Dr. D. Schwein
Dr. J. Anschütz **WS 2023/24**

Algebra II - Local fields

7. Exercise sheet

Exercise 1 (4 points):

Let K be a complete discretely valued field with perfect residue field k, and let L/K be a finite Galois extension with group G and lower ramification filtration $G_i \subseteq G$, $i \geq -1$. Let $\pi \in L$ be a uniformizer, and let $U_L^0 := \mathcal{O}_L^\times \supseteq U_L^i := 1 + \pi^i \mathcal{O}_L$, $i \geq 1$, be the filtration of the units.
1) Show that $U_L^0/U_L^1 \cong (\mathcal{O}_L/\pi)^\times,\ xU_L^1 \mapsto x + \pi\mathcal{O}_L$ and $U_L^i/U_L^{i+1} \cong \pi^i\mathcal{O}_L/\pi^{i+1}\mathcal{O}_L,\ xU_L^{i+1} \mapsto (x-1) + \pi^{i+1}\mathcal{O}_L$ are isomorphisms for $i \geq 1$.
2) Show that for $i \geq 0$ the map $G_i/G_{i+1} \to U_L^i/U_L^{i+1},\ \sigma G_{i+1} \mapsto \sigma(\pi)/\pi$ is a well-defined injective group homomorphism, which is independent of the choice of π. Conclude that every Galois extension of a local non-archimedean field is solvable.

Exercise 2 (4 points):

With notations from exercise 1 let $\theta_i \colon G_i/G_{i+1} \to \pi^i\mathcal{O}_L/\pi^{i+1}\mathcal{O}_L$ for $i \geq 1$ be the injective morphism constructed there.
1) Assume $i, j \geq 1$, $s \in G_i$ and $t \in G_j$. Show $sts^{-1}t^{-1} \in G_{i+j}$ and $\theta_{i+j}(sts^{-1}t^{-1}) = (j-i)\theta_i(s)\theta_j(t)$ by calculating consequently modulo π^{i+j+1}.
2) Assume $j \geq 1$ is maximal such that $G_j \neq \{1\}$. Show that $sts^{-1}t^{-1} = 1$.
3) Conclude that if $i, j \geq 1$ are jumps in the lower numbering filtration, i.e., $G_i \neq G_{i+1}$ and $G_j \neq G_{j+1}$, then $j \equiv i$ modulo $\mathrm{char}(k)$.

Exercise 3 (4 points):

Compute the *upper* numbering ramification filtration for the Galois extensions K_n/K, $n \geq 1$ prime to p, from Sheet 6, Exercise 2. In particular, compute the functions $\varphi_{K_n/K}$, $\psi_{K_n/K}$.

Exercise 4 (4 points):

Let L/K be a totally ramified extension with Galois group G isomorphic to the quaternion group $Q_8 := \langle i, j \mid i^4 = 1, j^2 = i^2, ij = ji^{-1} \rangle$.
1) Show that $H := \langle i^2 \rangle \subseteq Q_8$ is the unique subgroup of order 2, and that every subgroup of order 4 is cyclic.
2) Assume that the ramification subgroup G_4 is trivial. Use exercise 2 to show that the lower ramification filtration on G is given by

$$\{1\} = G_4 \subsetneq G_3 = G_2 = H \subseteq G_1 = G_0 = G.$$

3) Show that the upper ramification filtration on G is given by

$$G^v = \begin{cases} G, & 0 \leq v \leq 1, \\ H, & 1 < v \leq \frac{3}{2}, \\ \{1\}, & v > \frac{3}{2}. \end{cases}$$

To be handed in on: Thursday, 30.11.2023 (during the lecture, or via eCampus).