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WS 2023/24

Algebra II - Local fields

5. Exercise sheet

Exercise 1 (4 points):

1) Let I be a filtered poset and let $(K_i, \iota_{ij}\colon K_i \to K_j, i \leq j)$ be an inductive system of fields, i.e., the K_i are fields and the ι_{ij} are ring homomorphisms. Show that the filtered union $\bigcup_{i\in I} K_i$ is a field.
2) Set $K := \mathbb{F}_p((t))$ and let $F\colon K \to K,\ x \mapsto x^p$ be the Frobenius of K. Realize the filtered union of the inductive system $K \xrightarrow{F} K \xrightarrow{F} \ldots$ and the inverse limit of the projective system $\ldots \xrightarrow{F} K \xrightarrow{F} K$ as subfields of $\overline{K}$.

Exercise 2 (4 points):

1) Show $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p \text{ prime}} \mathbb{Q}/\mathbb{Z}_{(p)}$ and $\mathbb{Q}/\mathbb{Z}_{(p)} \cong \mathbb{Q}_p/\mathbb{Z}_p$.
2) Show that $\widehat{\mathbb{Z}} \cong \mathrm{End}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}) \cong \prod_{p \text{ prime}} \mathbb{Z}_p$ and $\mathbb{Z}_p \cong \mathrm{End}_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p)$ (as rings).

Exercise 3 (4 points):

Let K be a non-archimedean local field with residue field $\mathbb{F}_q$ and let K^{sep} be a separable closure of K. Let $K^{\mathrm{tr}} \subseteq K^{\mathrm{sep}}$ be the union of all finite tamely ramified extensions L/K in K^{sep}.
1) Show that K^{tr} is Galois over K.
2) Show that $\mathrm{Gal}(K^{\mathrm{tr}}/K) \cong (\prod_{\ell \neq p \text{ prime}} \mathbb{Z}_\ell) \rtimes \widehat{\mathbb{Z}}$ with $1 \in \widehat{\mathbb{Z}}$ acting on $\prod_{\ell \neq p \text{ prime}} \mathbb{Z}_\ell$ via multiplication by q.

Exercise 4 (4 points):

Let S be a topological space.
1) Assume that S is compact and Hausdorff and that $T \subseteq S$ is a connected component. Show that T is the intersection of all open and closed subsets containing T.
2) Show that S is a profinite set if and only if S is compact, Hausdorff and totally disconnected.
Hint: For the reverse direction map S to the inverse limit of its discrete quotients S_i.
3) Show that $\{0\} \cup \{1/n \mid n \in \mathbb{N}\} \subseteq [0,1]$ and the Cantor set $\mathcal{C} \subseteq [0,1]$ are profinite sets and that as topological spaces $\mathcal{C} \cong \mathbb{Z}_2$.
Remark: In fact, $\mathcal{C} \cong \mathbb{Z}_p$ for any prime p.

To be handed in on: Thursday, 16.11.2023 (during the lecture, or via eCampus).