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WS 2023/24

Algebra II - Local fields

10. Exercise sheet

Exercise 1 (4 points):

Let G be a group and A a G-module. Set $H := A \rtimes G$ with projection $\pi\colon H \to G$.
1) Let $\delta\colon G \to A$ be a 1-cocycle. Show that $\varphi\colon G \to H,\ g \mapsto (\delta(g), g)$ is a group homomorphism.
2) Show that $H^1(G, A)$ is in bijection to the set of equivalence classes of group homomorphisms $\varphi\colon G \to H$ with $\pi \circ \varphi = \mathrm{Id}_G$, with equivalence induced by conjugation by elements in A.
3) Let $\delta\colon G \to A$ be a 1-cocycle. Show that $\ker(\delta) := \delta^{-1}(1)$ is a subgroup of G, which is not necessarily normal.

Exercise 2 (4 points):

Let K be a field with separable closure $\overline{K}$ and $n \geq 0$ prime to the characteristic of K. Let $x \in K$ with some n-th root $y \in \overline{K}$. Show that $\delta_x\colon \mathrm{Gal}(\overline{K}/K) \to \mu_n(\overline{K}),\ \sigma \mapsto \frac{\sigma(y)}{y}$ is a 1-cocycle whose cohomology class only depends on x, and that $\ker(\delta_x) = \mathrm{Gal}(\overline{K}/K(y))$.

Exercise 3 (4 points):

Let G be a group and A a G-module (written additively). An extension of G by A is an exact sequence $0 \to A \overset{\iota_E}{\to} E \overset{\pi_E}{\to} G \to 1$ of groups, such that π_E is surjective and ι_E an isomorphism $A \cong \ker(\pi_E)$ of G-modules.
1) Show that the conjugation action of E turns $\ker(\pi_E)$ naturally into a G-module. Hence, the condition on ι_E is well-defined.
2) Let $s\colon G \to E$ be a map of sets with $\pi_E \circ s = \mathrm{Id}_G$. Show that

$$u_s\colon G \times G \to A,\ (\sigma, \tau) \mapsto \iota_E^{-1}(s(\sigma)s(\tau)s(\sigma\tau)^{-1})$$

is a 2-cocycle with $u_s(1, g) = u_s(g, 1) = 0$ for $g \in G$, i.e., u_s is normalized.
3) Conversely, assume that $u\colon G \times G \to A$ is a normalized 2-cocycle. Show that the set $E_u := A \times G$ with the multiplication

$$(a, g) *_u (b, h) := (a + g(b) + u(g, h), gh)$$

is a group, which is naturally an extension of G by A.
4) Show that the extensions E and E_{u_s} are isomorphic for any s as in 2).
Remark: The above constructions determine a natural bijection between $H^2(G, A)$ and isomorphism classes of extensions E of G by A.

Exercise 4 (4 points):

With notation as in exercise 3 write the following extensions of groups as E_u for some 2-cocycle u.
1) $0 \to \mathbb{Z}/2 \cong \langle i^2 \rangle \to Q_8 \to \mathbb{Z}/2 \times \mathbb{Z}/2 \cong \langle \bar{i}, \bar{j} \rangle \to 1$ with Q_8 as in sheet 7, exercise 4.
2) $0 \to \mathbb{Z}/2 \cong \langle \tau^2 \rangle \to D_8 \to \mathbb{Z}/2 \times \mathbb{Z}/2 \cong \langle \overline{\sigma}, \overline{\tau} \rangle \to 1$ with D_8 the dihedral group as in sheet 8, exercise 3.
3) $0 \to \mathbb{Z}/10 \to \mathbb{Z}/100 \to \mathbb{Z}/10 = \{0, 1, \ldots, 9\} \to 0$. Deduce that "carrying is a 2-cocycle".

To be handed in on: Thursday, 21.12.2023 (during the lecture, or via eCampus).