Dr. D. Schwein
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Algebra II - Local fields

13. Exercise sheet

As on sheet 12 we denote by $A_{a,b,K}$ for a, b units in a field K of characteristic $\neq 2$ the central simple K-algebra with presentation $\langle x, y \mid x^2 = a, y^2 = b, xy = -yx\rangle$.

Exercise 1 (4 points):

Let K be a field of characteristic $\neq 2$ and $a, b \in K^\times$.
1) Show that $\overline{(\lambda + \mu x + \nu y + \rho xy)} := \lambda - \mu x - \nu y - \rho xy$ for $\lambda, \mu, \nu, \rho \in K$ defines an automorphism of $A_{a,b,K}$ as a K-vector space, which satisfies $\overline{q_1 q_2} = \overline{q_2} \cdot \overline{q_1}$ for $q_1, q_2 \in A_{a,b,K}$.
2) For $q \in A_{a,b,K}$ set $N(q) := q\overline{q} \in K$. Show that $q \in A_{a,b,K}$ is a unit if and only if $N(q) \in K^\times$.
3) Assume that a not a square in K. Show that $A_{a,b,K}$ is a division algebra if and only if b is not a norm from the extension $K(\sqrt{a})/K$.
Hint: If $b = N_{K(\sqrt{a})/K}(\lambda + \mu\sqrt{a})$, *then* $u := \lambda y + \mu xy$ *satisfies* $u^2 = 1$. *Use the element* $v := (1 + a)x + (1 - a)ux$ *to show* $A_{a,b,K} \cong A_{1,4a^2,K}$.

Exercise 2 (4 points):

For each prime p find $a, b \in \mathbb{Q}_p^\times$ such that the algebra $A_{a,b,\mathbb{Q}_p}$ is a division algebra (and hence generates the 2-torsion in the Brauer group of $\mathbb{Q}_p$ as $A_{a,b,\mathbb{Q}_p}$ splits over a degree 2 extension).

As a preparation for the exam we recommend to solve the next two exercises without consulting the lecture notes during their solution.

Exercise 3 (4 points):

1) Define the ramification index $e_{L/K}$ for a finite extension L/K of local fields.
2) Calculate the ramification index $e_{K/\mathbb{Q}_2}$ for $K = \mathbb{Q}_2(\alpha)$ with $\alpha^4 - 16\alpha^2 + 16 = 0$. Calculate $\nu_K(\alpha)$.

Exercise 4 (4 points):

1) Show that the polynomial $f(x) = \frac{1}{3} + \frac{1}{14}x + \frac{1}{1}x^2 + \frac{1}{59}x^3 + \frac{1}{2}x^4 \in \mathbb{Q}[x]$ is irreducible.
2) Use Hensel's lemma to show that the equation $x^3 + 5x^2 - 20x + 15 = 0$ has a solution in $\mathbb{Q}_3$.

To be handed in on: Thursday, 01.02.2024 (during the lecture, or via eCampus). Please contact your tutor if you want to receive your corrected exercise sheet.