Noncommutative Geometry

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V5B8 – Selected Topics in Analysis
Summer semester 2019

This lecture course is intended for first-year Master students. The lectures will take place on Fridays, 8:00–10:00, in Seminarraum 1.008.

Introduction

For a classical 'space' there is a duality between the space itself and the (commutative) algebra of functions on the space. To give a precise statement: the category of compact Hausdorff topological spaces is dual to the category of unital commutative C^*-algebras. This duality suggests that the study of *non*commutative C^*-algebras should be thought of as 'noncommutative topology'. The goal of noncommutative geometry is to extend this idea further, describing not only the topology of a space but also its (differential) geometry in terms of corresponding algebraic objects, taking the following steps:

1. translate all topological and geometrical data into algebraic data;
2. show how to reconstruct the topological and geometrical data from the algebraic data;
3. generalise by allowing noncommutative 'coordinate algebras'.

This approach also allows to deal with for instance 'bad quotients', e.g. a quotient of a compact Hausdorff space by a discrete group action, for which the quotient topology may fail to separate orbits. In this case, rather than considering continuous functions on the quotient space, one may obtain more information by studying certain noncommutative algebras.

List of topics

This lecture course is intended as a first introduction to the field of noncommutative geometry (NCG). The aim is to cover the following topics:

Noncommutative topology and K-theory: Gelfand-Naimark duality between compact Hausdorff topological spaces and unital commutative C^*-algebras, Serre-Swan duality between vector bundles and finitely generated projective modules, and a description of (topological and C^*-algebraic) K-theory.

Spectral triples: these are the main objects thought of as 'noncommutative spaces'. The spectral triple of a Riemannian spin manifold (Dirac operators), the geodesic distance functional, and Connes' reconstruction theorem.

Noncommutative torus: the quintessential example of a noncommutative space, which is closely related to the linear foliation of the torus by lines with an irrational angle.

Quantised calculus: the Dixmier trace (the 'noncommutative integral'), the Wodzicki residue, and Connes' trace theorem.

Cyclic cohomology: a noncommutative analogue of de Rham cohomology.

Index pairings: the index map giving a pairing between classes in K-theory and K-homology, which can be 'computed' using cyclic cohomology via Connes' Chern character.

Recommended literature

[Bla98] B. Blackadar, *K-theory for operator algebras*, 2nd ed., Math. Sci. Res. Inst. Publ., Cambridge University Press, 1998.

[Con94] A. Connes, *Noncommutative Geometry*, Academic Press, San Diego, CA, 1994.

[GVF01] J. Gracia-Bondía, J. Várilly, and H. Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser Advanced Texts, 2001.

[HR00] N. Higson and J. Roe, *Analytic K-Homology*, Oxford University Press, New York, 2000.