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Algebraic Geometry I

3. Exercise sheet

Let k be an algebraically closed field and $n \geq 1$. On this exercise sheet we study the geometry of blow-ups. The blow-up of $\mathbb{A}^n_k(k)$ at the origin $t := (0, \ldots, 0)$ is

$$Z := \{((x_1, \ldots, x_n), [y_1 : \ldots : y_n]) \in \mathbb{A}^n_k(k) \times \mathbb{P}^{n-1}_k(k) \mid x_i y_j = x_j y_i \text{ for all } i, j \geq 1\}.$$

Set $V_i := Z \cap (\mathbb{A}^n_k(k) \times D^+(y_i))$ for $i = 1, \ldots, n$ and let $\pi \colon Z \to \mathbb{A}^n_k(k)$ be the projection. If $Y \subseteq \mathbb{A}^n_k(k)$ is an affine algebraic set with $t \in Y$, we define the blow-up $\mathrm{BL}_t(Y)$ of Y at t to be the closure of $\pi^{-1}(Y \setminus \{t\})$ in Z. The exceptional divisor is by definition $\pi^{-1}(t) \cap \mathrm{BL}_t(Y)$.

Exercise 1 (4 points):

1) Set $U := \mathbb{A}^n_k(k) \setminus \{t\}$. Show that $\pi_{|\pi^{-1}(U)} \colon \pi^{-1}(U) \to U$ is an isomorphism.
2) Show that $V_i \cong \mathbb{A}^n_k(k)$ with coordinate ring $k[\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, x_i, \frac{x_{i+1}}{x_i} \ldots, \frac{x_n}{x_i}]$ and $\pi^{-1}(U) \cap V_i = D(x_i)$.
3) Show that $\pi^{-1}(t) \cong \mathbb{P}^{n-1}_k(k)$ and that $\pi^{-1}(t) \cap V_i, i = 1, \ldots, n$, identifies with the standard cover of $\mathbb{P}^{n-1}_k(k)$.

Exercise 2 (4 points):

Assume $n = 2$. For Y below show that $\mathrm{BL}_t(Y) \cong \mathbb{A}^1_k(k)$ by calculating $\mathrm{BL}_t(Y) \cap V_i$ for $i = 1, 2$. In both cases determine the exceptional divisor in $\mathrm{BL}_t(Y)$.
1) $Y := V(x_1^2 - x_2^3)$.
2) $Y := V(x_1^2 - x_2^3 - x_2^2)$.

Exercise 3 (4 points):

Assume $n = 3$. Set $Y := V(x_1^2 - x_2 x_3)$ with $f := \pi_{|\widetilde{Y}} \colon \widetilde{Y} := \mathrm{BL}_t(Y) \to Y$.
1) Show that the projection $g \colon \widetilde{Y} \to \mathbb{P}^2_k(k)$ has image $V^+(y_1^2 - y_2 y_3) \cong \mathbb{P}^1_k(k)$.
2) Show that $f^{-1}(t) \cong \mathbb{P}^1_k(k)$.
3) Show that there exists an open covering $U_1 \cup U_2 = \mathbb{P}^1_k(k)$ such that $g^{-1}(U_i) \cong \mathbb{A}^1_k(k) \times U_i$ for $i = 1, 2$.
Remark: Thus, $\widetilde{Y}$ is a "line bundle" over $\mathbb{P}^1_k(k)$. We will eventually see that $\widetilde{Y} \neq \mathbb{A}^1_k(k) \times \mathbb{P}^1_k(k)$.

Exercise 4 (4 points):

Let X be a topological space. Define $V_U := \{Z \subseteq X \text{ closed, irreduzible} \mid Z \cap U \neq \emptyset\}$ for $U \subseteq X$ open and set $X^{\mathrm{sob}} := V_X$ with topology such that the opens are V_U for $U \subseteq X$ open.
1) Show that X^{sob} is sober, i.e., each closed irreducible subset has a unique generic point, and that $f^{-1}(-)$ for $f \colon X \to X^{\mathrm{sob}},\ x \mapsto \overline{\{x\}}$ induces a bijection between open subsets of X^{sob} and X.
2) Show that for any continuous map $g \colon X \to Z$ with Z sober, there exists a unique continuous map $h \colon X^{\mathrm{sob}} \to Z$ such that $g = h \circ f$.
3) Let k be an algebraically closed field, and $V \subseteq \mathbb{A}^n_k(k)$ an affine algebraic set with coordinate ring A. Show $V \cong \mathrm{MaxSpec}(A)$ and $V^{\mathrm{sob}} \cong \mathrm{Spec}(A)$ as topological spaces.

To be handed in on: Thursday, 02.11.2023 (during the lecture, or via eCampus).