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WS 2023/24

Algebraic Geometry I

9. Exercise sheet

Exercise 1 (4 Points):

Let S be a scheme and $n \geq 0$. Let $\mathcal{L}$ be a line bundle on S and $s \in \mathcal{L}^{\otimes n}(S)$ a section. We define the functor $F_{\mathcal{L},s}(g\colon Y \to S) := \{t \in g^*\mathcal{L}(Y) \mid t^n = g^*(s) \in g^*\mathcal{L}^{\otimes n}(Y)\}$ on the category of schemes over S.

1) If $\mathcal{L} \cong \mathcal{O}_S$ show that $F_{\mathcal{L},s}$ is representable by the pullback of $\mathbb{A}^1_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[u]) \to \mathbb{A}^1_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[v]), u \mapsto v := u^n$ along the map $S \to \mathbb{A}^1_{\mathbb{Z}} := \mathrm{Spec}(\mathbb{Z}[v])$ classified by s, and the pullback of the global section $u \in \mathcal{O}_{\mathbb{A}^1_{\mathbb{Z}}}$.

2) Show that the functor $F_{\mathcal{L},s}$ is representable by some $f\colon X \to S$, $\sqrt[n]{s} \in f^*\mathcal{O}_X$ in general by constructing a suitable open cover of $F_{\mathcal{L},s}$.

3) Show that $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_S$-module of rank n.

Exercise 2 (4 Points):

Let k be a field and let $n \geq 0$. Let $U_0 = \mathrm{Spec}(k[T])$, $U_1 = \mathrm{Spec}(k[T^{-1}])$ with intersection $U_0 \cap U_1 = \mathrm{Spec}(k[T^{\pm 1}])$ be the standard covering of $\mathbb{P}^1_k$.

i) Prove that the map

$$\begin{array}{ccc} \{\text{iso. classes. of rank n vector bundles}\} & \to & \mathrm{GL}_n(k[T]) \setminus \mathrm{GL}_n(k[T^{\pm 1}])/\mathrm{GL}_n(k[T^{-1}]), \\ \mathcal{E} & \mapsto & \alpha^{-1}_{|U_0\cap U_1} \circ \beta_{|U_0\cap U_1} \end{array}$$

where $\alpha\colon \mathcal{O}^n_{U_0} \to \mathcal{E}_{|U_0}$ and $\beta\colon \mathcal{O}^n_{U_1} \to \mathcal{E}_{|U_1}$ are two isomorphisms, is well-defined and bijective.

ii) For all $n \geq 0$ the map

$$\begin{array}{ccc} \{(d_1,\ldots,d_n) \in \mathbb{Z} \mid d_1 \geq \ldots \geq d_n\} & \to & \mathrm{GL}_n(k[T]) \setminus \mathrm{GL}_n(k[T^{\pm 1}])/\mathrm{GL}_n(k[T^{-1}]) \\ d = (d_1,\ldots,d_n) & \mapsto & \mathrm{GL}_n(k[T])T^d\mathrm{GL}_n(k[T^{-1}]), \end{array}$$

is a bijection, where T^d denotes the diagonal matrix with entries $T^{d_1},\ldots,T^{d_n}$. Show this if $n = 2$.

iii) Write down all vector bundles on $\mathbb{P}^1_k$ (up to isomorphism).

Exercise 3 (4 points):

Let R be a ring and define the blow-up of $\mathbb{A}^{n+1}_R$ at the zero section as

$$X := \{((x_0,\ldots,x_n),(y_0:\ldots:y_n)) \in \mathbb{A}^{n+1}_R \times_{\mathrm{Spec}(R)} \mathbb{P}^n_R \mid x_iy_j = x_jy_i \text{ for all } i,j\}.$$

1) Show that X is a closed subscheme in $\mathbb{A}^{n+1}_R \times_{\mathrm{Spec}(R)} \mathbb{P}^n_R$.

2) Let $\pi\colon X \to \mathbb{A}^{n+1}_R$ be the canonical projection. Show that

$$\pi_{|\pi^{-1}(\mathbb{A}^{n+1}_R\setminus\{0\})}\colon \pi^{-1}(\mathbb{A}^{n+1}_R \setminus \{0\}) \to \mathbb{A}^{n+1}_R \setminus \{0\}$$

is an isomorphism and that $\pi^{-1}(0) \cong \mathbb{P}^n_R$.

3) Prove that X is isomorphic (as a $\mathbb{P}^n_R$-scheme) to the total space $\mathbb{V}(\mathcal{O}_{\mathbb{P}^n_R}(-1))$ of the line bundle $\mathcal{O}_{\mathbb{P}^n_R}(-1)$.

Hint: For 3) use the collection $(x_i)_{i=0,\ldots,n}$ to get a global section of the pullback of $\mathcal{O}_{\mathbb{P}^n_R}(-1)$ to X.

Exercise 4 (4 points):

We continue with the notation from exercise 3.
1) Let $\mathcal{I} := (x_0, \dots, x_n) \subseteq \mathcal{O}_{\mathbb{A}_R^{n+1}}$ be the ideal sheaf of the zero section. Show that the ideal sheaf $\pi^{-1}(\mathcal{I})\mathcal{O}_X = \mathrm{Im}(\pi^*\mathcal{I} \to \mathcal{O}_X)$ of the closed subscheme $\pi^{-1}(\{0\}) \subseteq X$ is an invertible ideal sheaf.
2) Conversely, let $f\colon S \to \mathbb{A}_R^{n+1}$ be a morphism such that the ideal sheaf $f^{-1}(\mathcal{I})\mathcal{O}_S \subseteq \mathcal{O}_S$ is invertible. Prove that f factors uniquely through π.

To be handed in on: Thursday, 14.12.2023 (during the lecture, or via eCampus).