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WS 2023/24

Algebra II - Local fields

6. Exercise sheet

Exercise 1 (4 points):

1) Let $K/\mathbb{Q}_p(\zeta_p)$ be a finite extension, and let $a, b \in \mathcal{O}_K^\times$ with $a \equiv b^p \mod p(1-\zeta_p)$. Show that the polynomial

$$f(X) := \frac{((1-\zeta_p)X + b)^p - a}{(1-\zeta_p)^p}$$

has coefficients in $\mathcal{O}_K$, that $L := K(\sqrt[p]{a})$ is the splitting field of $f(X)$ and that L/K is unramified.
2) Assume $K = \mathbb{Q}_p(\zeta_p)$ and let L/K be the unique unramified extension of degree p. Realize L as a Kummer extension of K, and as the splitting field of some Artin-Schreier polynomial $X^p - X + b$, $b \in K$.

Exercise 2 (4 bonus points):

Let $K := \mathbb{F}_p((t))$ and for $n \geq 1$ *prime to* p let K_n/K be the Artin-Schreier extension defined by $X^p - X - t^{-n} = 0$. Set $G := \mathrm{Gal}(K_n/K)$. Show that the ramification filtration on G is

$$G = G_0 = \ldots = G_n \supsetneq G_{n+1} = \ldots = \{1\}.$$

Remark: In a previous version the assumption that n is prime to p was forgotten. Hence, the points for this exercise have been converted to bonus points. If $n = p^a n'$ with n' not divisible by p, then $K_n = K_{n'}$, and thus in general the jump in the lower numbering filtration appears at n'.

Exercise 3 (4 points):

1) Let $\Phi \colon \prod\limits_{n \text{ prime to } p} \mathbb{Z}_p \to \mathbb{F}_p((t))^\times,\ (a_n)_n \mapsto \prod\limits_{n \text{ prime to } p} (1+t^n)^{a_n}$. Show that Φ is well-defined and an isomorphism onto the 1-units $1 + t\mathbb{F}_p[[t]]$.
2) Show that $\mathbb{F}_p((t))^\times \cong \mathbb{Z} \times \mathbb{F}_p^\times \times \prod\limits_{\mathbb{N}} \mathbb{Z}_p$.

Hint: If $(a_n)_n \in \ker(\Phi)$, show that each a_n is divisible by arbitrary powers of p.

Exercise 4 (4 bonus points):

Set $K := \mathbb{Q}_2(i)$, $i^2 = -1$, with uniformizer $\pi := i - 1 \in \mathcal{O}_K$ and set $L = K(\alpha)$ with $\alpha^4 = \pi$. Determine the ramification subgroups $G_i \subseteq G := \mathrm{Gal}(L/\mathbb{Q}_2)$, $i \geq -1$.
Remark: The statement of the exercise is wrong as $L/\mathbb{Q}_2$ is not Galois. Hence, the points for this exercise have been converted to bonus points. The aim of this exercise was to ask for the feasible calculation of the valuation of $\sigma(\alpha) - \alpha$ for certain Galois automorphisms σ, which as $L/\mathbb{Q}_2$ is not Galois, should be automorphisms of its Galois closure.

To be handed in on: Thursday, 23.11.2023 (during the lecture, or via eCampus).