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WS 2023/24

Algebraic Geometry I

7. Exercise sheet

Exercise 1 (4 points):

Let $f\colon Y = \mathrm{Spec}(B) \to X = \mathrm{Spec}(A)$ be a morphism of schemes.
1) Let N be a B-module with underlying A-module $N_{|A}$. Show that $f_*(\widetilde{N}) \cong \widetilde{N_{|A}}$.
2) Let M be an A-module. Show that $f^*(\widetilde{M}) \cong \widetilde{M \otimes_A B}$. Conclude that for any morphism $g\colon Z \to S$ of schemes, g^* preserves quasi-coherence.

Exercise 2 (4 points):

For a ring A, $n \geq 0$ and $m \in \mathbb{Z}$ we denote by Φ the map $A[X_0, \ldots, X_n]_m \to \Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m))$ constructed in the lecture. Show that Φ is an isomorphism.

Exercise 3 (4 points):

Let k be an algebraically closed field and $n \geq 0$.
1) Show that $\mathbb{P}^n_k \cong (\mathbb{P}^n_k(k))^{\mathrm{sch}}$ with $(-)^{\mathrm{sch}}$ as in exercise 2, sheet 5.
2) Let $f \in k[X_0, \ldots, X_n]_d$ homogeneous of degree d. Set $s := \Phi(f) \in \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d))$ with vanishing locus $V(s) \subseteq \mathbb{P}^n_k$ as defined in the lecture. Show that as closed subschemes of $\mathbb{P}^n_k \cong (\mathbb{P}^n_k(k))^{\mathrm{sch}}$

$$V(s)_{\mathrm{red}} = V^+(f)^{\mathrm{sch}}.$$

Here $V^+(f) \subseteq \mathbb{P}^n_k(k)$ denotes the classical vanishing locus of f.

Exercise 4 (4 points):

Let $f(x, y) = a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6 \in \mathbb{R}[x, y]$ be a non-zero polynomial of degree 2. Show that $V(f) \subseteq \mathbb{A}^2_{\mathbb{R}}$ is isomorphic to one of the following:

1. a conic without $\mathbb{R}$-rational points $V(x^2 + y^2 + 1)$,
2. a circle $V(x^2 + y^2 - 1)$,
3. a parabola $V(y - x^2)$,
4. a hyperbola $V(yx - 1)$,
5. a double line $V(x^2)$,
6. two parallel lines $V(x(x - 1))$,
7. the coordinate cross $V(xy)$,
8. two complex lines with non-real slopes meeting at a real point $V(x^2 + y^2)$,
9. two non-intersecting non-real complex lines $V(x^2 + 1)$.

Compare this result to Sheet 1, Exercise 3 via the base change $V(f) \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C})$.
Hint: First show that wlog $a_1x^2 + a_2xy + a_3y^2 \in \{x^2, xy, x^2 + y^2\}$ *after some affine linear transformations. In case* $x^2 + y^2$ *it may help then to use that* $\mathbb{R}^2 \setminus \{0\}/\mathrm{O}_2(\mathbb{R}) \cong \mathbb{R}_{>0}(1, 0)$.

To be handed in on: Thursday, 30.11.2023 (during the lecture, or via eCampus).