Dr. D. Schwein **WS 2023/24**
Dr. J. Anschütz

Algebra II - Local fields

8. Exercise sheet

Exercise 1 (4 points):

Set $K := \mathbb{Q}_2(\zeta_8)$ with ζ_8 a primitive 8-th root of unity.
1) Show that $K = \mathbb{Q}_2(\sqrt{2}, i)$.
2) Let $G := \mathrm{Gal}(K/\mathbb{Q}_2)$ and $H := \mathrm{Gal}(K/\mathbb{Q}_2(\sqrt{2}))$. Then $G/H \cong \mathrm{Gal}(\mathbb{Q}_2(\sqrt{2})/\mathbb{Q}_2)$. Show that $(G/H)_i \neq G_i H/H$ for some $i \geq 0$.
Hint: Show first $\mathbb{Q}(e^{2\pi i/8}) = \mathbb{Q}(\sqrt{2}, i)$.

Exercise 2 (4 points):

Let $L := \mathbb{Q}_2(\alpha, i)$, $\alpha^4 = 2$, and $K := \mathbb{Q}_2(\zeta_8) \subseteq L$ with ζ_8 a primitive 8-th root of unity. Set $\pi_K := 1 - \zeta_8$ and $u := \frac{\sqrt{2}}{\pi_K^2} \in K$.
1) Show that $1 - u \equiv \pi_K \mod \pi_K^2$ and deduce that $1 - u$ is a uniformizer in $\mathcal{O}_K$.
2) Show that $L = K(\sqrt{u})$ with uniformizer $1 + \sqrt{u} \in \mathcal{O}_L$. Deduce that $\mathcal{O}_L \neq \mathbb{Z}_2[\alpha, i]$.

Exercise 3 (4 points):

We keep the notation from exercise 2.
1) Show that $L/\mathbb{Q}_2$ is Galois with Galois group $G = \langle \sigma, \tau \rangle$ the dihedral group with 8 elements. Here, $\sigma(i) = -i, \tau(i) = i, \sigma(\alpha) = \alpha, \tau(\alpha) = i\alpha$.
2) Show that

$$G = G_1 \supsetneq G_2 = G_3 = \langle \tau \rangle \supsetneq G_4 = \ldots = G_7 = \langle \tau^2 \rangle \supsetneq G_8 = \{1\}$$

is the lower ramification filtration.
3) Show that the jumps in the upper ramification filtration on G are $1, 2, 3$.

Exercise 4 (4 points):

Set $f(X) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots + \frac{1}{8}x^8$.
1) Draw the Newton polygon of $f \in \mathbb{Q}_p[X]$ for $p = 5, 7$.
2) Show that $f(X)$ factors in $\mathbb{Q}_7[X]$ in irreducible polynomials of degree 1 and 7, and over $\mathbb{Q}_5[X]$ in irreducible polynomials of degree 3 and 5. Deduce that $f(X) \in \mathbb{Q}[X]$ is irreducible.

To be handed in on: Thursday, 7.12.2023 (during the lecture, or via eCampus).