Hyperbololic Constant Mean Curvature One Surfaces with Compact Fundamental Domains

by H. Karcher, Bonn. Version December 2001

Abstract. We construct surfaces of constant mean curvature one (cmc1) in hyperbolic 3-space $\mathbb{H}^3$. They have the symmetry group of a Platonic tessellation of $\mathbb{H}^3$ and therefore compact fundamental domains.

Introduction.

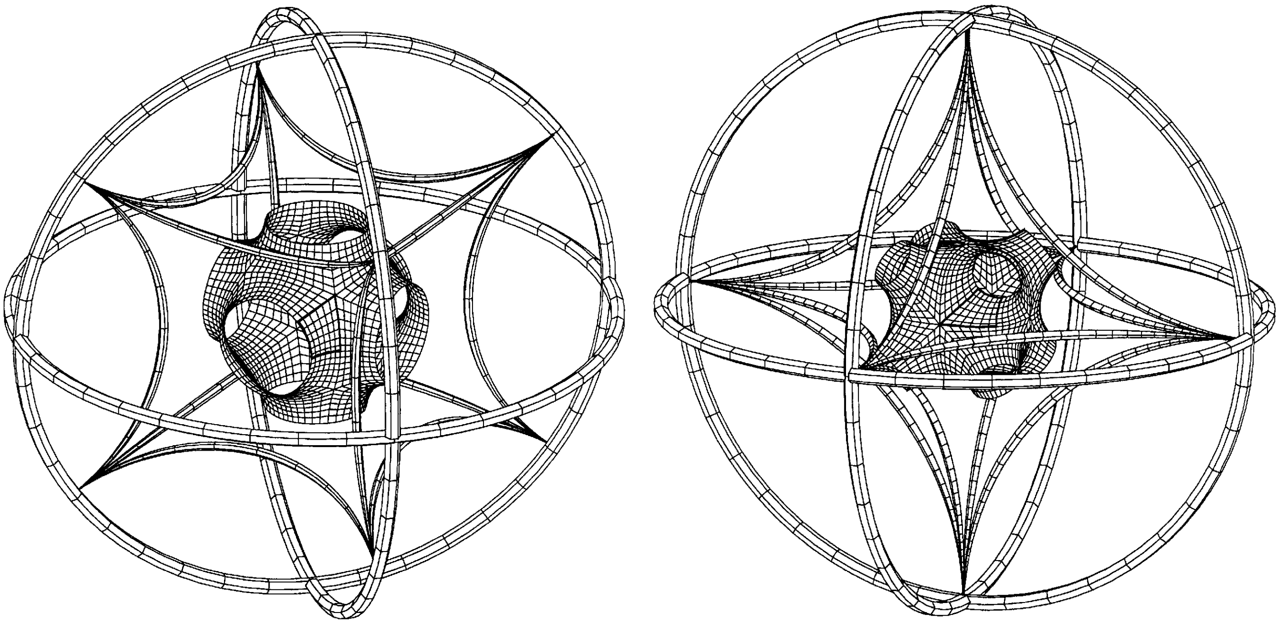
In 1987 Bryant [Br] gave a so called "Weierstraß" representation of cmc1 surfaces in $\mathbb{H}^3$: A holomorphic null immersion $F:\ M^2 \to SL(2,\mathbb{C})$ (i.e. $\langle F', F'\rangle_{\mathbb{C}} = 0$), gives a cmc1 immersion $f := FF^* : M^2 \to \mathbb{H}^3$, where $\mathbb{H}^3$ is the hypersurface $\{\det = 1\}$ in the fourdimensional space of positive definite hermitian symmetric 2×2-matrices. (Note that the determinant is a quadratic function on this space.) And vice versa, every cmc1 immersion f lifts to such a holomorphic null immersion into $SL(2,\mathbb{C})$, the double cover of the isometry group. The connection of cmc1 surfaces with holomorphic data is older; I learnt from Lawson [La] how simply connected minimal surfaces in $\mathbb{R}^3$ give isometric cmc1 surfaces in $\mathbb{H}^3$: from the surface data of any minimal surface, namely Riemannian metric $g(,)$ and shape operator S, one gets surface data of a cmc1 surface by taking the same Riemannian metric and a new shape operator $S_\pm := S \pm id$. This change does not affect the Codazzi equation, and the Gauss equation for $g(,), S$ in $\mathbb{R}^3$ becomes the Gauss equation for $g(,), S_\pm$ in $\mathbb{H}^3$. Both, Bryant's Weierstraß representation and the described Lawson correspondence, can be used to construct cmc1 surfaces in $\mathbb{H}^3$. The Weierstraß representation point of view was developed by Umehara and Yamada [UY1-5]. They emphasize the following: while Bryant's Weierstraß data, namely a so called secondary Gauss map g and the Hopf holomorphic quadratic differential Q, come from the minimal surface in $\mathbb{R}^3$ and correspond to a left invariant ODE for the immersion F into the group $SL(2,\mathbb{C})$, they prefer a right invariant ODE for F in terms of the hyperbolic Gauss map G and the Hopf differential Q, where G is defined as the map from the surface to the conformal sphere at infinity of $\mathbb{H}^3$ obtained by following the geodesics from the surface in the direction of the mean curvature vector to infinity. This map is conformal for cmc1 surfaces (the normal map in the other direction to infinity is not conformal). [UY] give a connection between the two representations by noting that the map $F \to F^{-1}$ in the group leads to a pair of cmc1 immersions, $FF^*, F^{-1}F^{-1*}$, where the role of the hyperbolic and the secondary Gauss maps are interchanged. This method leads to a large collection of examples: together with Rossman [RUY] they construct from many of the *finite total curvature* minimal surfaces in $\mathbb{R}^3$ corresponding cmc1 cousin surfaces in $\mathbb{H}^3$ which have, up to some parameter adjustment, the same meromorphic (hyperbolic) Gauss map and Hopf differential as the minimal surfaces they started from.

More recently F. Pacard constructed cmc1 surfaces by functional analysis techniques, he connected horospheres by tiny necks.

In this paper we use the Lawson correspondence to construct, from solutions of polygonal Plateau problems in $\mathbb{R}^3$, cmc1 surfaces in $\mathbb{H}^3$ that are bounded by planar arcs of symmetry. Reflections in the symmetry planes, repeatedly applied to such *compact* fundamental domains, yield the complete surfaces. This is similar to the construction [La, Ka, Gb] of cmc1 surfaces in $\mathbb{R}^3$ from solutions of Plateau problems in $\mathbb{S}^3$, except that in the older situation we could choose the Plateau contours explicitly so that all the angles between the symmetry planes of the cmc1 surfaces come out correctly. In the present situation we have to consider 2-parameter families of Plateau contours and show that the contour parameters can be chosen such that the angles between the symmetry planes of the cmc1 piece in $\mathbb{H}^3$ are correct. One of the two Plateau parameters simply scales the size of the polygonal contour and we have inequalities which control the effect of this change on the hyperbolic figure. Therefore we can find the correct parameter values with an iterated intermediate value argument (instead of a more complicated degree argument). In the [RUY] construction this scaling size appears as a real factor in front of the Hopf differential Q and leads to 1-parameter families of solutions; in our case, such 1-parameter families would exist if we were content with the solution of those free boundary value problems in which a cmc1 surface piece meets all faces of *some* Platonic polyhedron in convex curves. Since we look for surfaces in *tessellating* polyhedra, that is with dihedral angles π/m, only one, two or three parameter values in these families give the desired surfaces. While the hyperbolic Gauß map is a known meromorphic map at the beginning of the constructions of [RUY] we do not even consider it for our examples: Note that every orbit in the sphere at infinity of the isometry group of a Platonic tessellation is dense so that the orbit of any Gauß value under the isometries of the surface is dense.

The first section of this paper deals with the Platonic tessellations of hyperbolic space. Their isometry groups have (as in the Euclidean cubical tessellation) tetrahedral fundamental domains in $\mathbb{H}^3$ from which one can build (by reflections in the tetrahedron faces) Platonic polyhedra, polyhedra that is, on whose directed edges their symmetry group acts transitively. Further repeated reflections in the faces of the hyperbolic Platonic polyhedra give a tessellation of $\mathbb{H}^3$. The symmetry planes of the tessellation are also the symmetry planes of the looked for cmc1 surfaces. We want to construct these surfaces from the assumption that they are cut in a similar way as Polthier's minimal surfaces in the following pictures. Namely, we expect them to be cut into congruent tiles which are bounded by only *four* arcs of symmetry. The corresponding Plateau contours in $\mathbb{R}^3$ would then be polygons with only four edges. We therefore start our construction by considering the Plateau solutions of all such polygons in $\mathbb{R}^3$.

In the second section we explain the geometric relations between the minimal patches in $\mathbb{R}^3$ and their cmc1 cousins in $\mathbb{H}^3$. Then we prove two comparison lemmas which rely on the simplicity of the Plateau contours: the projection of a (nonplanar) quadrilateral in the direction of an edge is a triangle and the Plateau solution is a *graph* over this triangle; this implies that the tangent planes along the vertical edge rotates monotonically (and not back and forth since that contradicts the graph property) so that the principal curvature function of the corresponding symmetry arc of the conjugate patch *does not change sign.* The two lemmas control the angles between the symmetry planes of the cmc1 patches well enough so that we can prove existence of a quadrilateral in $\mathbb{R}^3$ which spans a minimal surface patch whose cousin in $\mathbb{H}^3$ is the fundamental piece of the cmc1 surface with Platonic symmetry.



Hyperbolic cube and octahedron with vertices at ∞.

Inside these solids are minimal surfaces that meet their faces orthogonal in symmetry lines that are close to circles. These minimal surfaces were constructed by K. Polthier, he also computed the numerical approximations and made these pictures. Note that the symmetry planes of the Platonic solids cut the minimal surfaces into fundamental domains which are bounded by four planar arcs of reflectional symmetry. The constant mean curvature surfaces that will be constructed in this paper have the same symmetries and meet the faces of the solids orthogonally, also in fairly circular curves.

Platonic tessellations of $\mathbb{H}^3$.

We describe the compact and noncompact hyperbolic Platonic solids that we shall use. View a Euclidean Platonic polyhedron from its center, assume that the vertices are on a sphere of radius r and let $\mathbb{R}^3$ be the tangent space at $p \in \mathbb{H}^3$. Then consider the hyperbolic geodesics of length r from p in the direction towards the vertices; the hyperbolic convex hull of the endpoints is a hyperbolic Platonic solid with circumsphere of radius r. The dihedral angles of this polyhedron are smaller than the Euclidean dihedral angles, they decrease with growing r and converge to the Euclidean limit as $r \to 0$. In the limit $r \to \infty$ we obtain a Platonic solid with vertices on the sphere at infinity; its dihedral angles can be seen on the intersection with a horosphere around a vertex. The intrinsic geometry of a horosphere is Euclidean, therefore this intersection is an equilateral triangle for the solids with trivalent vertices (tetrahedron, cube, dodecahedron), is a square for the octahedron and is a 108° regular pentagon for the icosahedron. Therefore we have the following tessellating platonic solids (the degrees refer to the dihedral angles [the first ones give the Euclidean case] and the last one of each kind has the vertices at infinity of $\mathbb{H}^3$):

[70.53°-tetrahedron,] 60°-tetrahedron

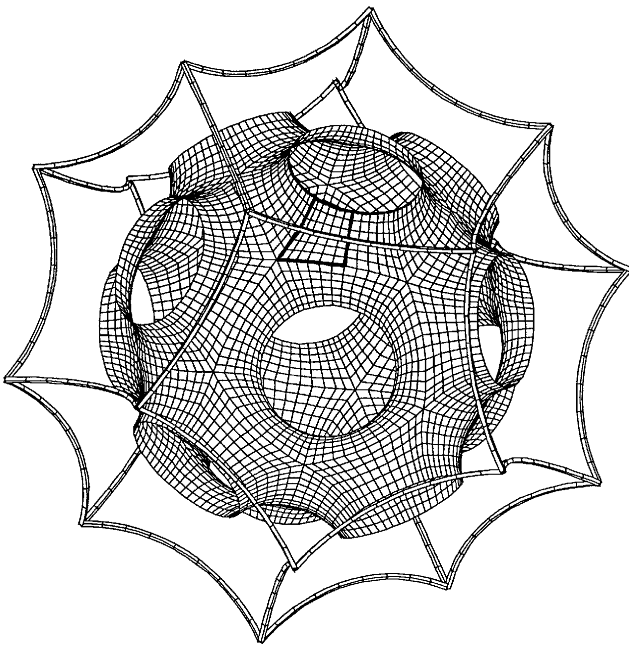
[90°-cube,] 72°-cube, 60°-cube

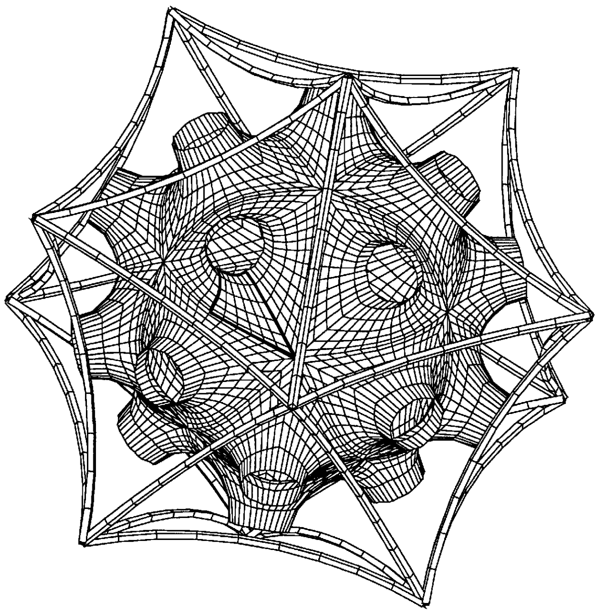
[116.57°-dodecahedron,] 90°-dodecahedron, 72°-dodecahedron, 60°-dodecahedron

[109.47°-octahedron,] 90°-octahedron

[138.19°-icosahedron,] 120°-icosahedron, [108°-icosahedron, not tessellating $\mathbb{H}^3$]

A Platonic solid is cut by its planes of symmetry into tetrahedra; all of them have one vertex at the center (C), the other three vertices are at the midpoint of a face (F), at the midpoint of an edge (E) and at a vertex of the solid (V). The dihedral angles of these tetrahedra are important for our construction, three of them, at the edges CE, FE, FV, are $\pi/2$; at the edge EV we have half the dihedral angle of the solid; at the edge CF the angle is π/k if the faces are k-gons; and at the edge CV the angle is π/n if n edges of the solid join to that vertex. (Compare the Euclidean cube.)





Hyperbolic 72°-dodecahedron and 120°-icosahedron.

The surfaces inside the solids are the minimal surfaces constructed and computed by K. Polthier. Quadrilateral fundamental domains for the symmetry groups are emphasized.

In Euclidean space the tetrahedron is called *selfdual* because the convex hull of the midpoints of its *faces* is again a tetrahedron; similarly, the Euclidean cube-octahedron or dodecahedron-icosahedron are called *dual pairs*. In hyperbolic space one also has selfdual Platonic solids and dual pairs. The convex hull of the midpoints of all those solids of a tessellation which have one vertex in common is again a tessellating Platonic solid. It gives the *dual* tessellation; if these two tessellations are congruent we also call them selfdual. In Euclidean space the tessellation by cubes is selfdual. In hyperbolic space the tessellations by 72°-dodecahedra or by 120°-icosahedra are selfdual, while the tessellations by 72°-cubes and by 90°-dodecahedra are a dual pair. (The dual partners of the solids with infinite vertices are Platonic solids with infinitely many vertices. They have no Euclidean analogue and we will not use them here.)
If one projects the edges of a Platonic solid from the *center* C onto a sphere around the center then one obtains a Platonic tessellation of $\mathbb{S}^2$, and dual polyhedra give dual tessellations of $\mathbb{S}^2$. Similarly, one obtains from a *tessellation* by Platonic solids another Platonic tessellation of $\mathbb{S}^2$, if one intersects a small sphere around a *vertex* V with the faces of the solids meeting at that vertex. These spherical tessellations are very helpful to imagine the neighbours of a solid in a tessellation.

Comparison between Euclidean minimal and their hyperbolic cousin surfaces.

We repeat some known facts (because we have to observe signs carefully) about

Conjugate minimal surfaces in constant curvature spaces.

The notion refers to minimal surfaces which are simply connected (i.e., pieces or coverings). Let J be the (parallel) almost complex structure (i.e., the oriented 90°-rotation in each tangent space). Take the Riemannian metric $g(,)$ and change the shape operator S of a given minimal surface to $S_* := J \cdot S$; then S_* is trace free, symmetric and together with $g(,)$ satisfies the Gauss- and Codazzi equations, hence these data define an isometric but usually not congruent minimal surface, that is refered to as the conjugate surface. For geodesics c we have an interesting relation between (normal) curvature and torsion on the original and conjugate immersion:

$$\kappa := g(c', S \cdot c') = g(J \cdot c', J \cdot S \cdot c') = g(J \cdot c', S_* \cdot c') = -\tau_*$$
$$\tau := g(c', J \cdot S \cdot c') = g(c', S_* \cdot c') = \kappa_* \ .$$

This is used to relate symmetry lines on a minimal surface and its conjugate: Geodesic curvature lines lie in a plane orthogonal to the surface and reflection in this plane is a congruence of the minimal surface. On the conjugate immersion this curve has $\kappa_* = \tau = 0$, it is a straight line (i.e. a geodesic in the three dimensional constant curvature space), and 180°-rotation around it is a congruence of the conjugate surface. The rotation speed of the tangent plane along such a straight line (segment) is $\tau_* = -\kappa$. And vice versa. We use this as follows: Consider a minimal surface which is the (simply connected) Plateau solution in a quadrilateral (non-planar) contour. Each projection in the direction of an edge is a (convex) triangle so that the Plateau solution is a *graph* over this triangle. The graph property implies that the tangent plane along the vertical edge cannot rotate back and forth, it can only rotate monotonically, i.e. the function $\tau = \kappa_*$ does *not change sign* along the edge. The total rotation of the tangent plane is the angle of the triangular projection below the vertical edge; of course this angle is also the total rotation of the normal on the conjugate immersion. This means that the arc conjugate to the edge bounds, together with the normals at its end points, a convex domain, provided the normals intersect with an interior angle $\leq \pi$. Finally, the curvature functions $\kappa_* = \tau$ of arcs conjugate to adjacent edges have opposite sign: Place these adjacent edges in a horizontal plane, then the two other edges go both either up or down; since one is at an initial point the other at an end point of its horizontal neighbour this means that the tangent planes along each pair of adjacent edges rotate in opposite directions.

Constant mean curvature cousins of minimal surfaces.

Since the sign of the second fundamental form can be changed by reversing the normal, we assume that the constant mean curvature is positive. Given a simply connected minimal surface in a space $M^3(K)$ of constant curvature K, take its Riemannian metric $g(,)$ and change its shape operator S to $S_+ := S + c \cdot \text{id}$. Note that S_+ is symmetric with the same eigenvectors (principal curvature directions) as S; together with $g(,)$ it satisfies the Gauss- and Codazzi equations for the space $M^3(K - c^2)$ and therefore defines a surface with constant mean curvature c in this space; it is referred to as a constant mean curvature cousin of the given minimal surface.

We will use this with $K = 0,\ c = 1$ and apply the construction to the conjugate of the Plateau solution of a quadrilateral contour, i.e, we use the surface data $\{g, \hat{S} = J\cdot S + c\cdot\text{id}\}$. This gives us cmc1 surfaces in $\mathbb{H}^3$ which are bounded by four planar symmetry arcs. The angles between adjacent symmetry arcs are given by the Platonic tessellation into which we plan to fit the cmc1 surface to be constructed. This determines the angles between adjacent edges of the quadrilateral Plateau contour. We also need that the angles between each pair of opposite symmetry planes (of the cmc1 surface piece) has the value needed for the chosen Platonic tessellation. To achieve this we need the remaining work (to find the correct quadrilateral contour).

The effect of scaling the minimal surface in $\mathbb{R}^3$.

Consider the conjugate piece of the Plateau solution of a quadrilateral contour in $\mathbb{R}^3$ and the angle between the normals at the endpoints of each of the four planar symmetry arcs that bound the conjugate minimal surface piece. (This angle is of course the angle between a pair of opposite symmetry planes.) We know that the curvature function κ of each of the four boundary arcs of the conjugate Plateau surface does not change sign (recall that the tangent planes of the Plateau solution rotate monotonically along each of the four edges).

We also observed above that adjacent arcs have curvature functions of opposite sign. If we scale the boundary arcs, $c \to \lambda c$, then their curvatures change as $\kappa \to \frac{1}{\lambda}\kappa$ and the corresponding curvatures of the hyperbolic cmc1 cousin boundary arcs are

$\kappa_h = \frac{1}{\lambda}\kappa + 1$, where κ equals the torsion function τ of the Plateau patch.

This means that the arcs with $\kappa \geq 0$ have $\kappa_h \geq 1$, i.e. they have a focal point along each normal, and independent of λ always on the same side (the limit infinity can be included). We call these the +arcs. The others, the −arcs, may still have focal points on the same side as the corresponding Euclidean arc for sufficiently small λ, but as λ increases, κ_h eventually changes sign along the whole arc. Curvature as a function of arc length determines a planar curve, and, in our case, the dependence on λ is continuous; moreover, for $\lambda \to 0$, the behaviour of the hyperbolic arc converges to that of the Euclidean arc from which we started (this is best seen if we scale $\mathbb{H}^3$ to $M^3(-\lambda^2)$).

As long as the normals of the hyperbolic arc intersect on the same side as for the Euclidean arc (which they do for sufficiently small λ), we can compute the angle ω between the normals at the endpoints with the Gauss-Bonnet theorem. Let l be the length of the hyperbolic arc and A the area bounded by the arc and the normals at its endpoints.

For +arcs we have

$$\omega_+ = 2\pi - (\pi/2 + \pi/2 + (\pi - \omega_+)) = \int \kappa_h \, ds - A = \int \kappa ds + l - A.$$

For −arcs the normals intersect for small λ on the negatively curved side, i.e. $-\kappa_h = |\kappa| - 1$, hence

$$\omega_- = \int -\kappa_h \, ds - A = \int |\kappa| ds - l - A.$$

Now we show that the angle ω_- decreases from its Euclidean value $\int \kappa ds$ to 0 as the scaling increases the hyperbolic length l from ~ 0 to a maximal length smaller than the obvious bound $\int |\kappa| ds$. And we show that the angle ω_+ increases from its Euclidean value $\int \kappa ds$ to $\pi/2$ as l is scaled up, but we do not obtain a *bound* on l that guarantees that $\omega_+ = \pi/2$ is reached before ℓ is increased to that bound. In both cases we will need that the normals at the endpoints of the arc *do not meet the arc* before they intersect each other. Consider a one quarter arc of a very elongated ellipse and extend it a bit beyond the vertex of maximal curvature; the normal line at this endpoint meets the arc again before it intersects the normal line at the other endpoint. This explains why we require *the additional hypothesis*, that we start with a Euclidean angle $\int |\kappa| ds \leq \pi/2$.

We deal with the −arcs first. For them, ω_- is at least by $-l$ smaller than the turning angle of the Euclidean arc. By making λ, hence l, larger we can indeed decrease ω_- to 0, even with the obvious a priori bound: $l < \int |\kappa| ds \leq \pi/2$. Note that the assumption $\int |\kappa| ds \leq \pi/2$ for the Euclidean turning angle is also sufficient to exclude (use Gauß-Bonnet) that the normals at the endpoints of the hyperbolic arc meet the arc before they intersect each other.

Next we deal with the +arcs. We will prove the inequality $A < l$ and we want to increase ω_+ to values well above the Euclidean turning angle $\int \kappa ds$ just by increasing λ. Unfortunately this means that we cannot reach $\omega_+ = \pi/2$ with an intermediate value argument; our following proof will reach $\pi/2$, but without an a priori bound on l. Such a bound would be convenient in view of such a bound for the $-$arcs. Therefore we cannot extend our construction from the above mentioned Platonic tessellations to tessellations by Platonic prisms (given by infinite orthogonal prisms over a Platonic tessellation of a hyperbolic plane).

Lemma 1

On the area A between a convex arc of hyperbolic length l and its end point normals.
ASSUMPTIONS. Consider a hyperbolic arc with curvature $\kappa_h \geq 1$ and such that the normals at the endpoints intersect each other with interior angle $\leq \pi$ (and do not meet the arc except at their foot point).
CLAIM.

$$A \leq \int_0^l (\kappa_h - \sqrt{\kappa_h^2 - 1}) ds = \int_0^l \frac{ds}{\kappa_h + \sqrt{\kappa_h^2 - 1}} \leq l \tag{1}$$

PROOF. By assumption the arc and its end point normals bound a convex domain. Connect any interior point p to the endpoints of the arc by shortest geodesics, both must meet the arc with an angle $< \pi/2$. The endpoints are therefore *not* the nearest points on the arc from p and a nearest point q to p must exist as an *interior* point of the arc. Then p is not beyond a focal point on the normal of the arc in q. This means that we can get an upper bound for the area if we integrate in parallel coordinates of the arc along each normal up to the focal point. The focal distance r_f at a point of curvature $\kappa_h \geq 1$ is given by $\kappa_h = \coth r_f$. The line element in parallel coordinates (t, r) is

$$ds^2 = dr^2 + (\cosh r - \kappa_h(t) \sinh r)^2 dt^2.$$

Therefore we get the following bound for the area:

$$A \leq \int_0^l \int_0^{r_f(t)} (\cosh r - \kappa_h(t) \sinh r) dr\, dt = \int_0^l (\sinh\ r_f(t) - \kappa_h(t)(\cosh\ r_f(t) - 1)) dt.$$

Now use the relation $\kappa_h = \coth r_f$ to simplify the integrand:

$$\begin{aligned}\sinh\ r_f(t) - \kappa_h(t)(\cosh\ r_f(t) - 1) &= \sinh\ r_f(t) - \cosh\ r_f(t)^2 / \sinh\ r_f(t) + \kappa_h(t) \\ &= \kappa_h(t) - 1/\sinh r_f(t) = \kappa_h - \sqrt{\kappa_h^2 - 1}.\end{aligned}$$

The integrand is strictly < 1 unless $\kappa_h(t) = 1$ and the focal distance is infinite; this limit is included in the proof.

Lemma 2

On the change of the normal angle of $+$arcs under scaling.
ASSUMPTIONS. Consider a fixed $+$arc on a minimal surface in $\mathbb{R}^3$ with curvature function $\kappa \geq 0$ and total turning angle $\int \kappa\ ds < \pi/2$. Take it so small that the normal angle ω_+ of its hyperbolic cousin is still $\leq \pi/2$. The $+$arc and its scalings define hyperbolic symmetry arcs on cmc1 surfaces with curvature $\kappa_h(s) = \frac{1}{\lambda}\kappa(s/\lambda) + 1$. As long as the normal angles

satisfy $\omega_+(\lambda) \le \pi/2$ we have the
CLAIM.

$$\omega_+(\lambda) - \int \kappa \;=\; l(\lambda) - A(\lambda) \ge \int \frac{2\kappa(t)\; dt}{1 + 2\kappa/\lambda}, \tag{2.1}$$

$$\omega_+(\lambda) - \int \kappa \;=\; l(\lambda) - A(\lambda) \ge \sqrt{2\lambda} \int \frac{\sqrt{\kappa(t)}dt}{1 + \sqrt{2\kappa/\lambda}}. \tag{2.2}$$

PROOF. Insert the expression $\kappa_h(s) = \frac{1}{\lambda}\kappa(s/\lambda) + 1$ in the upper bound for A of the preceeding lemma to get the lower bound

$$l(\lambda) - A(\lambda) \ge \int_0^{\lambda l} (1 - \left(\kappa_h(s) - \sqrt{\kappa_h(s)^2 \;-1}\right))ds = \int_0^l (\sqrt{1 + 2\lambda/\kappa(t)} - 1)\kappa(t)dt.$$

Then use $\sqrt{a} - 1 = (a-1)/(\sqrt{a}+1)$ with $a = 1 + 2\lambda/\kappa$ and simplify in the denominator with $\sqrt{1+2x} \le 1 + x$, respectively with $\sqrt{1+x} \le 1 + \sqrt{x}$ to get the two lower bounds. The lower bound (2.1) increases with λ to $\int 2\kappa dt$. If this makes ω_+ as large as we want (at most π), then we have an a priori upper bound for λ in terms of $\max \kappa$, i.e. a bound for the required length of the hyperbolic arc . The lower bound (2.2) increases with λ to infinity, but less explicitly. – In the first examples below one would get a better understanding if one had an upper bound better than $l(\lambda) - A(\lambda) \le l(\lambda)$; focal point arguments do not give that.

Illustrating examples, the symmetric n-noids.

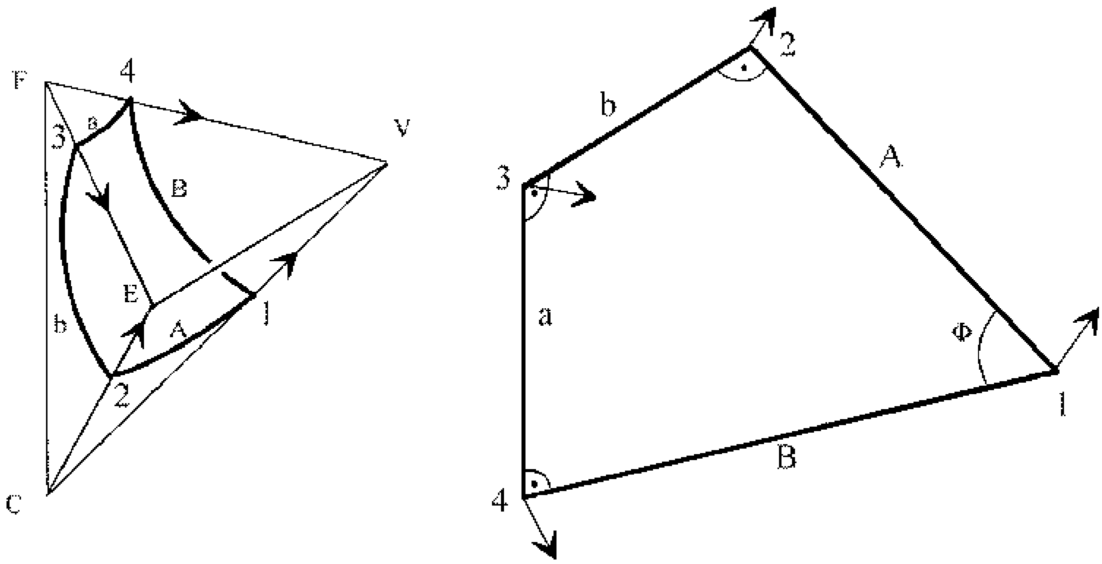
These surfaces have already been constructed in [UY]. With the conjugate cousin method they are simpler than the following *compact* fundamental domain examples. Therefore they are constructed here to explain this method. It will be sufficient to explain the cmc1 three-noid. For a qualitative picture place a small Euclidean minimal three-noid at the center of a ball model of $\mathbb{H}^3$ and let small horospheres around the limit points of the three axes of the half-catenoids grow until they touch the minimal three-noid; imagine that the three horospheres are connected by the central piece of the three-noid. Notice that the underlying Riemann surface is a 3-punctured sphere and the hyperbolic Gauss map is of degree two; the Weierstraß representation construction starts from here. Such a surface has the same planar symmetries as the minimal three-noid, one equator symmetry plane and three planes orthogonal to it which intersect in the normal line through the two umbilic points of the three-noid. These symmetry planes cut the surface into six simply connected (congruent) pieces, each of them bounded by three planar symmetry arcs. The first arc (a) comes from infinity to the umbilic point, the next (b) starts from there making a 60° angle with the first arc and meets the last one, the arc (c), in the equator plane orthogonally. Arc (c) then goes back to infinity. The length of the finite arc (b) we call $|b|$. The minimal surface in $\mathbb{R}^3$ of which the cmc1 surface patch is the conjugate cousin, is bounded by two half lines (a), (c) which are connected by a segment (b) which meets (a) under 60° and

(c) under 90°. If we project this contour in the direction of (b) then we obtain an infinite sector, the angle ω of which parametrizes a 1-parameter family of solutions. We need to assume (because our lemmas are not optimal) $\omega < \pi/2$. The Plateau solution is a graph, which implies that the tangent plane along (b) rotates only in one direction. We orient the normal so that the conjugate arc of (b) has curvature $\kappa \geq 0$, because we want to consider it as the +arc. (On the cmc1 surface we denote the corresponding boundary arcs also (a), (b) and (c).)
Now we apply our lemmas. As long as $|b| < \pi/2 - \omega$ we have for the angle between the end point normals of the corresponding hyperbolic arc $\omega_+ \leq \omega + |b| < \pi/2$. On the other hand, the lower bound (2.2) for $l - A$ (which grows with λ to ∞) shows that we can scale the Plateau contour enough to make $\omega_+ = \pi/2$. This says that the normal of the cmc1 surface rotates along the finite symmetry arc (b) by $\pi/2$, or in other words, the symmetry plane of (a) meets the symmetry plane of (c) orthogonally. This completes the existence proof since the intersection line (in $\mathbb{H}^3$) of the symmetry planes of (a),(b) meets the symmetry plane of (c) with the angle ω_+; consequently: the reflections in these symmetry planes make a smooth surface out of six copies of the fundamental piece. – If the Euclidean turning angle ω is very close to $\pi/2$, then the lower bound (2.1) for $l(\lambda) - A(\lambda)$ shows that κ/λ has to be large, i.e. $|b|$ is small. In the other direction, one could suspect that $\omega \to 0$ implies $|b| \to \infty$, but, as mentioned in the lemma, we miss a sufficiently good upper bound for $l - A$ and cannot prove that.

The quadrilateral Plateau contours needed for the Platonic examples.

The surfaces we want to construct are already vaguely suggested by the Schwarz P–surface, a triply periodic minimal surface in $\mathbb{R}^3$ which meets all the boundary squares of a cubical tessellation of $\mathbb{R}^3$ orthogonally in convex almost circular curves. The pictures of Polthier's minimal surfaces in each hyperbolic Platonic solid give a more precise idea. Analoguous constant mean curvature (> 1) surfaces can be described more easily: take a sphere inscribed in a Platonic solid and puncture it where it touches the faces in their midpoints; shrink the sphere a bit and replace the punctures by small necks which meet the faces orthogonally in convex curves around the face midpoints. This limit of surfaces with extremely small necks does not exist for cmc1 surfaces in $\mathbb{H}^3$ since the horospheres are too big for our Platonic solids. Therefore it is not a priori plausible that this picture will lead to an existence proof, and we will see at the end that it only succeeds because of explicit angle properties of our Euclidean Plateau contours.
The symmetry planes of any Platonic solid cut this solid into tetrahedral fundamental domains, described above; they also cut a Schwarz type surface inside the solid (see Polthier's pictures above) into fundamental pieces which lie in these tetrahedra and meet their faces orthogonally. I have indicated such quadrilateral fundamental domains on all of the preceeding pictures. The simplest such situation has been studied by Smyth [Sm] for minimal surfaces in Euclidean tetrahedra: these surface patches meet each face of the given

tetrahedron once; they miss two edges and they meet the other four edges orthogonally, so that the angle between the neighbouring boundary arcs of the patch is the dihedral angle at that tetrahedron edge. This description implies a 2-parameter family of quadrilateral contours for the conjugate minimal surface patch in $\mathbb{R}^3$ and the existence proof starts with these: one takes the Plateau solution to any such contour and proves that the symmetry planes of *its* conjugate patch indeed bound (up to scaling) the tetrahedron from which Smyth started.



Patch in a fundamental tetrahedron with angles $\pi/2$ at edges 2,3,4 and quadrilateral polygon contour of conjugate patch. The arrows are normal vectors.

Our initial situation is more special, since our tetrahedra are fundamental domains for the symmetry groups of (hyperbolic) Platonic solids. But the required control is more difficult since it cannot be done explicitly in terms of the conjugate contour. The fundamental surface patches do not meet two edges of the fundamental tetrahedron, not the edge CF from the center to the face nor the edge EV from an edge midpoint of the Platonic solid to a vertex. The surface patch meets the other edges orthogonally so that the dihedral angles of the tetrahedron are also the angles between the bounding arcs of the patch. The boundary arc (a) of the patch in the face FEV meets the boundary arc (b) in the face EFC with the dihedral angle $\pi/2$ of the edge FE. The boundary arc (A) of the patch in the face EVC and the boundary arc (B) in the face FVC meet with the dihedral angle $\phi = \pi/n$ of the edge VC (n is the number of edges of the solid that meet at the vertex V). The boundary arcs (b) and (A) meet with the dihedral angle $\pi/2$ of the edge CE, and the boundary arcs (a) and (B) meet with the dihedral angle $\pi/2$ of the edge FV. (To see the three dihedral angles $\pi/2$, note that the edge CF is orthogonal to the face FEV, and the edge VE is orthogonal to the face EFC.) The four angles at the vertices of the patch are also the angles of the quadrilateral conjugate contour, they are therefore easily achieved. However, the four planes of the boundary arcs of the patch also intersect in two lines which are *not met* by the patch: along the edge CF with a dihedral angle π/k (where k is the number of vertices of a face of the Platonic solid) and along the edge EV with the dihedral angle π/m, where $2\pi/m$ is the dihedral angle of the Platonic solid. The special form of this last angle is not needed for the conjugate construction, but it is needed for the solids to tessellate $\mathbb{H}^3$. This leaves us with the following two conditions: the normals at the endpoints of the arc (a) must intersect with the angle π/k, and the normals at the endpoints of the arc (b) must intersect with the angle π/m. Indeed, if we can construct such a patch then completion by repeated reflection in the planes of its planar bounding arcs gives the desired cmc1 surface.

Therefore we can now describe the families of conjugate contours: start with two half lines (A), (B) which meet under the angle ϕ, take a half line (a) which meets (B) under the angle $\pi/2$ and take the common perpendicular (b) of (a) and (A). For each given ϕ this gives a 2-parameter family of quadrilaterals. If we start with the Plateau solutions for these contours and take their conjugate cousin cmc1 patches then, indeed, these are bounded by planar symmetry arcs which meet under the correct angles $\pi/2,\ \pi/2,\ \pi/2,\ \phi$; their planes therefore intersect in a hyperbolic tetrahedron which has already four correct dihedral angles. It remains to show with lemmas 1 and 2 that we can choose the quadrilaterals to get also the other two dihedral angles correct. If the Platonic solid (see the above list) has k-gon faces then the correct angle α_{cor} between the end point normals of the arc (a) has to be $\pi/k,\ k=3,4,5$. If the Platonic solid has dihedral angles $2\pi/m$ then the correct angle β_{cor} between the end point normals of the arc (b) has to be $\pi/m,\ m=3,4,5,6$.

Explicit determination of the Euclidean data of the contour quadrilaterals.

Let α be the angle through which the tangent plane of the Plateau solution rotates along the (a)-edge (this is also the angle between the direction vectors of (b) and (-B)); let β be the corresponding rotation angle for the (b)-edge. – If we position (b) vertically, then we see a right triangle: (a), (A) are horizontal with angle β between them and the projection of (B) is orthogonal to (a). Similarly, if we position (a) vertically. This gives two relations

$$a = A\cos\beta, \quad b = B\cos\alpha.$$

Because of the right angles between (a),(B) and also between (b),(A) we can express the diagonal between the other two vertices as $D^2 = a^2 + B^2 = b^2 + A^2$. We eliminate a, b (or instead A, B) with the previous equations and obtain

$$B\sin\alpha = A\sin\beta, \quad b\tan\alpha = a\tan\beta.$$

The length of the other diagonal gives $d^2 = a^2 + b^2 = A^2 + B^2 - 2AB\cos\phi$; eliminate first a, b then A/B to obtain

$$\cos\phi = \sin\alpha \cdot \sin\beta.$$

Control of the angle between the end point normals.

We orient the normals to make the curvature of (a) positive, that of (b) negative. The strategy then is: For all a/b in a certain range we can, by simply scaling the quadrilateral from a very small size up, decrease β_h from the Euclidean value $\le \pi/2$ to the correct

value without increasing α_h beyond $\pi/2$. Moreover, for $a/b \ll 1$ the value of α_h stays below the correct value and for $a/b \approx 1$ the value of α_h stays above the correct value. The intermediate value argument with respect to the parameter a/b achieves the correct normal angles for (a) and (b), so that the Schwarz type cmc1 surfaces exist for all the tessellating Platonic solids of $\mathbb{H}^3$.
Now the details. Recall that $\phi = \pi/n$, n the number of edges at a polyhedron vertex, $\alpha_{cor} = \pi/k$, $k =$ the number of edges of a polyhedron face and $\beta_{cor} = \pi/m =$ half the dihedral angle of the polyhedron.

	tetrahedron	cube	dodecahedron	octahedron	icosahedron
$\pi/2 - \phi$	30	30	30	45	54
α_{cor}	60	45	36	60	60
β_{cor}	30	36,30	45,36,30	45	60

First, if $a/b \ll 1$, then we show how to achieve $\beta_h = \beta_{cor}$ with α_h staying below α_{cor}. In the limit $a/b \to 0$ we have the following rotation angles $\alpha_e = \pi/2 - \phi$, $\beta_e = \pi/2$ of the Euclidean quadrilateral. As we scale the quadrilateral down, the hyperbolic angles α_h, β_h converge to the Euclidean ones. We check in the above list of correct angles that β_e is well above all desired values for β_h and α_e is smaller by at least $6° = 36° - 30° = 60° - 54°$ than all the correct values α_{cor}. Therefore we start with a very small quadrilateral and scale it up at most to a length $l_b \le \pi/2 - \beta_{cor}$ to achieve $\beta_h = \beta_{cor}$. If we start with $a/b < 6/90$ then the trivial bound $\alpha_h \le \alpha_e + l_a$ shows that α_h does not increase above α_{cor}.
Secondly, we show how to achieve $\beta_h = \beta_{cor}$ while holding α_h between α_{cor} and $\pi/2$:
Consider the first three cases which have $\phi = 60°$. We start with quadrilaterals with $\alpha_e = \alpha_{cor}$ and check with $\sin\beta_e := \cos\phi/\sin\alpha_e$ that in all cases $\beta_e > \beta_{cor}$. Again, by scaling up from a very small size we can decrease β_h to β_{cor}, while α_h increases further and we only have to hold it below $\pi/2$ in (i) to (iii).
(i) The three dodecahedra. $\alpha_e = 36°$ hence $\sin\beta_e := \cos\phi/\sin\alpha_e = \sin 58.283°$, and the desired correct values are $\beta_{cor} = 45°,\ 36°,\ 30°$. Since $a < b$ we can decrease β_h to the smallest value $30°$ without α_h growing above $(36 + 28.283)°$. This proves existence for the dodecahedra.
(ii) The two cubes. $\alpha_e = 45°$ hence $\beta_e = 45°$ and $a = b$. Therefore we can decrease, by scaling, β_h from the Euclidean limit down to the interesting values $\beta_{cor} = 36°,\ 30°$ without α_h growing by more than the needed decrease, i.e. $\alpha_h \le 60°$. This proves existence for the cubes.
(iii) The $60°$-tetrahedron. $\alpha_e = 60°$ hence $\beta_e = 35.264°$ and $a/b = \tan\alpha/\tan\beta \le 2.45$. Again we can decrease, by scaling, β_h from the Euclidean limit β_e down to the desired value $\beta_{cor} = 30°$ without increasing α_h to more than $(60 + 2.5 \cdot 5.3)° < 75°$. This proves existence for the infinite tetrahedron.
The $120°$-icosahedron: Choose $\phi = 36°$ and $a = b$. Then $\cos 36° = (\sin 64.086°)^2$ shows that we need to scale only to a length $l_b \le 4.086/180 \cdot \pi$ to decrease β_h from the Euclidean value $64.086°$ for very small quadrilaterals to the desired $\beta_{cor} = 60°$. Here α_h increases at

most by the same amount because of $l_a = l_b$, in particular stays below $\pi/2$. q.e.d.
The 90°-octahedron: $\phi = 45°$; $\cos\phi = \sin\alpha \cdot \sin\beta = \sin 60° \cdot \sin 54.735°$ shows that we have to decrease β_h by less than 10° from 54.735° to 45°, and from $a/b = \tan\alpha / \tan\beta = 1.225$ follows that α_h does not increase more than 12.25°. q.e.d.

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