Dr. D. Schwein **WS 2023/24**
Dr. J. Anschütz

Algebra II - Local fields

1. Exercise sheet

Exercise 1 (4 points):

1) Find the 7-adic expansion of $2/5 \in \mathbb{Q}$ up to 7^3, i.e., the coefficients in $\{0, 1, \ldots, 6\}$ in front of $7^0, 7^1, 7^2$.
2) Find the full 7-adic expansion of $1/6 \in \mathbb{Q}$.
3) Find a solution of the equation $x^2 = 2$ in $\mathbb{Z}_7$ up to 7^3.

Exercise 2 (4 points):

1) Let p be a prime and $a \in \mathbb{Z}$ coprime to p. Show that $a^{p^n}, n \in \mathbb{N}$, is a Cauchy sequence in $\mathbb{Q}_p$ and hence converging to some element $x \in \mathbb{Q}_p$. Show that $x \in \mathbb{Z}_p$.
Hint: Prove first that $a^{(p-1)p^{n-1}} \equiv 1$ mod p^n.
2) For a, x as in 2) show $a \equiv x \bmod p\mathbb{Z}_p$ and that $x^{p-1} = 1$. In particular, deduce that the polynomial $X^{p-1} - 1$ splits completely in $\mathbb{Q}_p$.

Exercise 3 (4 points):

Let K be a field and let $|-|: K \to \mathbb{R}_{\geq 0}$ be a norm. Show that the completion $\widehat{K}$ of K with respect to $|-|$ is a topological field, i.e., $\widehat{K}$ is a field and addition/multiplication/inversion are well-defined and continuous. Here, $\widehat{K} \times \widehat{K}$ is equipped with the product topology and $\widehat{K} \setminus \{0\}$ with the subspace topology.

Exercise 4 (4 points):

1) Let p be a prime. Show that any continuous ring automorphism of $\mathbb{Q}_p$ is the identity.
Remark: We will eventually see that any ring automorphism of $\mathbb{Q}_p$ is continuous.
2) Show that any ring automorphism of $\mathbb{R}$ is the identity.
Hint: Show first that each ring automorphism must preserve the order on $\mathbb{R}$.

To be handed in on: Thursday, 19.10.2023 (during the lecture, or via eCampus).