.5. *Let $\mathcal{X}$ be a proper p-adic formal $\mathcal{O}_K$-scheme that in the étale topology may be covered by affines $\mathcal{U}$ each of which has an étale morphism*

$$\mathcal{U} \to \mathrm{Spf}(\mathcal{O}_K\{t_0, \dots, t_r, t_{r+1}^{\pm 1}, \dots, t_d^{\pm 1}\}/(t_0 \cdots t_r - \pi)) \quad \textit{for some nonunit} \quad \pi \in \mathcal{O}_K \setminus \{0\} \tag{9.5.1}$$

(where π, r, and d may depend on $\mathcal{U}$) and endow $\mathcal{X}$ with the log structure $\mathcal{O}_{\mathcal{X}} \cap (\mathcal{O}_{\mathcal{X}}[\frac{1}{p}])^\times$. There is the following G-equivariant natural isomorphism that is compatible with the action of φ and N:

$$R\Gamma_{\log \mathrm{cris}}(\mathcal{X}_{k_0}/W(k_0)) \otimes^{\mathbb{L}}_{W(k_0)} B_{\mathrm{st}} \cong R\Gamma_{\text{ét}}(\mathcal{X}^{\mathrm{ad}}_C, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} B_{\mathrm{st}}, \tag{9.5.2}$$

where $W(k_0)$ is endowed with the log structure associated to $\mathbb{N}_{\geq 0} \xrightarrow{0} W(k_0)$. In particular, for every $i \in \mathbb{Z}$, the G-representation $H^i_{\text{ét}}(\mathcal{X}^{\mathrm{ad}}_C, \mathbb{Q}_p)$ is semistable.

Proof. We set $\mathfrak{X} := \mathcal{X} \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_C$, so that $\mathfrak{X}$ meets the requirements of §1.5. By Claims 1.6.1 and 1.6.3 and [Kat89, 4.8], the base change $\mathcal{X}_{\mathcal{O}_K/p}$ is fine, log smooth, and of Cartier type over $\mathcal{O}_K/p$, so Proposition 9.2 applies to it and gives the G-equivariant (see §9.4) identification

$$R\Gamma_{\log \mathrm{cris}}(\mathcal{X}_{k_0}/W(k_0)) \otimes^{\mathbb{L}}_{W(k_0)} B^+_{\mathrm{st}} \cong R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) \otimes^{\mathbb{L}}_{A_{\mathrm{cris}}} B^+_{\mathrm{st}} \overset{(5.43.2)}{\cong} R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} B^+_{\mathrm{st}}$$

that is compatible with φ and N. In addition, by (2.3.1), we have a G-equivariant identification

$$R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} B_{\mathrm{st}} \cong R\Gamma(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} B_{\mathrm{st}} \cong R\Gamma(\mathcal{X}^{\mathrm{ad}}_C, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} B_{\mathrm{st}}$$

that is trivially compatible with φ and N. By combining these two displayed equations, we obtain the desired identification (9.5.2). $\square$

Remark 9.6. The isomorphism (9.5.2) is compatible with filtrations in the following sense: by [Fon94, §4.2], there is a (noncanonical) A_{cris}-algebra homomorphism $B_{\mathrm{st}} \to B_{\mathrm{dR}}$ and, by the proof above and Proposition 6.9, the B_{dR}-base change of the isomorphism (9.5.2) is identified with the de Rham comparison isomorphism (6.8.2) (with $X_0 = \mathcal{X}^{\mathrm{ad}}_K$) that is compatible with filtrations.

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