Dr. D. Schwein
Dr. J. Anschütz **WS 2023/24**

Algebra II - Local fields

12. Exercise sheet

Exercise 1 (4 points):

Let K be a field with $\mathrm{char}(K) \neq 2$. For $a, b \in K^\times$ we set $A_{a,b,K}$ as the K-algebra with presentation

$$A_{a,b,K} := \langle x, y \mid x^2 = a, y^2 = b, xy = -yx \rangle.$$

1) Show that $A_{a,b,K}$ has K-dimension 4 and that $A_{a,b,K} \cong A_{\lambda^2 a,b}$ as K-algebras for any $\lambda \in K$.

2) Show that $A_{1,b,K} \cong \mathrm{Mat}_2(K)$ with x mapping to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

3) For a field extension L/K show $L \otimes_K A_{a,b,K} \cong A_{a,b,L}$.

Remark: This implies that $A_{a,b,K}$ is a central simple K-algebra.

Exercise 2 (4 points):

Let G be a group, $H \subseteq G$ a normal subgroup of finite index and p a prime not dividing $[G : H]$. Show that the natural restriction map

$$H^i(G, \mathbb{F}_p) \cong H^0(G/H, H^i(H, \mathbb{F}_p))$$

is an isomorphism for any $i \geq 0$.

Hint: Use that $M \mapsto M^{G/H}$ is exact on the category of G/H-representations over $\mathbb{F}_p$, and the standard resolution for G.

Exercise 3 (4 points):

Set $G = S_3$ with subgroup $H := A_3 \subseteq S_3$. Let $\alpha \in H^1(H, \mathbb{F}_3), \beta \in H^2(H, \mathbb{F}_3)$ be generators.

1) Use that $\beta \cup (-)$ induces the isomorphism $H^i(H, \mathbb{F}_3) \cong H^{i+2}(H, \mathbb{F}_3)$ for any $i \geq 1$ to show that as (commutative) $\mathbb{F}_3$-algebras $H^*(H, \mathbb{F}_3) \cong \mathbb{F}_3[\alpha, \beta]$ with only relation $\alpha^2 = 0$.

2) Let $\sigma \in G$ be of order 2. Show that $\sigma(\alpha) = -\alpha$ and $\sigma(\beta) = -\beta$.

3) Use exercise 2 to show $H^*(G, \mathbb{F}_3) \cong \mathbb{F}_3[x, y]/x^2$ with $x = \alpha\beta$ of degree 3 and $y = \beta^2$ of degree 4.

Hint: For 2) use that $0 \to \mathbb{F}_3 \to \mathbb{Z}/9 \to \mathbb{F}_3 \to 0$ yields a G/H-equivariant isomorphism $H^1(H, \mathbb{F}_3) \cong H^2(H, \mathbb{F}_3)$.

Exercise 4 (4 points):

Set $K := \mathbb{Q}_2(\sqrt{2})$. Calculate $H := N_{K/\mathbb{Q}_2}(K^\times) \subseteq \mathbb{Q}_2^\times$.

Hint: First argue that $\mathbb{Q}_2^{\times 2} \subseteq H$ and that H has index 2 in $\mathbb{Q}_2^\times$ using Artin reciprocity.

To be handed in on: Thursday, 25.01.2023 (during the lecture, or via eCampus).