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WS 2023/24

Algebraic Geometry I

12. Exercise sheet

Let k be a field.

Exercise 1 (4 Points):
Let $X = \mathbb{A}^2_k$ with k-rational points $x = (0,0), y = (1,1)$. Show the existence of a pushout $S := X \cup_{\{x\} \coprod \{y\}} \operatorname{Spec}(k)$ in the category of k-schemes. Show that S is not normal using sheet 11, exercise 4.

Exercise 2 (4 Points):
Let X be a scheme of finite type over k and let $x \in X(k)$ be a k-rational point. Assume $X = V(f_1, \ldots, f_r) \subseteq \mathbb{A}^n_k$ and write $x = (x_1, \ldots, x_n) \in X(k)$. We define the Jacobi matrix $J_x \in k^{r\times n}$ at x as the $r \times n$-matrix

$$J_x := \begin{pmatrix} \frac{\partial f_1}{\partial X_1}(x_1,\ldots,x_n) & \ldots & \frac{\partial f_1}{\partial X_n}(x_1,\ldots,x_n) \\ \ldots & \ldots & \ldots \\ \frac{\partial f_r}{\partial X_1}(x_1,\ldots,x_n) & \ldots & \frac{\partial f_r}{\partial X_n}(x_1,\ldots,x_n) \end{pmatrix}$$

where

$$\frac{\partial f}{\partial X_i} := \sum j_i a_{j_1,\ldots,j_n} X_1^{j_1} \cdots X_i^{j_i-1} \cdots X_n^{j_n}$$

denotes the i-th partial derivative of a polynomial $f = \sum a_{j_1,\ldots,j_n} X_1^{j_1} \cdots X_n^{j_n} \in k[X_1, \ldots, X_n]$.
1) Let $h\colon \mathbb{A}^n_k \to \mathbb{A}^r_k,\ (y_1,\ldots,y_n) \mapsto (f_1(y_1,\ldots,y_n),\ldots,f_r(y_1,\ldots,y_n))$. Show that there exists an exact sequence

$$0 \to T_xX \to T_x\mathbb{A}^n_k \cong k^n \xrightarrow{J_x} T_{h(x)}\mathbb{A}^r \cong k^r.$$

2) Prove that x is a smooth point of X (as defined on sheet 11) if and only if J_x has rank $n - \dim \operatorname{Spec}(\mathcal{O}_{X,x})$.

Exercise 3 (4 points):
Assume k algebraically closed of characteristic $\neq 2$ and let $Y \subseteq \mathbb{A}^2_k$ be defined by the polynomial $f(x,y) := y^2 - x^3 + x$.
1) Show that every k-rational point of Y is smooth. Deduce that $A := k[x,y]/f$ is a normal domain.
2) Let $R := k[x] \subseteq A$. Show that R is a polynomial ring and that A is integral over R.
3) Show that there exists an automorphism $\sigma\colon A \to A$ which sends y to $-y$, but leaves x fixed.
4) For $a \in A$ set $N(a) := a\sigma(a)$. Show $N(a) \in R$ and that N is multiplicative.
5) Show that $A^\times = k^\times$ and that y, x are irreducible elements in A. Deduce that A is not factorial and that Y is not isomorphic to an open subset of $\mathbb{A}^1_k$.

Exercise 4 (4 points):
Assume $k = \mathbb{F}_p(t)$ with $p \neq 2$. Let $f(x,y) := y^{p-1}x + x^p - t$ and $X := V(f) \subseteq \mathbb{A}^2_k$.
1) Show that $\overline{X} := X \times_{\operatorname{Spec}(k)} \operatorname{Spec}(\overline{k})$ has the single non-smooth $\overline{k}$-rational point $\overline{x_0} := (t^{1/p}, 0)$
2) Show that $V(y) = \{x_0\}$ with $k(x_0) \cong k(t^{1/p})$ and that $\mathcal{O}_{X,x_0}$ is regular while $\mathcal{O}_{\overline{X},\overline{x_0}}$ not.

To be handed in on: Thursday, 25.01.2024 (during the lecture, or via eCampus).