Dr. D. Schwein
Dr. J. Anschütz **WS 2023/24**

Algebra II - Local fields

3. Exercise sheet

Exercise 1 (4 points):

Let K be a complete non-archimedean field with norm $|-|$ and let $\pi \in K$ with $0 < |\pi| < 1$.
1) Let $S \subseteq \mathcal{O}_K$ be a system of representatives of $\mathcal{O}_K/\pi\mathcal{O}_K$. Show that the map

$$S^{\mathbb{N}} \to \mathcal{O}_K,\ (a_0, a_1, \ldots) \mapsto \sum_{n=0}^{\infty} a_n \pi^n$$

is a well-defined homeomorphism (with S given the discrete topology).
2) Let L be one of the fields $\bigcup_{n\geq 1} \mathbb{Q}_p(\mu_n)$, $\bigcup_{n\geq 1} \mathbb{Q}_p(p^{1/n}), \overline{\mathbb{Q}_p}$ with $\mu_n \subseteq \overline{\mathbb{Q}_p}^{\times}$ the subgroup of n-th roots of unity. Let $\widehat{L}$ be the completion of L for the unique norm extending $|-|_p$ on $\mathbb{Q}_p$. In each case construct an element $x \in \widehat{L} \setminus L$.

Exercise 2 (4 points):

1) Let p be a prime. Construct a sequence $a_n \in \mathbb{Q}, n \geq 0$, and $x \neq y \in \mathbb{Q}$ such that $\sum\limits_{n=0}^{\infty} a_n$ converges in $\mathbb{Q}_p$ to x, while $\sum\limits_{n=0}^{\infty} a_n$ converges in $\mathbb{R}$ to y.
2) Let K be a complete non-archimedean field. Show that an infinite product $\prod\limits_{n=0}^{\infty} a_n$ with $a_n \in K^{\times}$ converges to some $a \in K^{\times}$ if and only if $a_n \to 1, n \to \infty$.

Exercise 3 (4 points):

Let K be a complete non-archimedean field with residue field $k := \mathcal{O}_K/\mathfrak{m}_K$ of characteristic $p \geq 0$. Let $n \geq 1$ and assume that p does not divide n.
1) Show that the map $\{y \in \mathcal{O}_K \mid y^n = 1\} \to \{z \in k \mid z^n = 1\},\ y \mapsto y \bmod \mathfrak{m}_K$ is bijective.
2) Show that the map $x \mapsto x^n$ is bijective on the principal units $U_K := \{y \in \mathcal{O}_K \mid y \equiv 1 \bmod \mathfrak{m}_K\}$.
3) Let p be an odd prime. Show that the degree 2 extensions of $\mathbb{Q}_p$ are exactly $\mathbb{Q}_p(\sqrt{p}), \mathbb{Q}_p(\zeta_{p^2-1})$ or $\mathbb{Q}_p(\zeta_{p^2-1} \cdot \sqrt{p})$. Here, ζ_{p^2-1} denotes a p^2-1-th primitive root of unity.

Exercise 4 (4 points):

1) Show that $\mathbb{Q}_p^{\times} \cong \mathbb{Z} \times \mathbb{F}_p^{\times} \times U_{\mathbb{Q}_p}$ as topological groups, and that $U_{\mathbb{Q}_p}$ is the maximal $\mathbb{Z}_{(p)}$-module of $\mathbb{Q}_p^{\times}$. Here, $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at the multiplicative set $p\mathbb{Z} \setminus \{0\}$.
2) Let p_1, p_2 be primes and let $\alpha\colon \mathbb{Q}_{p_1} \cong \mathbb{Q}_{p_2}$ be any isomorphism of fields. Show that $p_1 = p_2$ and $\alpha = \mathrm{Id}_{\mathbb{Q}_{p_1}}$.
Hint: First show $p_1 = p_2$. By sheet 1, exercise 1 it is then sufficient to show that α is continuous.

To be handed in on: Thursday, 02.11.2023 (during the lecture, or via eCampus).