Dr. I. Gleason
Dr. J. Anschütz **WS 2023/24**

Algebraic Geometry I

6. Exercise sheet

Exercise 1 (4 points):

1) Let $X \xrightarrow{f} S$, $Y \xrightarrow{g} S$ be two morphisms of locally ringed spaces. Let $X = \bigcup_i U_i$, $Y = \bigcup_j V_j$, $S = \bigcup_{i,j} S_{i,j}$ be open coverings with $f(U_i), g(V_j) \subseteq S_{i,j}$. We view $U_i, V_j, S_{i,j}$ as locally ringed spaces via the restriction of the structure sheaves on X, Y, S. Using the universal property of fiber products (which exist in locally ringed spaces) show that for all i, j the map $U_i \times_{S_{i,j}} V_j \to X \times_S Y$ identifies the source as an open subspace of the latter and

$$\bigcup_{i,j} U_i \times_{S_{i,j}} V_j = X \times_S Y.$$

2) Assume that X, Y, S are schemes. Show that the natural map $|X \times_S Y| \to |X| \times_{|S|} |Y|$ is surjective, but not injective in general.

Exercise 2 (4 points):

Let $f\colon X \to S, g\colon S' \to S$ be morphisms of schemes. Let $f'\colon X' := X \times_S S' \to S'$ be the projection.
1) Show that if f is an open (resp. closed) immersion, then f' is an open (resp. closed) immersion.
2) Assume $S' = \mathrm{Spec}(k(s)) \to S$ is the canonical morphism for some $s \in S$. Show that $|X'| \to |X| \times_{|S|} \{s\}$ is a homeomorphism.

Exercise 3 (4 points):

Let k be a field. Describe the fibers in all points of the following morphisms $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ corresponding in each case to the canonical morphism $A \to B$.
i) $\mathrm{Spec}(k[T,U]/(TU-1)) \to \mathrm{Spec}(k[T])$
ii) $\mathrm{Spec}(k[T,U]/(T^2-U^2)) \to \mathrm{Spec}(k[T])$
iii) $\mathrm{Spec}(k[T,U]/(T^2+U^2)) \to \mathrm{Spec}(k[T])$
iv) $\mathrm{Spec}(k[T,U]/(TU)) \to \mathrm{Spec}(k[T])$

Exercise 4 (4 points):

Let A be a ring and let M be an A-module. Show that

$$D(f) \mapsto M[f^{-1}] = M \otimes_A A[f^{-1}]$$

is a sheaf on the basis of principal opens in $X := \mathrm{Spec}(A)$.

To be handed in on: Thursday, 23.11.2023 (during the lecture, or via eCampus).