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WS 2023/24

Algebraic Geometry I

8. Exercise sheet

Exercise 1 (4 Points):

Let A be a unique factorization domain.
1) Let $I \subseteq A$ be an ideal, which is finite locally free. Show that I is principal.
2) Show that $\mathrm{Pic}(A) = 0$.
Hint: Write a non-zero $f \in I$ in prime factorization $f = up_1^{a_1} \dots p_r^{a_r}$ with pairwise non-associated prime elements p_i and $u \in A^\times$. Show $I_{(p_i)} = (p_i^{c_i})$ for some $c_i \geq 0$. Argue that $I = (p_1^{c_1} \dots p_r^{c_r})$ by localizing on $\mathrm{Spec}(A)$.

Exercise 2 (4 Points):

Let A be a unique factorization domain, e.g., a field or $\mathbb{Z}$. Prove that $\mathbb{Z} \cong \mathrm{Pic}(\mathbb{P}^n_A)$ via $1 \mapsto \mathcal{O}_{\mathbb{P}^n_A}(1)$.
Hint: Use Exercise 1 and that $A[X_1, \dots, X_n]$ is a unique factorization domain.

Exercise 3 (4 points):

Let k be a field and $n, m \geq 0$. Let $f \colon \mathbb{P}^n_k \to \mathbb{P}^m_k$ be a morphism.
1) Show $f^*\mathcal{O}_{\mathbb{P}^m_k}(1) \cong \mathcal{O}_{\mathbb{P}^n_k}(d)$ for some $d \geq 0$.
2) Let $g_i := f^*(y_i) \in \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d)) \cong k[x_0, \dots, x_n]_d$ be the preimages of $y_0, \dots, y_m \in \Gamma(\mathbb{P}^m_k, \mathcal{O}_{\mathbb{P}^m_k}(1)) = k[y_0, \dots, y_m]_1$. Show that $V(g_0, \dots, g_m) \subseteq \mathbb{A}^{n+1}_k$ is contained (set-theoretically) in $\{0\}$.
3) If $m < n$, conclude $d = 0$ and that f is constant with image some $y \in \mathbb{P}^m_k(k)$.

Exercise 4 (4 points):

i) Let X be a scheme. Assume that $U_1, U_2 \subseteq X$ are two open, affine subschemes. Show that $U_1 \cap U_2$ can be covered by open subsets $\tilde{U}$ such that $\tilde{U}$ is simultaneously a principal open in U_1 and U_2.
ii) Let $f \colon X \to S$ be a morphism of schemes. Assume there exist open covers $X = \bigcup_{i \in I} \mathrm{Spec}(B_i), S = \bigcup_{i \in I} \mathrm{Spec}(A_i)$ with $f(\mathrm{Spec}(B_i)) \subseteq \mathrm{Spec}(A_i)$ such that $A_i \to B_i$ is of finite presentation. Let $\mathrm{Spec}(B) \subseteq X, \mathrm{Spec}(A) \subseteq S$ be open with $f(\mathrm{Spec}(B)) \subseteq \mathrm{Spec}(A)$. Show that $A \to B$ is of finite presentation.

To be handed in on: Thursday, 07.12.2023 (during the lecture, or via eCampus).