Dr. I. Gleason
Dr. J. Anschütz **WS 2023/24**

Algebraic Geometry I

4. Exercise sheet

In the lectures and exercises we will start using categorical language as for example explained in Saunders MacLane's "Categories for the working mathematician". In case more background knowledge is needed or wanted, please inform the assistant or your tutor.

Exercise 1 (4 points):

1) Show that any finite irreducible topological space admits a generic point. In particular, show that a finite T_0-space is spectral.
2) Write $\mathrm{Spec}(\mathbb{Z})$ as an inverse limit of finite T_0-spaces.

Exercise 2 (4 points):

Let I be a filtered partially ordered set. Show that for each I-indexed inductive system

$$0 \to A_i \xrightarrow{\alpha_i} B_i \xrightarrow{\beta_i} C_i \to 0$$

of short exact sequences of abelian groups, the sequence

$$0 \to \varinjlim_{i\in I} A_i \to \varinjlim_{i\in I} B_i \to \varinjlim_{i\in I} C_i \to 0$$

of colimits is again exact.

Exercise 3 (4 points):

Let X be a topological space.
1) Show that the category $\mathrm{Sh}_{\mathrm{ab}}(X)$ of sheaves of abelian groups on X is an abelian category with all colimits and limits.
Hint: First construct colimits and limits of presheaves, and then sheafify the presheaf colimits.
2) Show that a morphism $f\colon \mathcal{F} \to \mathcal{G}$ of sheaves of abelian groups is surjective on each stalk if and only if f is an epimorphism in $\mathrm{Sh}_{\mathrm{ab}}(X)$.
3) Let $X = S^1 \subseteq \mathbb{R}^2$ with upper and lower hemisphere $i_+\colon D_+ \to S^1$, $i_-\colon D_- \to S^1$. Set $i\colon D_+ \cap D_- \to S^1$, and $\mathcal{F} := i_{-,*}\underline{\mathbb{Z}} \oplus i_{+,*}\underline{\mathbb{Z}}$, $\mathcal{G} := i_*(\underline{\mathbb{Z}})$. Construct an epimorphism $\mathcal{F} \to \mathcal{G}$ such that $\mathcal{F}(X) \to \mathcal{G}(X)$ is not surjective. Here, $\underline{\mathbb{Z}} = C(-,\mathbb{Z})$ denotes the constant sheaf with value on $\mathbb{Z}$ on the respective topological spaces.

Exercise 4 (4 points):

Let X be a topological space and let $\mathcal{B}$ be a basis of the topology of X, stable under finite intersections. Let $\mathrm{Sh}_{\mathcal{B}}(X)$ be the category of sheaves on the basis $\mathcal{B}$ as defined in the lecture. Prove that the functors

$$\mathrm{Sh}(X) \to \mathrm{Sh}_{\mathcal{B}}(X),\ (\mathcal{F}\colon \mathrm{Ouv}(X)^{\mathrm{op}} \to \mathrm{Sets}) \mapsto (\mathcal{F}_{|\mathcal{B}^{\mathrm{op}}}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Sets})$$

and

$$\mathrm{Sh}_{\mathcal{B}}(X) \to \mathrm{Sh}(X),\ \mathcal{F} \mapsto (U \mapsto \varprojlim_{V\subseteq U, V\in\mathcal{B}} \mathcal{F}(V))$$

are inverse equivalences of categories.

To be handed in on: Thursday, 09.11.2023 (during the lecture, or via eCampus).