Stress Energy Tensor

Simple Examples and Geometric Consequences, a Schur Theorem

Notational conventions. For the Ricci Tensor I will use different names for its bilinear version: $ric(v, w)$ and its 1-1-Tensor version: $Ric(v)$, and of course: $ric(v, w) = g(Ric(v),\ w)$. The divergence free part of the Ricci Tensor is the Einstein Tensor G:

$$G := Ric - \frac{1}{2}(\mathrm{trace}\, Ric) \cdot \mathrm{id}\,, \qquad \mathrm{trace}\, G = -\mathrm{trace}\, Ric.$$

The Einstein Equation

$$8\pi T = G + \Lambda \mathrm{id}\,.$$

Of course, without further words this means nothing: One could take any Lorentz manifold, compute $(G + \Lambda \mathrm{id}\,)/(8\pi)$ and call the result the stress energy of the matter in that universe. This is not the intended use of the equations. Rather one should have an opinion what kind of matter is in the universe one intends to model, one should understand this matter well enough to be able to write down its stress energy tensor and finally look for a Lorentz manifold such that the Einstein Equation is satisfied. For how much complication should we be prepared? First, of course, there are the stars. It turned out that for modeling ordinary stars one does not need General Relativity. And the more exotic stars, imploding ones for example, require so broad a background in physics that they are out of my reach. We have seen the Schwarzschild geometry and glimpses of Kerr as models of the outside of a star. The next larger structures are galaxies and eventually the cosmology. I want to recall a very successful continuous model of an obviously discrete situation: *the kinetic theory of gases* in terms of differentiable functions called volume, pressure and temperature. A gas consists of molecules of diameter $10^{-10}m$ and up, and their mean distance is about a factor 30 larger. Our galaxy has a diameter of about 50.000 light years and the distance to the Andromeda galaxy is about 20 times that large. It will turn out that a cosmological model in which the matter is a dust of mass density ρ and the dust grains are the galaxies (in other words: a very oversimplifying assumption) is surprisingly successful. And for the galaxies themselves, the ratio of distances between stars to star diameters is more like 10^7 and therefore maybe too large for a continuous approximation. (I have been told that the shuttle reentry computations in the very thin high atmosphere do not describe the "gas" by using a very small continuous density, but really deal with individual molecules.) Very recently I obtained the following reference:

1995 Phys. Rev. Letters 75, 3046 , Neugebauer, G.; Meinel, R.: General Relativistic Gravitational Field of a Rigidly Rotating Disk of Dust: Solution in Terms of Ultraelliptic Functions .

I did not have time to see what one can learn from it, the words *"rigidly rotating"* do exclude that it is a galactic model. Concerning galaxies. I know that really huge numerical simulations have been made, but I do not know any details. Therefore, with obvious regret, I cannot discuss relativistic models of galaxies in these notes.

The remaining goal therefore is to discuss a family of cosmological models that are filled with a very simple type of matter. We will not meet complicated stress energy tensors, but in the same way as the detailed discussion of our first vacuum solution (Schwarzschild) turned out to be very educational we will gain insight about the interplay between matter and geometry on a cosmological scale even though we work with the simplest kind of matter that can be imagined. Before turning to that goal I end this section with definitions and with some more local arguments.

A matter is called a **perfect fluid** if it has just two physical properties called *pressure* p and *mass density* ρ (p, ρ are differentiable functions) and if at every point in the rest system of the matter (this makes sense only where $\rho \neq 0$) the stress energy 1-1-tensor T has the rest space as 3-dim eigenspace with eigenvalue p and the time like unit vector U of the rest frame is an eigenvector with eigenvalue $-\rho$. Since U is defined everywhere, it is a time like unit vector field whose integral curves are the world lines of the matter particles. Note that the infinitesimal rest spaces $U^\perp$ in general are **not** an integrable distribution. This means that in general there are no natural space slices. This phenomenon will be obscured by our examples: additional simplicity assumptions make $U^\perp$ integrable and therefore lead to natural space slices. I find it important to emphasize that even with all the specifics above we do not yet have some physically specific perfect fluid. In addition one needs a

matter equation or *equation of state*: $F(p, \rho) = const,\ \frac{\partial}{\partial p} F \neq 0.$

We shall mainly work with the equation $p = 0$ that specifies a **dust**.
We shall mention $3p - \rho = 0$ specifying a perfect fluid called **photon gas**.
In the absence of a matter equation the following inequalities are required: $0 \leq 3p \leq \rho$.
My knowledge of continuum mechanics is insufficient for comments about these inequalities.

Next we translate the given information about T, using the Einstein equation, in information about Ric:

For arbitrary vectors W holds:
$$T \cdot W = (p \cdot W + (\rho + p) g(U, W) \cdot U$$
$$8\pi \cdot \mathrm{trace}\,(T) = \mathrm{trace}\,(Ric) - 2\mathrm{trace}\,(Ric) - 4\Lambda$$
$$Ric = 8\pi (T - \frac{1}{2}\mathrm{trace}\,(T)\mathrm{id}\,) + \Lambda \mathrm{id}$$
$$Ric(U) = (\Lambda - 4\pi(\rho + 3p)) \cdot U, \quad Ric\big|_{U^\perp} = (\Lambda + 4\pi(\rho - p)) \cdot \mathrm{id}\,\big|_{U^\perp}.$$

By looking at the Ricci tensor we can now recognize whether some Lorentz manifold has as its matter content a perfect fluid. The quadratic examples of lecture 2 do not model such type of matter.

Recall that, when Einstein wrote down the above field equation, physicists had already met stress energy tensors of materials and they were convinced that T would be divergence free for all materials. Therefore Einstein constructed the right side of the equation to be divergence free. We learn some facts about perfect fluids by computing the divergence of T:

$$\mathrm{div}\,(T) := \sum_i \frac{(D_{e_i} T) \cdot e_i}{g(e_i,\ e_i)} \Longrightarrow g(\mathrm{div}\,(T), W) = \sum_i \frac{g((D_{e_i} T) \cdot W, e_i)}{g(e_i,\ e_i)}$$
$$g(\mathrm{div}\,(T), W) = T_W p + (p + \rho) g(W, D_U U) + g(W, U) \mathrm{div}\,((p + \rho) U).$$

If we *use* $\operatorname{div}(T) = 0$ and apply this computation for $W \perp U$, then we get

$$D_U U = -(\operatorname{grad} p)/(p+\rho), \quad \operatorname{grad} = \operatorname{grad}^{Restspace}$$

in particular, in the case of dust, we get *geodesic world lines* for the dust particles. In general the acceleration is caused by the pressure gradient (in the rest space).
If we use the computation for $W = U$ in the dust case, we get $\operatorname{div}(\rho \cdot U) = 0$, a conservation of mass result. This shows that quite basic facts about the behavior of the perfect fluid follow from the Einstein field equation without prior knowledge of these facts from classical physics.

What is $\operatorname{div} T = 0$ good for?

If in some field theory a vector field V with $\operatorname{div}(V) = 0$ occurs then Gauß' theorem implies that the flow of V carries some conserved quantity around. However, there is no Gauß' theorem for 1-1-tensors and therefore: why is $\operatorname{div} T = 0$ important? A celebrated fact from classical mechanics is the observation that symmetry groups, or Killing fields, lead to conserved quantities. And Killing fields X (characterized by the skew-symmetry of their covariant differential, $DX = -DX^{tr}$) are similarly useful in our context:

Claim: $\operatorname{div} T = 0$ and $DX = -DX^{tr} \implies V := T \cdot X$ satisfies $\operatorname{div}(V) = 0$.

Proof: $DV = (DT) \cdot X + T \cdot DX,$

$\operatorname{div}(V) = \operatorname{trace}(DV),$

$\operatorname{trace}(T \cdot DX) = 0$ since T is symmetric and DX is skew,

$$\operatorname{trace}((DT) \cdot X) = \sum_i \frac{g((D_{e_i}T) \cdot X, e_i)}{g(e_i, e_i)} = \sum_i \frac{g((D_{e_i}T) \cdot e_i, X)}{g(e_i, e_i)}$$
$$= g(\operatorname{div}(T), X) = 0.$$

This shows that the divergence free stress energy tensor T together with any Killing field X leads to a divergence free vector fields $V = T \cdot X$, i.e. to vector fields V whose flow transports some conserved quantity. This observation makes $\operatorname{div} T = 0$ important, if there are Killing fields. Not surprisingly do our simplified models carry Killing fields, but on a real cosmology with all its individual features there won't be Killing fields. Is $\operatorname{div} T = 0$ still important? I will argue "yes, and for almost the same reason".

First recall that in Euclidean space and in Minkowsky's Special Relativity Killing fields are explicitly determined by value and derivative at one point:

$$X(x) = X(p) + DX\big|_p \cdot (x - p).$$

Secondly, an observing physicist, of course, cannot leave his world line. Moreover we have by now some experience in viewing physicists as infinitesimal observers who perform their experiments in the tangent spaces of the Lorentz manifold, along their world line. This means that for observing conserved quantities they do not really need globally defined Killing fields, what they need are "almost" Killing fields defined on a tube around their world line. Recall that a Killing field satisfies along any geodesic γ(i.e. along the world line of any unaccelerated observer) and for any *parallel* field v along γ the following PDE:

$$D_{\gamma'}(D_v X) + R(X,\gamma')v = 0.$$

This says: X and DX is determined along γ by its initial value $X(\gamma(0))$ and its initial derivative $DX\big|_{\gamma(0)}$, just as in the Euclidean/Minkowski case. Of course $DX\big|_{\gamma(0)}$ needs to be skew-symmetric, but if this initial constraint is met then $DX\big|_{\gamma(s)}$ continues to be skew-symmetric:

$$\frac{d}{ds}g(D_{v(s)}X, v(s)) = -g\Big(R\big(X(s),\gamma'(s)\big)v(s),\ v(s)\Big) = 0.$$

We can therefore construct as many almost Killing fields X on an infinitesimal tube around γ as we have in Special Relativity and $\operatorname{div} T = 0$ allows us to observe the conserved quantities of the flows of the fields $V := T\cdot X$, so that $\operatorname{div} T = 0$ is really responsible for observable conserved quantities.

Interplay with Conformal Flatness.

We are interested in conformally flat Lorentz manifolds because then we get solutions of Maxwell's equation for free. A (pseudo)-Riemannian metric is (locally) conformally flat iff its Weil conformal curvature tensor vanishes. In such a case one can write the full curvature tensor in terms of the Ricci tensor. In the case of a perfect fluid we saw that the Ricci tensor does not distinguish any space like directions in the rest spaces of the matter. Taking the two facts together shows:

A conformally flat perfect fluid is curvature isotropic.

We write more explicitly what we mean by "curvature isotropic with respect to U", i.e., by the property that *the curvature tensor distinguishes no directions in the rest spaces $U^\perp$ of the matter.* Clearly, such a curvature tensor has to have the following properties:

$$X,Y,Z \perp U \Longrightarrow R(X,Y)Z = k(p)(g(Y,Z)X - g(X,Z)Y),$$
$$R(X,U)U = \mu(p)\cdot X,$$

with the immediate consequences:

$$R(X,Y)U = 0,$$
$$R(U,X)Y = -\mu(p)\cdot g(X,Y)\cdot U.$$

(Note that $g(R(U,X)Y,Z) = 0$ for all $Z \perp U$ and $g(R(U,X)Y,U) = g(R(X,U)U,\ Y)$.)
This is enough information about the curvature tensor to check that any curvature isotropic curvature tensor has its Weyl conformal curvature tensor vanish, so that the manifold is locally conformally flat. Moreover, we find for the Ricci tensor (of such a curvature tensor):

$$\begin{aligned}
ric(U,U) &= 3\mu(p) && = -\lambda_U = (-\Lambda + 4\pi(\rho+3p))\\
ric(U,Y) &= 0 &&\\
ric(X,Y) &= (2k-\mu)g(X,Y) && = \lambda_{U^\perp} = (\Lambda + 4\pi(\rho - p)).
\end{aligned}$$

This shows that the eigenspace decomposition is the correct one for a perfect fluid (we also need to satisfy $0 \le 3p \le \rho$), so that, essentially, "conformally flat perfect fluid" and "curvature isotropic space" describe the same Lorentz manifolds.
Note:

$$6k - 2\Lambda = 16\pi\rho, \quad 4\mu - 2k + 2\Lambda = 16\pi p, \quad \mu + k = 4\pi(p + \rho).$$

After introducing the concepts and show immediate relations we come to a real theorem:

Theorem of Schur type. Let M^4 be curvature isotropic for a time like unit vector field U so that M^4 models a perfect fluid. We also assume $\rho > 0$, since otherwise one cannot everywhere define the local rest frame of the matter, namely $U, U^\perp$. Then:

a) $U^\perp$ is an **integrable** distribution.
b) The 3-dim integral manifolds have intrinsically constant curvature.
c) A matter equation $F(p,\rho) = 0$, $\frac{\partial}{\partial p}F \neq 0$ implies $D_U U = 0$ so that extrinsically these integral manifolds are parallel hypersurfaces with the matter world lines as the orthogonal geodesics.

The **proof** is modeled after Schur's theorem for Riemannian manifolds that states: *If the sectional curvatures are constant at each point then they are constant.* The argument relies on the 2nd Bianchi identity, we will use

$$0 = (D_U R)(X,Y)Z + (D_X R)(Y,U)Z + (D_Y R)(U,X)Z.$$

(Other combinations of arguments do not contain additional information.) Our curvature assumptions are such that the orthogonal splitting $T_pM = U(p)\mathbb{R} \oplus U^\perp$ is essential. Therefore we will use the induced covariant derivative $D^\perp$ on the 3-dim bundle $U^\perp$ over M. By X, Y, Z we will always denote vector fields from that bundle.

$$D^\perp X := DX + g(DX, U) \cdot U \perp U.$$
$$D^\perp_{\dot c} X = 0 \;\Rightarrow\; D_{\dot c} X = -g(D_{\dot c} X, U) \cdot U = g(X, D_{\dot c} U) \cdot U.$$

Clearly, $D^\perp$-parallel vector fields have constant scalar products. For the evaluation of the terms in the Bianchi sum we may assume that the vector fields $X, Y, Z \perp U$ are $D^\perp$-parallel in the direction of the differentiation field. Now compute the Bianchi sum terms:

First: $D_U(R(X,Y)Z) = dk(U)(g(Y,Z)X - g(X,Z)Y) + k(g(Y,Z)D_U X - g(X,Z)D_U Y)$.

Since $D_U X, D_U Y, D_U Z$ are proportional to U we have

$$R(X,Y)D_U Z = 0, \;\; R(D_U X, Y)Z = -\mu g(Y,Z) D_U X. \;\; R(X, D_U Y)Z = \mu g(X,Z) D_U Y$$

(1) $$(D_U R)(X,Y)Z = dk(U)(g(Y,Z)X - g(X,Z)Y) \quad \perp U$$
$$+ (k+\mu)(g(Y,Z)D_U X - g(X,Z)D_U Y) \quad \in U\mathbb{R}$$

Second: $$D_X(R(U,Y)Z) = -d\mu(X) g(Y,Z) \cdot U - \mu g(Y,Z) D_X U.$$

Again, the derivatives of the arguments are either parallel or orthogonal to U, hence

$$R(D_X U, Y)Z = k(g(Y,Z)D_X U - g(D_X U, Z)Y), \quad R(U, D_X Y)Z = 0,$$
$$R(U,Y)D_X Z = -\mu g(Z, D_X U)Y \quad \text{(recall } D_X Z = g(Z, D_X U)U \text{)}$$
$$\begin{aligned}(2)\quad (D_X R)(Y,U)Z &= -(D_X R)(U,Y)Z \\ &= d\mu(X)g(Y,Z)U \quad \in U\mathbb{R} \\ &+ (k+\mu)(g(Y,Z)D_X U - g(Z, D_X U)Y) \quad \perp U.\end{aligned}$$

And similarly (interchange X and Y and a sign)

$$\begin{aligned}(3)\quad (D_Y R)(U,X)Z &= -d\mu(Y)g(X,Z)U \quad \in U\mathbb{R} \\ &- (k+\mu)(g(X,Z)D_Y U - g(Z, D_Y U)X) \quad \perp U.\end{aligned}$$

Using the 2nd Bianchi identity in (1)+(2)+(3) gives two equations, one in $U\mathbb{R}$, one in $U^\perp$:

In $U\mathbb{R}$ $\; d\mu(X)g(Y,Z)U - d\mu(Y)g(X,Z)U = -(k+\mu)(g(Y,Z)D_U X - g(X,Z)D_U Y)$,

In $U^\perp$ $\; dk(U)(g(Y,Z)X - g(X,Z)Y) =$

$$= (k+\mu)\left(g(Y,Z)D_X U - g(X,Z)D_Y U + g(D_Y U, Z)X - g(D_X U, Z)Y\right).$$

If we use unit vectors $X \perp Y = Z$ in the first equation we get

$$d\mu(X) = -(k+\mu)g(X, D_U U),$$

we computed earlier

$$\begin{aligned}\operatorname{div}(T) = 0 \quad \Longrightarrow \quad 8\pi dp(X) &= -8\pi(\rho+p)g(X, D_U U) = \\ &= d(2\mu - k)(X) = -2(\mu+k)g(X, D_U U),\end{aligned}$$

and both equations together give

$$dk(X) = 0 \quad \text{for all} \quad X \in U^\perp$$

This shows: if ρ, hence k, are not konstant then the levels of ρ are the integral manifolds of the distribution $U^\perp$.

We still have to consider the case of constant k since the absence of matter equations makes still many examples possible. Therefore we need another proof of the integrability of the distribution $U^\perp$. We claim, the vector field $(k+\mu)U$ has a symmetric covariant differential and therefore is (locally) the gradient of a function, and since $(k+\mu) > 0$ is implied by our assumption $\rho > 0$, this proves integrability of $U^\perp$. To see the claim, first put orthonormal vectors X, Y, Z in the second part of the above Bianchi equation to obtain

$$0 = (k+\mu)(g(D_Y U, Z)X - g(D_X U, Z)Y), \quad \text{hence} \quad g(D_Y U, Z) = 0.$$

This says that for any orthonormal basis in $U^\perp$ the matrix of $DU\big|_{U^\perp}$ is diagonal, in particular symmetric. It remains to check, with the above equations, the remaining symmetry:

$$g(D_X((k+\mu)U), U) = g(d\mu(X)U, U) = (k+\mu)g(X, D_U U) = g(D_U((k+\mu)U), X),$$

and thus prove the integrability of $U^\perp$ in all cases. We emphasize that this integrability was deduced from strong assumptions, it is normally false.

Next we determine the *intrinsic curvature* of the integral submanifolds of $U^\perp$. We also refer to them as *space slices*. The unit (time like) vector field U is of course normal along them. The Weingarten map (=shape operator) of the space slices therefore is $S := DU$ and we proved already that DU is diagonal in any orthonormal basis, i.e., is proportional to id. We use unit vectors $X \perp Y = Z$ in the Bianchi equation involving $dk(U)$. Taking a scalar product with X we obtain:

$$-\frac{dk(U)}{k+\mu} \cdot g(Y,Y) = g(D_X U, X) + g(D_Y U, Y) = 2 \cdot \text{eigenvalue of } S.$$

We use this in the Gauss equation:

$$\begin{aligned} R(X,Y)Z &= k(g(Y,Z)X - g(X,Z)Y) \quad (\text{ assumption about } M^4) \\ &\overset{(Gauss)}{=} R^{Hyp}(X,Y)Z - ((g(SY,Z)SX - g(SX,Z)SY) \cdot g(U,U)^{-1} \\ R^{Hyp}(X,Y)Z &= \left(k - \frac{1}{4}(\frac{dk(U)}{k+\mu})^2\right) \cdot (g(Y,Z)X - g(X,Z)Y). \end{aligned}$$

This shows that the space slices satisfy the assumptions of the Riemannian Schur theorem so that the *curvature value is indeed constant* on each space slice.

Finally we assume a matter equation $F(\rho, p) = 0$, $\frac{\partial}{\partial p} F \neq 0$. Recall that we proved for all $X \perp U$ that $dk(X) = 0$. This says that $\mathrm{grad}\,\rho$ is proportional to U (including 0). Differentiation of the matter equation gives that $\mathrm{grad}\,p$ is proportional to $\mathrm{grad}\,\rho$ (again including 0). Therefore we have for all $X \perp U$ that $0 = dp(X)$, hence

$$0 = dp(X) = -(\rho + p) g(X, D_U U).$$

The integral curves of U, the world lines of the matter particles, are therefore *geodesics* with integrable orthogonal complements $U^\perp$ and these space slices are a family of *geodesically parallel hypersurfaces*. Q.E.D.

Summary of Conformal Changes

Given $\bar{g} = \lambda^{-2} g$

then $\overline{D}_Y Z = D_Y Z + \Gamma(Y, Z)$

with $\Gamma(Y, Z) = -\frac{T_Z \lambda}{\lambda} Y - \frac{T_Y \lambda}{\lambda} Z + g(Y, Z) \mathrm{grad}\, \lambda.$

$$\overline{R}(X, Y)Z = R(X, Y)Z + (D_X \Gamma)(Y, Z) - (D_Y \Gamma)(X, Z) + \Gamma(X, \Gamma(Y, Z)) - \Gamma(Y, \Gamma(X, Z))$$

gives with the abbreviation $B := D \mathrm{grad}\, \lambda$

$$\begin{aligned}\overline{R}(X, Y)Z = R(X, Y)Z &+ \frac{1}{\lambda}\big(g(Y, Z)BX - g(X, Z)BY + (g(BY, Z)X - g(BX, Z)Y\big) \\ &- \frac{1}{\lambda^2} g(\mathrm{grad}\, \lambda, \mathrm{grad}\, \lambda) \cdot (g(Y, Z)X - g(X, Z)Y).\end{aligned}$$

This gives the new Einstein tensor as

$$\overline{G} = \lambda^2 \Big(G + \frac{2}{\lambda} B + \big(\frac{3}{\lambda^2} g(\mathrm{grad}\, \lambda, \mathrm{grad}\, \lambda) - \frac{2 \Delta \lambda}{\lambda}\big) \cdot \mathrm{id}\Big).$$

We have done no computations with the *Weyl conformal curvature tensor*, we list it as a reference:

$$\begin{aligned}&\overline{C}(X, Y)Z = C(X, Y)Z = \\ &= R(X, Y)Z - \frac{1}{n-2}\big(ric(Y, Z)X - ric(X, Z)Y + g(Y, Z)Ric(X) - g(X, Z)Ric(Y)\big) \\ &\qquad + \frac{\mathrm{trace}\,(Ric)}{(n-2)(n-1)}\big(g(Y, Z)X - g(X, Z)Y\big).\end{aligned}$$