Dr. D. Schwein **WS 2023/24**
Dr. J. Anschütz

Algebra II - Local fields

2. Exercise sheet

Let $\nu_p\colon \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$ be the p-adic valuation and let $|-|_p := p^{-\nu_p(-)}$ be the p-adic norm.

Exercise 1 (4 points):

Prove that every non-trivial non-archimedean norm $|-|$ on $\mathbb{Q}$ is equivalent to the p-adic norm $|-|_p$ for some prime number p.

Exercise 2 (4 points):

Let $(X, d\colon X \times X \to \mathbb{R}_{\geq 0})$ be a metric space. We assume that d is non-archimedean, i.e., for $x, y, z \in X$ we have $d(x, z) \leq \max(d(x, y), d(y, z))$.
1) For $x, y, z \in X$ show that one of the following equalities holds: $d(x, y) = d(y, z)$, $d(x, z) = d(z, y)$ or $d(x, y) = d(z, x)$. In particular, each triangle in X has two sides of the same length.
2) Let $x \in X$ and $r \geq 0$. Show that each point $y \in B(x, r) := \{z \in X \mid d(x, z) \leq r\}$ is a center of the ball, i.e., $B(y, r) = B(x, r)$.
3) Show that if $x_n \to x, n \to \infty$, is a converging sequence, then for all $y \in X, y \neq x$, the sequence $d(y, x_n)$ is eventually constant (with value $d(y, x)$).

Exercise 3 (4 points):

Let K be a complete non-archimedean[1] field with norm $|-|\colon K \to \mathbb{R}_{\geq 0}$. Let $f(X) = \sum_{n=0}^{\infty} a_n X^n \in K[[X]]$ and define $r_f := \frac{1}{\limsup\limits_{n\to\infty} |a_n|^{1/n}}$, called the radius of convergence of f.
1) Let $x \in K$. Show that if $|x| < r_f$, then $f(x)$ converges, and if $|x| > r_f$, then $f(x)$ diverges.
2) Show that $r_{f+g} \geq \min\{r_f, r_g\}$, $r_{fg} \geq \min\{r_f, r_g\}$ for $g \in K[[X]]$.
3) Show that $f(X)$ converges for all $x \in T_r := \{y \in K \mid |y| = r\}$ if and only if it converges for some $x \in T_r$.

Exercise 4 (4 points):

Let K be a complete non-archimedean extension of $\mathbb{Q}_p$ with norm $|-|$ extending $|-|_p$. Set $\exp(X) := \sum\limits_{n=0}^{\infty} \frac{X^n}{n!} \in K[[X]]$ and $\log(1+X) := \sum\limits_{n=1}^{\infty} (-1)^{n-1} \frac{X^n}{n} \in K[[X]]$.
1) For an integer $n \geq 1$, write $n = a_0 + a_1 p + \ldots + a_r p^r$ with $0 \leq a_i \leq p-1$, and set $s_p(n) := a_0 + a_1 + \ldots + a_r$. Prove that $\nu_p(n!) = \frac{n - s_p(n)}{p-1}$.
2) Show that $\exp(X)$ has radius of convergence $p^{-\frac{1}{p-1}}$. If $|x| = p^{-\frac{1}{p-1}}$, determine whether $\exp(x)$ converges or not.
3) Show that $\log(1+X)$ has radius of convergence 1. If $|x| = 1$, determine whether $\log(x)$ converges or not.

To be handed in on: Thursday, 26.10.2023 (during the lecture, or via eCampus).

[1] In the literature it is common to define a non-archimedean field as a complete topological field whose topology is induced by a non-trivial non-archimedean norm (with values in $\mathbb{R}_{\geq 0}$). For clarity we don't follow this terminology and explicitly mention completeness.