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Algebraic Geometry I

11. Exercise sheet

Let X be a scheme of finite type over a field k. A k-rational point $x \in X(k)$ is called smooth if its *tangent space*

$$T_x X := \mathrm{Hom}_k(\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}, k)$$

has dimension equal to the Krull dimension of $\mathcal{O}_{X,x}$, i.e., $\mathcal{O}_{X,x}$ is regular. Here $\mathfrak{m}_{X,x} \subseteq \mathcal{O}_{X,x}$ denotes the maximal ideal of $\mathcal{O}_{X,x}$.

Exercise 1 (4 Points):

Let k be a field and let X be a normal scheme of finite type over k with $\dim X \leq 1$. Prove that every $x \in X(k)$ is a smooth point of X.
Hint: Use that a 1-dimensional normal local noetherian domain is a discrete valuation ring.

Exercise 2 (4 Points):

1) Let X be a scheme and let $f\colon Z \to X, g\colon Y \to X$ be two closed subschemes with (scheme-theoretic) union $h\colon Z \cup Y \to X$ and (scheme-theoretic) intersection $i\colon Z \cap Y \to X$. Show that the morphism $h_*\mathcal{O}_{Z\cup Y} \to g_*\mathcal{O}_Y \times_{i_*\mathcal{O}_{Z\cap Y}} f_*\mathcal{O}_Z$ is an isomorphism of sheaves of rings.
2) Let $A \to R, B \to R$ be morphisms of rings, and assume that $A \to R$ is surjective. Show that the map

$$\mathrm{Spec}(A) \cup_{\mathrm{Spec}(R)} \mathrm{Spec}(B) \to \mathrm{Spec}(A \times_R B)$$

is a homeomorphism, where the LHS denotes the pushout in topological spaces.

Exercise 3 (4 points):

Let k be an algebraically closed field and let $f \in k[x,y]$ be a non-zero polynomial such that $f(0,0) = 0$. Write

$$f = f_r + f_{r+1} + \ldots + f_n + \ldots$$

with f_n homogeneous of degree n and $f_r \neq 0$. Let $X := V(f) \subseteq \mathbb{A}^2_k$ and define $Z := V(f_r) \subseteq \mathbb{A}^2_k$.
1) Prove that $f_r = l_1 \cdots l_r$ with $l_i \in k[x,y]$ homogeneous of degree 1. Deduce that Z is a union of lines.
2) Show that X is smooth in $(0,0)$ if and only if $r = 1$.
3) Solve exercise I.5.1 in Hartshorne's "Algebraic Geometry". (A non-smooth point $z \in X(k)$ is also called a singular point.)
Remark: The variety Z is called the "tangent cone" of X in $(0,0)$.

Exercise 4 (4 points):

Let A be a noetherian normal domain, and let $U \subseteq X := \mathrm{Spec}(A)$ be an open subset with complement $Z := X \setminus U$. Assume that for every $z \in Z$ the local ring $\mathcal{O}_{X,z}$ has Krull dimension ≥ 2. Show that for every vector bundle $\mathcal{E}$ on X the morphism

$$\mathcal{E}(X) \to \mathcal{E}(U),\ s \mapsto s_{|U}$$

is an isomorphism.
Hint: Proposition B.70 in the book "Algebraic Geometry I" by Görtz-Wedhorn.

To be handed in on: Thursday, 18.01.2024 (during the lecture, or via eCampus).