Dr. I. Gleason
Dr. J. Anschütz

WS 2023/24

Algebraic Geometry I

10. Exercise sheet

Exercise 1 (4 Points):

Let k be a field.
1) Let $X \subseteq \mathbb{A}^n_k$ be a closed, irreducible subscheme of dimension d. Let $f_1, \dots, f_r \in k[x_1, \dots, x_n]$ and let $Z \subseteq X \cap V(f_1, \dots, f_r)$ be an irreducible component. Use Krull's principal ideal theorem to show that Z has dimension at least $d - r$.
2) Let $X, Y \subseteq \mathbb{A}^n_k$ be two closed irreducible subschemes of dimensions d, e. Use that $X \cap Y \cong \Delta \cap (X \times Y)$ for the diagonal $\Delta\colon \mathbb{A}^n_k \to \mathbb{A}^n_k \times \mathbb{A}^n_k$ to show that each irreducible component of $X \cap Y$ has dimension at least $d + e - n$.
3) Let $X, Y \subseteq \mathbb{P}^n_k$ be two closed subschemes of dimensions d and e. If $d + e \geq n$, show that $X \cap Y \neq \emptyset$.
Hint: For 2) you may use that each irreducible component of $X \times Y$ has dimension $d+e$. For 3) let $f\colon \mathbb{A}^{n+1}_k \setminus \{0\} \to \mathbb{P}^n_k$ be the natural map, and let $\tilde{X}, \tilde{Y} \subseteq \mathbb{A}^{n+1}_k$ be the closures of $f^{-1}(X), f^{-1}(Y)$. Now show that any irreducible component Z containing $0 \in \tilde{X} \cap \tilde{Y}$ has dimension ≥ 1.

Exercise 2 (4 Points):

Let $f\colon Y \to X$ be a morphism of schemes.
1) Assume $|f|\colon |Y| \to |X|$ is a topological immersion with closed image. Show that f is affine.
2) Show that f is a universal homeomorphism if and only if it is surjective, universally injective and integral.
Hint: For 1) show that each $x \in X$ has an open, affine neighborhood U_x such that $f^{-1}(U_x)$ is affine by using arguments as in sheet 8, exercise 4.i).

Exercise 3 (4 points):

Let k be an algebraically closed field, and $X \to \operatorname{Spec}(k)$ a proper morphism with X reduced and connected.
1) Let $f \in \Gamma(X, \mathcal{O}_X)$ with associated function $g\colon X \to \mathbb{A}^1_k$. Show that $g(X)$ does not contain the generic point of $\mathbb{A}^1_k$.
2) Deduce $\Gamma(X, \mathcal{O}_X) \cong k$
Hint: For 1) embed $\mathbb{A}^1_k$ into $\mathbb{P}^1_k$.

Exercise 4 (4 points):

Let $f\colon X \to S$ be a morphism. The schematic image $\operatorname{Im}(f)$ of f is defined as the minimal closed subscheme $Z \subseteq S$ such that f factors through the inclusion $Z \to S$.
1) Prove that the schematic image $\operatorname{Im}(f)$ of f exists, and that its ideal sheaf is the largest quasi-coherent ideal contained in the kernel of $f^\#\colon \mathcal{O}_S \to f_*(\mathcal{O}_X)$.
2) If f is quasi-compact (but not necessarily quasi-separated!), show that the the kernel of $f^\#\colon \mathcal{O}_S \to f_*(\mathcal{O}_X)$ is quasi-coherent, thus defining the schematic image.
3) Let p be a prime. Determine the schematic image of $\coprod_{n\geq 0} \operatorname{Spec}(\mathbb{Z}/p^n) \to \operatorname{Spec}(\mathbb{Z})$.

To be handed in on: Thursday, 21.12.2023 (during the lecture, or via eCampus).