Introduction to C^*-algebras

Matthias Lesch & Koen van den Dungen

Graduate Seminar on Global Analysis
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Preliminary meeting: Thursday 19 July 2018 at 9:30 in Room 1.008.

During this seminar we will develop the basic theory of C^*-algebras. Among the main goals are two theorems by Gelfand and Naimark, which state:

1. that every commutative C^*-algebra is given by continuous functions on a (locally compact, Hausdorff) space;
2. that every C^*-algebra is isometrically isomorphic to a closed $*$-subalgebra of bounded operators on a Hilbert space.

The first statement suggests that a noncommutative C^*-algebra can be thought of as continuous functions on some 'noncommutative space'. Thus, C^*-algebras can be thought of as describing 'noncommutative topology', and as such they form the first stepping stone towards a description of 'noncommutative geometry'.

Seminar outline

A rough plan of the seminar talks is as follows. First, the basic theory will be developed in a number of talks:

Talk 0: introductory overview of the seminar; preliminaries on Banach spaces and linear operators.
Talk 1: Banach algebras: ideals, unitisations, spectrum, holomorphic functional calculus.
Talk 2: the Gelfand transform for Banach algebras: maximal regular ideals, characters.
Talk 3: C^*-algebras: definition and basic properties; Gelfand-Naimark duality between commutative C^*-algebras and locally compact Hausdorff spaces, continuous functional calculus.
Talk 4: positive elements, approximate units, positive linear functionals.
Talk 5: representations, the Gelfand-Naimark-Segal (GNS) construction, existence of faithful representations, strictly positive elements and σ-unital C^*-algebras.
Talk 6: von Neumann algebras: definition and basic properties, double commutant theorem, Borel functional calculus.
Talk 7: pure states and irreducible representations, relation to characters, Krein-Milman theorem.

Next, the remaining talks will be selected from the following topics:

Talk a: basic examples: continuous functions on topological spaces, essentially bounded functions on measure spaces; every finite-dimensional C^*-algebra is a matrix algebra.
Talk b: multiplier algebras: equivalence of various definitions (via concrete representation, via double centralisers, via Hilbert modules).
Talk c: Morita equivalence and stable isomorphism.
Talk d: tensor products: cross-norms for Banach spaces, minimal/maximal C^*-norms for C^*-algebras.
Talk e: group C^*-algebras.
Talk f: universal C^*-algebras (given by generators and relations).
Talk g: crossed products.
Talk h: classification of von Neumann algebras.
Talk i: classification of certain classes of C^*-algebras.
Talk j: Gelfand-Naimark duality as an equivalence of categories.

Recommended literature

[Bla06] B. Blackadar, *Operator algebras: Theory of C^*-algebras and von Neumann algebras*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer, 2006.

[Dav96] K. Davidson, *C^*-algebras by example*, Fields Institute for Research in Mathematical Sciences Toronto: Fields Institute monographs, American Mathematical Soc., 1996.

[Mur90] G. Murphy, *C^*-algebras and Operator Theory*, Academic Press, 1990.

[Ped89] G. Pedersen, *Analysis now*, Graduate texts in mathematics, vol. 118, Springer-Verlag, 1989.

[Tak01] M. Takesaki, *Theory of operator algebra I*, Encyclopaedia of Mathematical Sciences, vol. 124, Springer, 2001.