Dirac Operators and Spin Geometry

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Seminar outline

In the 1920s of the last century the physicist Paul Dirac was looking for a relativistic quantum mechanical theory of the electron. He was lead to the problem of finding a first order differential operator $D = \sum_{j=1}^{n} \gamma_j \frac{\partial}{\partial x_j}$ in $\mathbb{R}^n$ whose square equals the Laplacian,[1] i.e.

$$\frac{1}{2} \sum_{1 \le i < j \le n} (\gamma_i \gamma_j + \gamma_j \gamma_i) \frac{\partial^2}{\partial x_i \partial x_j} = D^2 = \Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

This immediately leads to the set of equations

$$\gamma_i \gamma_j + \gamma_j \gamma_i = -2 \cdot \delta_{ij}, \quad 1 \le i, j \le n. \tag{1}$$

It is immediately clear that except in dimension one these equations do not have a scalar solution. However, matrix solutions exist, e.g. in dimension 2 the famous Pauli spin matrices.
A thorough analysis of the solutions of the equations (1) leads to the representation theory of the *Clifford algebras* and further to a very elegant description of the *Spin groups*, which are the natural universal covering groups of the orthogonal group SO(n).
On a Riemannian manifold the concepts sketched before lead to a natural class of geometrically defined first order elliptic differential operators which are also called *Dirac operators*. These operators are of fundamental importance in geometry, topology, global analysis and mathematical physics. In particular the Atiyah–Singer Index Theorem for Dirac operators is one of the cornerstones of modern mathematics. Many classical first order elliptic differential operators (e.g. de Rham, Dolbeault) are examples of Dirac operators.
In this seminar we will establish the fundamentals of Dirac operators on Riemannian manifolds.

Prerequisites: Global Analysis I / Geometry I.
Basic knowledge of vector bundles will be assumed.

Talks

1. **Clifford algebras.** Review of quadratic spaces (we restrict attention to characteristic different from two), Clifford algebras, $\mathbb{Z}_2$-grading on Clifford algebras, description of linear structure of Clifford algebras, center of Clifford algebras, Clifford algebras of direct sums of quadratic spaces, Clifford algebras and complexifications.
   References: §§1.1-1.2 and §1.3 up to the first proposition in [Fri00] (see also [LM89, I.1]).

[1] I am simplifying and cheating here for the sake of a concise motivation.

2. **Classification of Clifford algebras and their representations.** Clifford algebras of standard non-degenerate (real or complex) quadratic spaces, explicit identifications in low dimensions, periodicity results, description of these Clifford algebras via matrix algebras, irreducible representations of these Clifford algebras, spinor representations.
References: [Fri00, §1.3], [LM89, I.3-I.5].
3. **The groups $\mathrm{Spin}(n)$ and the spin representations.** The groups $\mathrm{Pin}(n)$ and $\mathrm{Spin}(n)$, $\mathrm{Spin}(n)$ as universal covering space of $SO(n)$ for $n \geq 3$, description of Lie algebra of $\mathrm{Spin}(n)$, spin representation of $\mathrm{Spin}(n)$, outlook on $\mathrm{Spin}(r,s)$ and the more general $\mathrm{Spin}(V,q)$ associated to a quadratic space (V,q), optional: discuss in outline the groups $\mathrm{Spin}^{\mathbb{C}}(n)$.
References: [Fri00, §§1.4-1.5], [LM89, I.2] (optional: [Fri00, §1.6]).
4. **First applications.** Construction of linearly independent vector fields on spheres and projective spaces, quote result of Adams on geometric dimension of tangent bundles of spheres, construction of exceptional isomorphisms of low-dimensional Lie groups, Cayley numbers $\mathbb{O}$ and S^7 as a homogenous space $\mathrm{Spin}(7)/\mathrm{Aut}(\mathbb{O})$.
References: [LM89, I.7-I.8] (optional: for additional inspiration take a look at [Har90]).
5. **Topological K-theory.** Classification of vector bundles, stable isomorphism classes of vector bundles, real and complex topological K-theory, Bott periodicity theorems, K-theory as a generalized cohomology theory.
References: [Hat] and [Hus94] (or any of the many introductory books on bundle theory and topological K-theory).
6. **The Atiyah–Bott–Shapiro construction.** Alternative approach to K-theory via 'finite length exact sequences of vector bundles', Euler characteristics, the Atiyah–Bott–Shapiro isomorphisms relating Clifford modules to KO_* and K_*, some explicit descriptions of generators in KO_* in terms of Clifford modules.
References: [LM89, I.9], [ABS64, parts 2-3].
7. **Principal fibre bundles and connections.** Definition of principal fibre bundles, description via non-abelian Cech cocycles, examples, construction of associated fibre bundles, illustration via frame bundles and associated vector bundles, connections, reduction of structure groups, orientability and reduction to $SO(n)$, Whitney sums and reduction of structure group, reduction to trivial group.
References: [Fri00, B.1, B.3], [LM89, Appendix A; II.3, II.4], [Roe98, Chap. 2].
8. **Spin structures.** Spin structures on vector bundles, criteria for existence in terms of Stieffel-Whitney classes, relation to orientability, optional $\mathrm{Spin}^{\mathbb{C}}$ structures, examples of spin manifolds, examples of (orientable!) manifolds which do not admit a spin structure, spinor bundles, associated bundles.
References: [LM89, II.1, II.2], [Roe98, relevant sections in Chap. 4], [Fri00, Chap. 2].
9. **Dirac operators.** Connections on spinor bundles, Dirac and Dirac-Laplace operators, Lichnerowicz formula.
References: [Fri00, Chap. 3], [LM89, II.3, II.4] (only material complementing talk 7), [LM89, II.5] (only selected material, cf. [Fri00]), [Roe98, Chap. 3].
10. **Analytical properties of Dirac operators.** Essential selfadjointness on complete manifolds, compact resolvent on compact manifolds, Fredholm properties on compact manifolds.
References: [Fri00, Chap. 4], [Roe98, Chap. 5], [LM89, II.5].
11. **Fundamental elliptic operators.** Spin Dirac operator, Gauß Bonnet and signature operator, Dolbeault operator, optional Dirac operators of gauge theory.
References: [LM89, II.6], [Fri00, Appendix A] (for gauge theory), [BGV92, 3.6].
12. **Vanishing Theorems.** This talk basically discusses various applications of (variants of) Lichnerowicz's formula. This formula leads to obstructions against the existence of metrics fulfilling certain curvature inequalities (e.g. nonnegative scalar curvature). Refined methods require the so called $C\ell_k$-linear Dirac operator, which is constructed using the representation theory of real Clifford algebras.
References: [LM89, II.7, II.8]

Literature

[ABS64] M. Atiyah, R. Bott, and A. Shapiro, *Clifford modules*, Topology **3** (1964), 3 – 38.

[BGV92] N. Berline, E. Getzler, and M. Vergne, *Heat Kernels and Dirac Operators*, Springer-Verlag, 1992.

[Fri00] T. Friedrich, *Dirac operators in Riemannian geometry*, Graduate studies in mathematics, American Mathematical Society, 2000.

[Har90] F. Harvey, *Spinors and calibrations*, Perspectives in mathematics, Academic Press, 1990.

[Hat] A. Hatcher, *Vector bundles & K-theory*, available online via `https://pi.math.cornell.edu/~hatcher/VBKT/VBpage.html` (version of 2017).

[Hus94] D. Husemoller, *Fibre bundles*, Graduate Texts in Mathematics, Springer, 1994.

[LM89] H. Lawson and M. Michelsohn, *Spin Geometry*, Princeton mathematical series, Princeton University Press, 1989.

[Roe98] J. Roe, *Elliptic operators, topology and asymptotic methods*, second ed., Pitman Research Notes in Mathematics, vol. 395, Addison Wesley Longman Limited, 1998.

Warning: Note that the main references [LM89] and [Fri00] use different sign conventions in the definition of Clifford algebras.