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WS 2023/24

Algebraic Geometry I

2. Exercise sheet

Let k be an algebraically closed field. We consider $\mathbb{A}^n_k(k) \subseteq \mathbb{P}^n_k(k),\ (x_1, \ldots, x_n) \mapsto [x_1 : \ldots : x_n : 1]$ for $n \geq 0$ and call the complement $V^+(x_{n+1}) \cong \mathbb{P}^{n-1}_k(k)$ the "projective hyperplane at infinity".

Exercise 1 (4 points):

Let $I := (f_1, \ldots, f_r) \subseteq k[x_1, \ldots, x_n]$ be an ideal, and $X := V(I) \subseteq \mathbb{A}^n_k(k)$ be its vanishing locus.
1) Show that the closure $\overline{X} \subseteq \mathbb{P}^n_k(k)$ of X is the vanishing locus of the homogenizations $\widetilde{g}$ for *all* $g \in I$. If $r = 1$, show that $\overline{X} = V^+(\widetilde{f_1})$.
Reminder: If $g \in k[x_1, \ldots, x_n]$ is of degree m, its homogenization $\widetilde{g}$ is $x^m_{n+1} g(x_1/x_{n+1}, \ldots, x_n/x_{n+1})$.
2) For each of the 5 affine algebraic sets $X \subseteq \mathbb{A}^2_k(k)$ of exercise 3 on sheet 1 calculate the intersection of $\overline{X} \subseteq \mathbb{P}^2_k(k)$ with the line $\mathbb{P}^1_k(k) = \mathbb{P}^2_k(k) \setminus \mathbb{A}^2_k(k)$ at infinity.
3) Construct $f_1, \ldots, f_r$ such that $\overline{X} \neq V^+(\widetilde{f_1}, \ldots, \widetilde{f_r})$.
Hint: Use suitable examples from 2) and arrange that $\overline{V(f_1)}$ and $\overline{V(f_2)}$ meet at a point at infinity.

Exercise 2 (4 points):

Let $F \in k[x, y, z]$ be a non-zero homogeneous polynomial of degree 2.
1) Show that $V^+(F) \subseteq \mathbb{P}^2_k(k)$ is isomorphic to

1. a conic $V^+(x^2 - yz)$,
2. two projective lines meeting at a point $V^+(xy)$,
3. or a projective line $V^+(x) = V^+(x^2)$.

2) Show that $V^+(x^2 - yz) \setminus \{V^+(z)\} \to \mathbb{P}^1_k(k),\ [x : y : z] \mapsto [x : y]$ can uniquely be extended to an isomorphism $V^+(x^2 - yz) \cong \mathbb{P}^1_k(k)$.
Hint for 1): Show that each affine linear transformation of $\mathbb{A}^2_k(k)$ extends to an automorphism of $\mathbb{P}^2_k(k)$ and use exercise 1 from this sheet and exercise 3 from sheet 1.

Exercise 3 (4 points):

1) Let $f, g \colon X \to Y$ be morphisms of affine algebraic sets with X irreducible. Assume that there exists a non-empty open subset $U \subseteq X$ such that $f_{|U} = g_{|U}$. Show that $f = g$.
2) Use exercise 2 from sheet 1 to prove the Cayley-Hamilton theorem: Let L be any field, let A be any $n \times n$-matrix with coefficients in L and let $\chi_A(X)$ be its characteristic polynomial. Then $\chi_A(A) = 0$.

Exercise 4 (4 points):

Let X be a quasi-projective variety.
1) Show that X is T_1, i.e., for each $x, y \in X$, $x \neq y$ there exists an open subset $U \subseteq X$ such that $x \in U$ and $y \notin U$.
2) Show that X is T_2 (or Hausdorff), i.e., for each $x, y \in X$, $x \neq y$ there exists open sets $U, V \subseteq X$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$, if and only if X is a finite set.

To be handed in on: Thursday, 26.10.2023 (during the lecture, or via eCampus).