Dr. D. Schwein **WS 2023/24**
Dr. J. Anschütz

Algebra II - Local fields

11. Exercise sheet

Exercise 1 (4 points):

Let G be a group and let A, B be G-modules. Show that for $i \geq 0$ the $\cup$-product

$$\cup\colon \hat{H}^0(G,A) \times \hat{H}^i(G,B) \to \hat{H}^i(G, A \otimes B)$$

sends $(\overline{a}, b)$ to $\hat{H}^i(\varphi_a)(b)$, where $\varphi_a\colon B \to A \otimes B,\ c \mapsto a \otimes c$ for $a \in A^G$.
Hint: Use naturality of the $\cup$-product in short exact sequences to reduce to the claim for $i = 0$.

Exercise 2 (4 points):

Let G be a group and $0 \to A \to B \xrightarrow{\alpha} C \to 0$ a short exact sequence of G-modules. Let $\varphi\colon G \to C$ be a 1-cocycle, and let $\delta\colon H^1(G,C) \to H^2(G,A)$ be the connecting homomorphism. Show that under the bijection from sheet 10, exercise 3 the element $\delta([\varphi])$ is given by fiber product $B \times_C G = \{(b,g) \in B \times G \mid \alpha(b) = \varphi(g)\}$ with group structure $(b,g) \cdot (b',g') := (bg(b'), gg')$.

Exercise 3 (4 points):

Let p be a prime and $G := \mathbb{Z}/p$.
1) If p is odd, show that the $\cup$-product $H^1(G,\mathbb{Z}/p) \times H^1(G,\mathbb{Z}/p) \to H^2(G,\mathbb{Z}/p)$ is trivial.
2) If $p = 2$, show that the $\cup$-product $H^1(G,\mathbb{Z}/2) \times H^1(G,\mathbb{Z}/2) \to H^2(G,\mathbb{Z}/2) \cong \mathbb{Z}/2$ is surjective.
Hint: In 2) use Exercise 1, Exercise 2 and naturality of the $\cup$-product in short exact sequences.

Exercise 4 (4 points):

Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}ij$ be the Hamiltonians (thus $i^2 = j^2 = -1, ij = -ji$). We set $\mathbb{C} := \mathbb{R} \oplus \mathbb{R}i$. Show that $H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}, \mathbb{C}^\times)$ is generated by the class of the so-called Weil group $W_\mathbb{R} := \mathbb{C}^\times \cup \mathbb{C}^\times j \subseteq \mathbb{H}^\times$ of $\mathbb{R}$.

To be handed in on: Thursday, 18.01.2023 (during the lecture, or via eCampus).