LECTURES ON THE FARGUES-FONTAINE CURVE

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Abstract. The topic of these lecture notes is the (schematic) Fargues-Fontaine curve, following [10]. We aim to prove its basic properties (e.g. that it is a Dedekind scheme), to sketch the classification of its vector bundles and finally to deduce the theorem "weakly admissible implies admissible" of p-adic Hodge theory.

Contents

1. General notations and remarks

Nearly all proofs are taken from [10]. Of course, all errors or inaccuracies are on my side. Any comments/corrections are welcome!

The author wants to thank Ben Heuer for replacing him in two lectures and for his detailed reading of the manuscript.

Some material has been revisited by the author and differs now from the original lecture (e.g. Section 3, Section 4, Section 5). Moreover, the manuscript contains some additional details which were not presented in the lectures.

The following notation will be used frequently.

- p a fixed prime
- $E/\mathbb{Q}_p$ a finite extension
- $\mathcal{O}_E \subseteq E$ the ring of integers
- $\pi \in \mathcal{O}_E$ a uniformizer
- $\mathbb{F}_q = \mathcal{O}_E/(\pi)$ the residue field of $\mathcal{O}_E$
- $F/\mathbb{F}_q$ a non-archimedean[1], algebraically closed extension
- $\mathcal{O}_F := \{x \in F \mid |x| \leq 1\} \subseteq F$ the ring of integers of F[2]
- $\mathfrak{m}_F := \{x \in \mathcal{O}_F \mid |x| < 1\} \subseteq \mathcal{O}_F$ the maximal ideal of $\mathcal{O}_F$
- $k := \mathcal{O}_F/\mathfrak{m}_F$ the residue field of $\mathcal{O}_F$
- $\mathbb{A}_{\mathrm{inf}} = \mathbb{A}_{\mathrm{inf}\,E,F} = W_{\mathcal{O}_E}(\mathcal{O}_F)$ the ring of ramified Witt vectors of $\mathcal{O}_F$

The ring $\mathcal{O}_F$ is a non-noetherian local integral domain with exactly two prime ideals, $\{0\}$ and $\mathfrak{m}_F$, its ideals are linearly ordered and each finitely generated ideal is principal.

2. Lecture of 16.10.2019: Introduction to p-adic Hodge theory

This lecture is meant to give a short motivational overview of p-adic Hodge theory and the theorem of Colmez/Fontaine ([7]) that "weakly admissible" implies "admissible", cf. Theorem 2.11. Only in this lecture we will use more theory from arithmetic geometry (such as étale cohomology theory, ...). For understanding the construction of the Fargues-Fontaine curve, knowledge of valuation theory, local fields and (basic) scheme theory is sufficient.

Fix a prime p, let $\mathbb{Q}_p$ be field of p-adic numbers and let K be a finite extension of $\mathbb{Q}_p$[3]. The usual p-adic norm

$$|-|_p\colon \mathbb{Q}_p \to \mathbb{R}_{\geq 0}$$

on $\mathbb{Q}_p$ extends uniquely to a norm

$$|-|\colon \overline{K} \to \mathbb{R}_{\geq 0}$$

on some fixed algebraic closure $\overline{K}$ of K. Let

$$C := \widehat{\overline{K}}$$

[1] By definition this means that F is a complete topological field whose topology is induced by a non-trivial non-archimedean norm $|-|\colon F \to \mathbb{R}_{\geq 0}$.

[2] The subring $\mathcal{O}_F$ does not depend on the choice of a norm $|-|$ on F as it consists precisely of the subset of powerbounded elements, i.e., those elements $x \in F$ such that $\{x^n \mid n \geq 0\}$ is bounded, where a subset $A \subset F$ is bounded if for all open neighborhoods U of 0 there exists an open neighborhood V of 0 such that $A \cdot V \subseteq U$.

[3] It is sufficient to assume that K is a discretely valued extension of $\mathbb{Q}_p$ with perfect residue field in the following discussions.

be the completion of $\overline{K}$ with respect to the norm $|-|$. Then C is again algebraically closed [4] and the action of the Galois group

$$G_K := \mathrm{Gal}(\overline{K}/K)$$

on $\overline{K}$ extends by continuity to an action of G_K on C.

Let $X \to \mathrm{Spec}(K)$ be a proper, smooth morphism.

A basic theorem of p-adic Hodge theory is the "Hodge-Tate decomposition".[5]

Theorem 2.1 (Faltings[8])**.** *For $n \geq 0$ there exists a natural G_K-equivariant isomorphism*

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K}) \otimes_K C(-j),$$

where $\Omega^j_{X/K} := \Lambda^j(\Omega^1_{X/K})$ is the sheaf of j-forms on X.

Remark 2.2.

- Here $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ denotes the n-th p-adic étale cohomology of X, which is a finite dimensional $\mathbb{Q}_p$-vector space equipped with a continuous action of G_K.
- If M is a $\mathbb{Z}_p$-module with an action of G_K, then the j-th Tate twist of M is defined by
$$M(j) := M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)^{\otimes j},\ j \in \mathbb{Z},$$
with the diagonal G_K-action, where
$$\mathbb{Z}_p(1) := \varprojlim_k \mu_{p^k}(\overline{K})$$
is the Tate module of the p^∞-roots of unity in $\overline{K}$ (with its canonical Galois action). As $\mathbb{Z}_p$-modules, $\mathbb{Z}_p(1) \cong \mathbb{Z}_p$.
- In Theorem 2.1 G_K acts diagonally on the LHS, and via $C(-j)$ on the RHS.
- The theorem has a precursor in complex Hodge theory: If Y is a compact Kähler manifold, then
$$H^n(Y, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \bigoplus_{i+j=n} H^i(Y, \Omega^j_Y),$$
where Ω^j_Y is the sheaf of holomorphic j-forms on Y and $H^*(Y, \mathbb{Z})$ the singular cohomology of Y.
- The Theorem 2.1 holds by work of Scholze (cf. [21]) for proper, smooth *rigid-analytic varieties* as well.

The Tate twists on the RHS in Theorem 2.1 are necessary to get G_K-equivariance: Set $X = \mathbb{P}^1_K$ and $n = 2$. Then the LHS of Theorem 2.1 is G_K-equivariantly isomorphic to

$$C(-1)$$

as

$$H^2_{\text{ét}}(X, \mathbb{Z}_p) \cong H^1_{\text{ét}}(\mathbb{G}_{m,\overline{K}}, \mathbb{Z}_p) \cong \mathbb{Z}_p(-1),$$

while the RHS is isomorphic to $C(-1)$ as

$$H^1(X, \Omega^1_{X/K}) \cong H^1(X, \mathcal{O}(-2)) \cong K.$$

[4]By Krasner's lemma.

[5]We recommend [2] for an approach to the Hodge-Tate decomposition via perfectoid spaces.

To see that C and $C(-1)$ are not isomorphic as C-modules with a semilinear G_K-action, we cite the following fundamental theorem of Tate.

Theorem 2.3. *The continuous group cohomology of G_K with coefficients in $C(j)$ is given by*

(1) $H^i_{\mathrm{cts}}(G_K, C(j)) = 0$ *if* $j \neq 0$ *or* $i \geq 2$,
(2) $K \cong H^0_{\mathrm{cts}}(G_K, C) \cong H^1_{\mathrm{cts}}(G_K, C)$.

Here by definition, continuous group cohomology of G_K relates to the usual Galois cohomology (with discrete coefficients) by the formula

$$H^*_{\mathrm{cts}}(G_K, C(j)) := H^*(R\varprojlim_k R\Gamma(G_K, \mathcal{O}_C/p^k(j)))[1/p]$$

where $\mathcal{O}_C \subseteq C$ is the ring of integers.

Even the statement that $H^0_{\mathrm{cts}}(G_K, C) = C^{G_K} = K$ is non-obvious as the completion C of $\overline{K}$ contains much more elements than $\overline{K}$.

The statement in Theorem 2.3 that $H^0_{\mathrm{cts}}(G_K, C(j)) = 0$ for $j \neq 0$ implies that

$$C \ncong C(j)$$

as G_K-modules.

The combination of Remark 2.2 and Theorem 2.3 yields an interesting corollary.

Corollary 2.4. *For $n \geq 0, j \geq 0$*

$$H^{n-j}(X, \Omega^j_{X/K}) \cong (H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C(j))^{G_K}.$$

That is, the geometric information $H^{n-j}(X, \Omega^j_{X/K})$ is encoded in the arithmetic of the Galois action on $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$. As a slogan: "$p$-adic étale cohomology knows Hodge cohomology".

The converse ("Hodge cohomology knows p-adic étale cohomology") is not true:

- If X is an elliptic curve[6], then $H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ with its Galois action can detect whether X has good or semistable reduction, but the Hodge cohomology $H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega^1_{X/K})$ can't.
- More concretely, if $X = \mathrm{Spec}(L)$ with L/K finite, then the Galois action on $H^0_{\text{ét}}(X, \mathbb{Q}_p) \cong \prod_{L \hookrightarrow \overline{K}} \mathbb{Q}_p$ determines L (by Galois theory), but the vector space $H^0(X, \mathcal{O}_X)$ only determines the degree of L over K.

Corollary 2.4 has a nice application to complex geometry, cf. [15]. Recall that a projective, smooth scheme Y over $\mathrm{Spec}(\mathbb{C})$ is called a smooth minimal model if the canonical bundle ω_Y is nef(=numerically effective), i.e., $\omega_Y \cdot Z \geq 0$ for any curve $Z \subseteq Y$.

Theorem 2.5. *[Veys, Wang, Ito] Let Y, Y' be two smooth birational minimal models, then*

$$\dim_{\mathbb{C}} H^i(Y, \Omega^j_Y) = \dim_{\mathbb{C}} H^i(Y', \Omega^j_{Y'})$$

for $i, j \geq 0$.

[6] or an abelian variety

Proof. (Sketch) The schemes Y, Y' being birational and smooth minimal models implies that Y, Y' are K-equivalent, i.e., that there exists a diagram

$$\begin{array}{ccc} & Z & \\ {}^{f}\swarrow & & \searrow^{g} \\ Y & & Y' \end{array}$$

with Z proper and smooth over $\mathbb{C}$, f, g proper and birational, such that

$$f^*K_Y \cong g^*K_{Y'}.$$

Here $K_Y, K_{Y'}$ denote the canonical bundles on Y and Y'. This situation can be spread out over a finitely generated $\mathbb{Z}$-algebra $A \subseteq \mathbb{C}$. As Hodge numbers are locally constant for proper, smooth morphisms of schemes over $\mathbb{Q}$[7] one can reduce to the case $A = \mathcal{O}_F[\frac{1}{N}]$ for $F/\mathbb{Q}$ finite and $N \in \mathbb{N}$ large. We arrive in the situation of a diagram of proper, smooth A-schemes

$$\begin{array}{ccc} & \mathcal{Z} & \\ {}^{\tilde{f}}\swarrow & & \searrow^{\tilde{g}} \\ \mathcal{Y} & & \mathcal{Y}' \end{array}$$

with $\tilde{f}, \tilde{g}$ birational and $\tilde{f}^*K_{\mathcal{Y}} \cong \tilde{g}^*K_{\mathcal{Y}'}$. The theory of p-adic integration then implies that

$$|\mathcal{Y}(\mathbb{F}_{l^k})| = |\mathcal{Y}'(\mathbb{F}_{l^k})| \tag{1}$$

for all primes l not dividing N and all $k \geq 0$. Let us fix a prime p, not dividing N. The equality (1) and the Weil conjectures imply that the semisimplified[8] Galois representations

$$H^*_{\text{ét}}(\mathcal{Y}_{\overline{\mathbb{F}}_l}, \mathbb{Q}_p)^{\text{ss}} \cong H^*_{\text{ét}}(\mathcal{Y}'_{\overline{\mathbb{F}}_l}, \mathbb{Q}_p)^{\text{ss}}$$

are isomorphic for any l not dividing pN. By Chebotarev this implies an isomorphism

$$H^*_{\text{ét}}(\mathcal{Y}_{\overline{F}}, \mathbb{Q}_p)^{\text{ss}} \cong H^*_{\text{ét}}(\mathcal{Y}'_{\overline{F}}, \mathbb{Q}_p)^{\text{ss}}$$

of semisimplified global Galois representations. Now pick a place $\mathfrak{p}|p$ and set $K := F_{\mathfrak{p}}$. Then the semisimplified local Galois representations

$$H^*_{\text{ét}}(\mathcal{Y}_{\overline{K}}, \mathbb{Q}_p)^{\text{ss}} \cong H^*_{\text{ét}}(\mathcal{Y}'_{\overline{K}}, \mathbb{Q}_p)^{\text{ss}}$$

are isomorphic, too. The Hodge-Tate comparison Theorem 2.1 or rather Corollary 2.4 (plus a small argument handling the passage to the semisimplification) imply that

$$\dim_K H^i(\mathcal{Y}_K, \Omega^j_{\mathcal{Y}_K/K}) = \dim_K H^i(\mathcal{Y}'_K, \Omega^j_{\mathcal{Y}'_K/K})$$

for all $i, j \geq 0$. This was the desired statement. □

Another application of Remark 2.2 is an algebraic proof of the degeneration of the Hodge-de Rham spectral sequence.

[7] This can be tested after base change to $\mathbb{C}$ where usual Hodge theory applies.

[8] The Weil conjectures only imply that the traces of (geometric) Frobenius agree, but this allows to conclude the equivalence on semisimplifications as the coefficients are of characteristic 0.

Theorem 2.6. *Let $Y \to \operatorname{Spec}(\mathbb{C})$ be a proper, smooth scheme.*[9] *Then the Hodge-de Rham spectral sequence*

$$E_1^{i,j} = H^j(Y, \Omega_Y^i) \Rightarrow H_{\mathrm{dR}}^{i+j}(Y)$$

degenerates, where

$$H_{\mathrm{dR}}^*(Y) := H^*(R\Gamma(Y, 0 \to \mathcal{O}_Y \xrightarrow{d} \Omega_Y^1 \xrightarrow{d} \Omega_Y^2 \to \ldots))$$

denotes the de Rham cohomology of Y.

Proof. (Sketch) First reduce to the case that $Y = X \times_{\operatorname{Spec}(K)} \operatorname{Spec}(\mathbb{C})$ for some $K/\mathbb{Q}_p$ finite[10] and some embedding $K \hookrightarrow \mathbb{C}$. It suffices to show

$$\dim_{\mathbb{C}} H_{\mathrm{dR}}^n(Y) = \sum_{i+j=n} \dim_{\mathbb{C}} H^j(Y, \Omega_Y^i)$$

for all $n \geq 0$. We now see that

$$\begin{aligned} & \dim_C H_{\mathrm{dR}}^n(Y) \\ = \ & \dim_{\mathbb{Q}_p} H^n(Y(\mathbb{C}), \mathbb{Q}_p) \\ = \ & \dim_{\mathbb{Q}_p} H_{\text{ét}}^n(X_{\overline{K}}, \mathbb{Q}_p) \\ = \ & \sum_{i+j=n} \dim_K H^j(X, \Omega_{X/K}^i) \\ = \ & \sum_{i+j=n} \dim_{\mathbb{C}} H^j(Y, \Omega_Y^i), \end{aligned}$$

using various comparison theorems (de Rham vs singular, singular vs étale, coherent cohomology over K vs coherent cohomology over $\mathbb{C}$) and the Hodge-Tate decomposition Remark 2.2. □

The de Rham cohomology $H_{\mathrm{dR}}^n(X)$ of a proper, smooth scheme over K together with its filtration[11] is a slightly finer invariant than the Hodge cohomology

$$\bigoplus_{i+j=n} H^j(X, \Omega_{X/K}^i).$$

This leads to the following question:

Does the G_K-representation $H_{\text{ét}}^n(X_{\overline{K}}, \mathbb{Q}_p)$ determine $H_{\mathrm{dR}}^n(X)$ together with its filtration?

Again the answer is yes. However, the result is more complicated to state than the Hodge-Tate comparison as it involves Fontaine's field B_{dR} of p-adic periods.

Theorem 2.7 ("de Rham comparison")**.** *For $n \geq 0$, there exists a natural G_K-equivariant, filtered isomorphism*

$$H_{\text{ét}}^n(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \cong H_{\mathrm{dR}}^n(X) \otimes_K B_{\mathrm{dR}}.$$

Numerous authors have proven the de Rham comparison, Faltings, Scholze, Beilinson, ...

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[9] The statement holds true (by similar arguments) if $\mathbb{C}$ is replaced by any field L of characteristic 0, but it fails over fields of positive characteristic.

[10] and some prime p

[11] The abutment filtration of the Hodge-de Rham spectral sequence.

Remark 2.8.
- Here B_{dR} is Fontaine's field of p-adic periods, which is the fraction field of a complete discrete valuation ring B_{dR}^+ with residue field C, cf. 4.6.[12] As such B_{dR} is naturally filtered by
$$\mathrm{Fil}^j B_{\mathrm{dR}} := \xi^j B_{\mathrm{dR}}^+, j \in \mathbb{Z},$$
where $\xi \in B_{\mathrm{dR}}^+$ is a uniformizer.
- The G_K-action is diagonally on LHS, via B_{dR} on RHS. The filtration is via B_{dR} on the LHS, and diagonally on the RHS.
- There exists a canonical isomorphism
$$\mathrm{gr}^\bullet B_{\mathrm{dR}} \cong B_{\mathrm{HT}} := \bigoplus_{j \in \mathbb{Z}} C(j),$$
which implies that the de Rham comparison recovers the Hodge-Tate decomposition by passing to the associated graded. Theorem 2.3 therefore implies that $B_{\mathrm{dR}}^{G_K} \cong K$.
- The case $X = \mathbb{P}^1_K$, $n = 2$ in Theorem 2.7 yields a canonical G_K-equivariant isomorphism
$$\alpha\colon \mathbb{Q}_p(-1) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \cong B_{\mathrm{dR}}.$$
Thus, we see that B_{dR} contains a canonical G_K-stable line $\mathbb{Q}_p t \subseteq B_{\mathrm{dR}}$ (where t is not canonical), on which G_K acts via the cyclotomic character
$$\chi_{\mathrm{cycl}}\colon G_K \to \mathbb{Z}_p^\times,$$
i.e., $\mathbb{Q}_p t \cong \mathbb{Q}_p(1)$. Fontaine gave a concrete description for such t, namely for $\varepsilon \in T_p(\mu_{p^\infty}(\overline{K}))$ a generator, set
$$t := \log([\varepsilon]) \in B_{\mathrm{dR}}.$$
The analogue of t in complex geometry is $2\pi i$. The element $t \in B_{\mathrm{dR}}^+$ is a uniformizer.

Assume from now on that X has good reduction, i.e., $X = \mathcal{X}_K$ is the generic fiber for $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$ a proper smooth morphism. Let X_0 be the special fiber of $\mathcal{X}$.

In this situation one gets a great refinement of Theorem 2.7, called the crystalline comparison.

Theorem 2.9 ("crystalline comparison")**.** *For $n \geq 0$ there exists a natural G_K-equivariant, filtered φ-equivariant isomorphism*
$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}} \cong H^n_{\mathrm{cris}}(X_0/\mathcal{O}_{K_0}) \otimes_{\mathcal{O}_{K_0}} B_{\mathrm{cris}}.$$

Again many people have proven the crystalline comparison, Faltings, Tsuji, Nizioł, Bhatt/Morrow/Scholze... .

Remark 2.10.
- Here $K_0 \subseteq K$ denotes the maximal unramified subextension. This implies that p is a uniformizer in the ring of integers $\mathcal{O}_{K_0}$ of K_0 and there exists a canonical Frobenius lift φ on $\mathcal{O}_{K_0}$[13]

[12]This implies that abstractly $B_{\mathrm{dR}}^+ \cong C[[t]]$, but there exists no such isomorphism which is G_K-equivariant: there exists Hodge-Tate representations, which are not de Rham.

[13]There is a canonical isomorphism of $\mathcal{O}_{K_0}$ to the Witt vectors $W(k)$ of the residue field k of $\mathcal{O}_K$.

- $H^n_{\mathrm{cris}}(X_0/\mathcal{O}_{K_0})$ denotes the crystalline cohomology of X_0 with respect to $\mathcal{O}_{K_0}$ which is, roughly, the de Rham cohomology of a smooth lift of X_0 to $\mathcal{O}_{K_0}$.[14] By functoriality there exists a natural Frobenius φ endomorphism on $H^n_{\mathrm{cris}}(X_0/\mathcal{O}_{K_0})$ (which is semilinear over the Frobenius on $\mathcal{O}_{K_0}$).
- The data $H^n_{\mathrm{cris}}(X_0/\mathcal{O}_{K_0})$ with its Frobenius and the Hodge filtration over K (coming from the crystalline-de Rham comparison) is an example of a filtered φ-module $(D, \varphi_D, \mathrm{Fil}^\bullet(D_K))$ over K, i.e. a finite dimensional K_0-vector space D together with an isomorphism $\varphi_D \colon \varphi^* D \cong D$ and a decreasing, separated and exhaustive filtration on the base change $D_K := D \otimes_{K_0} K$.
- B_{cris} denotes Fontaine's ring of crystalline p-adic periods. Firstly, define

$$\mathbb{A}_{\mathrm{cris}} := H^0_{\mathrm{cris}}((\mathcal{O}_C/p)/\mathbb{Z}_p),\ B^+_{\mathrm{cris}} := \mathbb{A}_{\mathrm{cris}}[1/p].$$

  By functoriality, there exists a natural Frobenius φ on $\mathbb{A}_{\mathrm{cris}}$. It turns out that B^+_{cris} embeds into B^+_{dR} with image stable by G_K and containing $t = \log[\varepsilon]$. Finally,

$$B_{\mathrm{cris}} := B^+_{\mathrm{cris}}[1/t]$$

  and $\varphi(t) = pt$.
- Inverting t in Theorem 2.9 is necessary as can already be seen in the case $n = 2, X = \mathbb{P}^1_K$.
- The analogous statement in ℓ-adic étale cohomology, where $\ell \neq p$, is the following: If $f \colon \mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$ is a proper, smooth morphism, then $R^* f_*(\mathbb{Q}_\ell)$ is a local system on $\mathrm{Spec}(\mathcal{O}_K)$ and thus in particular, there exists a natural G_K-equivariant isomorphism[15]

$$H^*_{\acute{e}t}(\mathcal{X}_{\overline{\eta}}, \mathbb{Q}_\ell) \cong H^*_{\acute{e}t}(\mathcal{X}_{\overline{s}}, \mathbb{Q}_\ell)$$

  where $\eta, s \in \mathrm{Spec}(\mathcal{O}_K)$ are the generic resp. special point. Similarly, for a proper, smooth morphism $f \colon Y \to Y'$ of smooth complex manifolds, the pushforward $R^* f_*(\mathbb{Q})$ is a local system.
- The linear algebra related to the crystalline comparison is more mysterious than that of its ℓ-adic counterpart, i.e. when $H^*_{\acute{e}t}(\mathcal{X}_\eta, \mathbb{Q}_\ell)$ is replaced by $H^*_{\acute{e}t}(\mathcal{X}_\eta, \mathbb{Q}_p)$ and $H^*_{\acute{e}t}(\mathcal{X}_s, \mathbb{Q}_\ell)$ by $H^*_{\mathrm{cris}}(\mathcal{X}_s/\mathcal{O}_{K_0})$. How can one pass from a continuous G_K-representation on a finite dimensional $\mathbb{Q}_p$-vector space to a finite dimensional K_0-vector space with a Frobenius and a filtration (over K)? This was Grothendieck's question on the "mysterious functor". This question was resolved by Fontaine, who introduced the functors

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{Q}_p}(G_K) & \to & \{\text{filtered } \varphi-\text{modules}\} \\ V & \mapsto & D_{\mathrm{cris}}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{G_K} \\ V_{\mathrm{cris}}(D) = \mathrm{Fil}^0(D \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=1} & \leftarrow\!\shortmid & D \end{array}$$

In analogy with the ℓ-adic case, one should expect that $H^*_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p)$ and $H^*_{\mathrm{cris}}(X_0/\mathcal{O}_{K_0})[1/p]$ contain "the same information". That this is the case is the content of the theorem "weakly admissible implies admissible" of Colmez and Fontaine, cf. [7]

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[14] Note that we can't take $\mathcal{X}$ here as $\mathcal{X}$ is just a smooth lift of X_0 to $\mathcal{O}_K$.

[15] This has the interesting corollary that the G_K-action on $H^*_{\acute{e}t}(\mathcal{X}_{\overline{\eta}}, \mathbb{Q}_\ell)$ is unramified, which yields a cohomological obstruction for a scheme over K to admit good reduction. By 2.9, the analogous statement for $\ell = p$ is that $H^*_{\acute{e}t}(\mathcal{X}_{\overline{\eta}}, \mathbb{Q}_p)$ is crystalline. But note that crystalline representations are unramified if and only if the inertia acts with finite image, which is usually not the case (e.g., the cyclotomic character).

Theorem 2.11 ("weakly admissible implies admissible")**.** *The functors D_{cris}, V_{cris} restrict to equivalences between*

$$\{crystalline\ G_K - representations\}$$

and

$$\{weakly\ admissible\ filtered\ \varphi - modules\ over\ K\}.$$

Remark 2.12.
- A representation $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is called crystalline if $\dim_{K_0}(D_{\mathrm{cris}}(V)) = \dim_{\mathbb{Q}_p} V$.
- The condition "weakly admissible" is related to the statement that the "Newton polygon lies above the Hodge polygon".

A sketch of proof of this theorem will be the aim of this course, cf. 14. The essential ingredient will be the Fargues-Fontaine curve, cf. 8.4,

$$X_{\mathrm{FF}} := \mathrm{Proj}(\bigoplus_{d\geq 0}(B^+_{\mathrm{cris}})^{\varphi=p^d})$$

over $\mathbb{Q}_p$ (a Dedekind scheme!) together with the relation of its (G_K-equivariant) vector bundles to $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ resp. to filtered φ-modules. The rings $B^{(+)}_{\mathrm{dR}}, B^{(+)}_{\mathrm{cris}}, \ldots$ are closely related to functions on X_{FF} (or related objects). For example, B^+_{dR} will be isomorphic to the completion of X_{FF} at some closed point $\infty \in X_{\mathrm{FF}}$.

3. Lecture of 23.10.2019: Witt vectors (by Ben Heuer)

In our discussion of ramified Witt vectors and perfectoid rings we follow [10, 1.2.1.] and [3, Section 3].[16]

The following innocent lemma, or rather "key lemma for everything", is the starting point for many constructions in p-complete rings. It expresses the fact that the q-th power map is contracting for the p-adic (or π-adic) topology. Recall that we follow the notations in 1.

Lemma 3.1. *Let A be a $\mathcal{O}_E$-algebra, $I \subseteq A$ an ideal such that $\pi \in I$. Let $a, b \in A$ be two elements such that $a \equiv b \bmod I$. Then*

$$a^{q^k} \equiv b^{q^k} \ mod\ I^{k+1}$$

for any $k \geq 0$.

Proof. It suffices to prove that if $a \equiv b \bmod I^k$ with $k \geq 1$, then

$$a^q \equiv b^q \bmod I^{k+1}.$$

Write $b = a + c$ with $c \in I^k$. Then

$$b^q = a^q + \binom{q}{1} a^{q-1} + \ldots + c^q$$

and the terms different from a^q on the right hand side lie in I^{k+1} as $c \in I^k$ and $\pi \in I$. □

We now introduce the "tilt" of a ring.

[16] The presentation follows roughly the lecture, which was given by Ben Heuer.

Definition 3.2. Let A be a π-complete $\mathcal{O}_E$-algebra. Then we set

$$A^\flat := \varprojlim_{x\mapsto x^q} A/\pi = \{(a_0, a_1, \ldots) \in \prod_{\mathbb{N}} A/\pi \mid a_{i+1}^q = a_i\},$$

the "tilt" of A.

The ring $A^\flat$ is always a perfect $\mathbb{F}_q = \mathcal{O}_E/\pi$-algebra. Namely, the q-Frobenius on $A^\flat$ is has as inverse the map

$$(a_0, a_1, \ldots) \mapsto (a_1, a_2, \ldots).$$

Thus the tilt defines a functor[17]

$$(-)^\flat \colon \{\pi - \text{complete } \mathcal{O}_E - \text{algebras}\} \to \{\text{perfect } \mathbb{F}_q - \text{algebras}\}.$$

The tilt can be "rather small", e.g., $\mathcal{O}_E^\flat \cong \mathbb{F}_q$. If A is a perfectoid $\mathcal{O}_E$-algebra, cf. Definition 3.12, the tilt is however "rather large".

As another application of Lemma 3.1 we mention the following invariance of q-power compatible systems of elements under pro-infinitesimal thickenings.

Proposition 3.3. *Let A be a π-complete $\mathcal{O}_E$-algebra, $I \subseteq A$ an ideal such that $\pi \in I$ and A is I-adically complete. Then the canonical morphism (of multiplicative monoids)*

$$\varprojlim_{x\mapsto x^q} A \to (A/I)^\flat, \ (a_0, a_1, \ldots) \mapsto (\overline{a_0}, \overline{a_1}, \ldots)$$

is bijective.

In particular, the LHS side acquires naturally a ring structure. Explicitly, if $(a_0, a_1, \ldots), (b_0, b_1, \ldots) \in \varprojlim_{x\mapsto x^q} A$, then

$$(a_0, a_1, \ldots) + (b_0, b_1, \ldots) = (\lim_{n\to\infty} (a_n + b_n)^{q^n}, \lim_{n\to\infty} (a_n + b_n)^{q^{n-1}}, \ldots).$$

Proof. Let $x = (x_0, x_1, \ldots) \in (A/I)^\flat$ and lift each x_i to some $\tilde{x}_i \in A$. We claim that the sequence $\{\tilde{x}_i^{q^i}\}_{i\geq 0} \subseteq A$ is a Cauchy sequence for the I-adic topology. To see this let $j \geq i$. Then by the key lemma Lemma 3.1

$$\tilde{x}_j^{q^j} \equiv \tilde{x}_i^{q^i} \mod I^{i+1}$$

as

$$\tilde{x}_j^{q^{j-i}} \equiv x_j^{q^{j-i}} = x_i \equiv \tilde{x}_i \mod I.$$

Thus, setting

$$x^\sharp := \varinjlim_{i\to\infty} \tilde{x}_i^{q^i} \in A$$

is well-defined. A similar application of Lemma 3.1 implies that $x^\sharp$ is independent of the choice of lift $\tilde{x}_i$. Thus

$$(-)^\sharp \colon (A/I)^\flat \to A$$

is a well-defined, and multiplicative, map and

$$(A/I)^\flat \to \varprojlim_{x\mapsto x^q} A, \ x \mapsto (x^\sharp, (x^{1/q})^\sharp, \ldots).$$

is the desired inverse. ∎

[17] The π-adic completeness is not necessary, we only put it as we will only consider the tilt of π-complete rings.

If A/π is perfect, then $A/\pi \cong \varprojlim_{\mathrm{Frob}} A/\pi,\ a \mapsto (a, a^{1/q}, \ldots)$ is an isomorphism and the multiplicative map

$$[-]\colon A/\pi \cong \varprojlim_{\mathrm{Frob}} A/\pi \xrightarrow{(-)^\sharp} A$$

is classically called the Teichmüller lift. The Teichmüller lift defines a natural, non-additive (!), section of the projection $A \to A/\pi$.

In particular, if A is a π-complete, π-torsion free $\mathcal{O}_E$-algebra, then we can write each $a \in A$ uniquely in the form

$$a = \sum_{i\geq 0}^{\infty} [x_i]\pi^i$$

with $x_i \in A/\pi$ and thus as sets

$$A \cong (A/\pi)^{\mathbb{N}},\ a \mapsto (a_i)_{i\geq 0}.$$

But what can be said about the ring structure on A? As a motivation let's try for given $x = \sum\limits_{i=0}^{\infty}[x_i]\pi^i, y = \sum\limits_{i=0}^{\infty}[y_i]\pi^i \in A$ to find the sequence $(z_i)_{i\geq 0}$ such that

$$x + y = \sum_{i=0}^{\infty}[z_i]\pi^i.$$

Calculating modulo π shows

$$[x_0] + [y_0] \equiv [z_0] \bmod \pi$$

and thus necessarily

$$z_0 = x_0 + y_0.$$

Calculating modulo π^2 we find

$$[z_1]\pi \equiv [x_0] + [y_0] - [x_0 + y_0] + \pi[x_1 + y_1] \bmod \pi^2$$

and thus we are seeking to divide $[x_0] + [y_0] - [x_0 + y_0]$ by π. Now

$$[x_0^{1/q} + y_0^{1/q}] \equiv [x_0^{1/q}] + [y_0^{1/q}] \bmod \pi$$

and thus by Lemma 3.1

$$[x_0 + y_0] = [x_0^{1/q} + y_0^{1/q}]^q \equiv ([x_0^{1/q}] + [y_0^{1/q}])^q = \sum_{i=0}^{q}\binom{q}{i}[x_0^{i/q}][y_0^{q-i/q}] \bmod \pi^2$$

But $\pi|\binom{q}{i}$ for $1 \leq i \leq q-1$ (as $\pi|p$) and thus

$$\frac{[x_0 + y_0] - [x_0] - [y_0]}{\pi} \equiv \sum_{i=1}^{q-1}\frac{\binom{q}{i}}{\pi}[x_0^{i/q}][y_0^{q-i/q}] \bmod \pi$$

and we can set

$$z_1 := x_1 + y_1 - \sum_{i=1}^{q-1}\frac{\binom{q}{i}}{\pi}[x_0^{i/q}][y_0^{q-i/q}].$$

The upshot is that there exists universal formulas[18] for the $z_i, i \geq 0$, although these are rather useless and complicated (cf. Example 3.6).

[18] In particular, the $\mathcal{O}_E$-algebra A is uniquely determined by the requirements that A is π-adically complete, π-torsion free and A/π is perfect.

Fortunately, the strange ring structure on $(A/\pi)^{\mathbb{N}}$, making it isomorphic to A, can also be introduced by more abstract reasoning. This works as follows and yields the ring of (ramified) Witt vectors.

Set

$$\mathcal{W}_{n,\pi} := \sum_{i=0}^{n} \pi^i X_i^{q^{n-i}} \in \mathcal{O}_E[X_0, \ldots, X_n].$$

(If $E = \mathbb{Q}_p, \pi = p$ these are the classical Witt polynomials, leading to the classical Witt vectors as, for example, discussed in [26].) Define the functor

$$\mathcal{F} \colon (\mathcal{O}_E - \mathrm{alg}) \to (\mathrm{Sets}),\ \mathrm{A} \mapsto \mathrm{A}^{\mathbb{N}}.$$

Note that we consider $\mathcal{F}$ as a functor on *all* $\mathcal{O}_E$-algebras even though in the end we will only be interested in the case that A is perfect.

Lemma 3.4. *There exists a unique factorization*

$$\begin{array}{ccc} (\mathcal{O}_E - \mathrm{alg}) & \xrightarrow{\mathcal{F}} & (\mathrm{Sets}) \\ \downarrow{\scriptstyle W_{\mathcal{O}_E,\pi}} & \nearrow & \\ (\mathcal{O}_E - \mathrm{alg}) & & \end{array}$$

such that for any $\mathcal{O}_E$-algebra A the natural transformation

$$W_{\pi,A} \colon W_{\mathcal{O}_E,\pi}(A) \to A^{\mathbb{N}},\ (a_0, a_1, \ldots) \mapsto (\mathcal{W}_{n,\pi}(a_0, \ldots, a_n))_{n\geq 0} \tag{2}$$

is a morphism of $\mathcal{O}_E$-algebras.

Remark 3.5. In other words, for any $\mathcal{O}_E$-algebra there exists a natural ring structure on

$$W_{\mathcal{O}_E,\pi}(A) = A^{\mathbb{N}}$$

such that Equation (2) is a homomorphism of rings. The ring $W_{\mathcal{O}_E,\pi}(A)$ is called the ring of ramified Witt vectors of A.[19]

Note that if $\pi A = 0$, then

$$W_{\pi,A}(a_0, a_1, \ldots) = (a_0, a_0^q, a_0^{q^2}, \ldots).$$

Thus, even if one is only interested in $\mathcal{O}_E$-algebras A with $\pi A = 0$, it is important to consider the functor $\mathcal{F}$ on *all* $\mathcal{O}_E$-algebras. Lemma 3.4 is taken from [10, Lemme 1.2.1].

Proof. We claim that if A is a π-torsion free $\mathcal{O}_E$-algebra with a lift $\varphi \colon A \to A$ of the q-Frobenius, i.e., $\varphi(a) = a^q \bmod \pi$, then the natural transformation

$$W_{\pi,A} \colon W_{\mathcal{O}_E,\pi}(A) \to A^{\mathbb{N}},\ (a_0, a_1, \ldots) \mapsto (\mathcal{W}_{n,\pi}(a_0, \ldots, a_n))_{n\geq 0}$$

is injective with image

$$\{(b_i)_{i\geq 0} \in A^{\mathbb{N}} \mid b_{i+1} \equiv \varphi(b_i) \bmod \pi^{i+1}\}.$$

This in particular implies that $W_{\pi,A}$ is bijective if π is a unit in A. The injectivity follows from the definition of the polynomials $\mathcal{W}_{n,\pi}$ and π-torsion freeness of A (and does not require the existence of a Frobenius lift). Moreover,

$$\mathcal{W}_{i+1,\pi}(a_0, a_1, \ldots, a_{i+1}) = \sum_{j=0}^{i+1} \pi^j a_j^{q^{i+1-j}} \equiv \mathcal{W}_{i,\pi}(a_0, \ldots, a_i) \bmod \pi^{i+1}.$$

[19] Up to canonical isomorphism, it does not depend on π, cf. [10, Definition 1.2.2.].

Let $(b_i)_{i\geq 0} \in A^{\mathbb{N}}$ be a sequence of elements in A such that $b_{i+1} \equiv \varphi(b_i) \bmod \pi^{i+1}$ for all $i \geq 0$. We have to construct a sequence $a_0, a_1, \ldots$ of elements in A such that

$$W_{\pi,A}(a_0, a_1, \ldots) = (a_0, a_0^q + \pi a_1, \ldots) \overset{!}{=} (b_0, b_1, \ldots).$$

That is we have to solve inductively the equations

$$\pi^{i+1} a_{i+1} = b_{i+1} - \sum_{j=0}^{i} \pi^j a_j^{q^{i+1-j}}$$

for $i \geq 0$, i.e., we have to show that the RHS is 0 modulo π^{i+1}. For this we calculate, using the assumption on the b_i and induction,

$$b_{i+1} \equiv \varphi(b_i) = \varphi(\mathcal{W}_{i,\pi}(a_0, \ldots, a_i)) = \sum_{j=0}^{i} \pi^j \varphi(a_j)^{q^{i-j}}$$

modulo π^{i+1}. Thus it suffices to see (set $k = i - j$) that for each $a \in A$ and any $k \geq 0$

$$a^{q^{k+1}} \equiv \varphi(a)^{q^k}$$

modulo π^{k+1}. This follows from the following lemma Lemma 3.1 as

$$a^q \equiv \varphi(a) \bmod \pi.$$

As φ is a homomorphism we see that for a π-torsion free $\mathcal{O}_E$-algebra A with Frobenius lift φ the image of $W_{\pi,A}$ in $A^{\mathbb{N}}$ is stable under (coordinatewise) addition and multiplication. In particular, by transport of structure the lemma follows when $\mathcal{F}$ is restricted to the full subcategory of π-torsion free $\mathcal{O}_E$-algebras A which admit a lift of the q-Frobenius on A/π. The case of general A now follows by considering the universal cases which are polynomial rings (and these admit a lift of Frobenius). We leave the details as an exercise. □

Example 3.6. We spell out the formulas for addition and multiplication in low degrees (just to convince the reader that they are rather complicated).

$$(a_0, a_1, \ldots) + (b_0, b_1, \ldots) = (c_0, c_1, c_2, \ldots)$$
$$(a_0, a_1, \ldots) \cdot (b_0, b_1, \ldots) = (d_0, d_1, d_2, \ldots)$$

with

$$\begin{array}{rcl} c_0 &=& a_0 + b_0 \\ c_1 &=& a_1 + b_1 + \frac{a_0^q + b_0^q - (a_0+b_0)^q}{\pi} \\ c_2 &=& a_2 + b_2 + \frac{a_0^{q^2} + b_0^{q^2} - (a_0+b_0)^{q^2} + \pi(a_1^q + b_1^q - c_1^q)}{\pi^2} \\ c_3 &=& a_3 + b_3 + \ldots \\ d_0 &=& a_0 \cdot b_0 \\ d_1 &=& a_0^q b_1 + a_1 b_0^q + \pi a_1 b_1 \\ d_2 &=& \frac{a_0^{q^2} b_1^q + a_1^q b_0^{q^2} - d_1^q}{\pi} + a_0^{q^2} b_2 + a_1^q b_1^q + a_2 b_0^{q^2} + \pi(a_2 b_1^q + a_1^q b_2) + \pi^2 a_2 b_2 \\ d_3 &=& \ldots \end{array}$$

where the division by π is meant to be the one in the universal case. If $p = \pi = 2$ and $E = \mathbb{Q}_p$, then c_1 and c_2 are more explicitly

$$\begin{array}{rcl} c_1 &=& a_1 + b_1 - a_0 b_0 \\ c_2 &=& a_2 + b_2 - a_0^3 b_0 - 2a_0^2 b_0^2 - a_0 b_1^3 - a_1 b_1 + (a_1 + b_1) a_0 b_0. \end{array}$$

In the general case, the formulas for the $\mathcal{O}_E$-linear structure on $W_{\mathcal{O}_E}(A)$ are also computable. First of all, for $a \in A$ and $(a_0, a_1, a_2, \ldots)$

$$(a, 0, 0, \ldots) \cdot (a_0, a_1, a_2, \ldots) = (aa_0, a^q a_1, a^{q^2} a_2, \ldots)$$

(the element $[a] = (a, 0, 0, \ldots)$ is the Teichmüller lift of a from Proposition 3.7) and

$$\pi \cdot (a_0, a_1, a_2, \ldots) = (\pi a_0, a_0^q + a_1 - \pi^{q-1} b_0^q, \ldots)$$

This makes explicit the $\mathcal{O}_E = W_{\mathcal{O}_E}(\mathbb{F}_q)$-linear structure.

We list some properties of the functor of Witt vectors (all of these can be proven similarly as in Lemma 3.4), by reducing to the case of polynomial algebras over $\mathcal{O}_E$, cf. [26].

Proposition 3.7. (1) *There exists the natural multiplicative, non-additive Teichmüller lift*

$$[-]\colon A \to W_{\mathcal{O}_E,\pi}(A),\ a \mapsto [a] := (a, 0, \ldots).$$

(2) *There exists a natural ring homomorphism*

$$F\colon W_{\mathcal{O}_E,\pi}(A) \to W_{\mathcal{O}_E,\pi}(A)$$

lifting the q-Frobenius.[20]. *If $\pi A = 0$, then $F = W_{\mathcal{O}_E,\pi}(\varphi)$ is induced by the q-Frobenius on A/π by functoriality and we will write again φ for F.*

(3) *There exists the natural $\mathcal{O}_E$-linear morphism*

$$V_\pi\colon W_{\mathcal{O}_E,\pi}(A) \to W_{\mathcal{O}_E,\pi}(A),\ (a_0, a_1, \ldots) \mapsto (0, a_0, a_1, \ldots)$$

which furthermore satisfies

$$FV_\pi = \pi \text{ and } V_\pi(F(x).y) = x.V_\pi(y).$$

(4)

$$W_{\mathcal{O}_E,\pi}(A) \cong \varprojlim_n W_{\mathcal{O}_E,\pi}(A)/V_\pi^n W_{\mathcal{O}_E,\pi}(A)$$

and any element $a \in W_{\mathcal{O}_E,\pi}(A)$ can be written uniquely in the form

$$a = \sum_{n \geq 0} V_\pi^n [a_n]$$

for some $a_n \in A$, $n \geq 0$.

(5) *If $\pi A = 0$, then $V_\pi F = \pi$ and $F([a]) = [a^q]$.*

(6) *If $\pi A = 0$ and A is perfect (i.e., the Frobenius $A \to A$, $a \mapsto a^p$ on A is bijective), then $V_\pi^n W_{\mathcal{O}_E,\pi}(A) = \pi^n W_{\mathcal{O}_E,\pi}(A)$, $W_{\mathcal{O}_E,\pi}(A)$ is π-adically complete, π-torsion free with $W_{\mathcal{O}_E,\pi}(A)/\pi \cong A$ and every element $a \in W_{\mathcal{O}_E,\pi}(A)$ can uniquely written as*

$$a = \sum_{n \geq 0} \pi^n [a'_n]$$

with $a'_n \in A$.

[20]E.g. $F((a_0, a_1, \ldots)) = (a_0^q + \pi a_1, \ldots)$ and $F([a]) = [a^q]$.

The case (6) of a perfect $\mathbb{F}_q$-algebra will be the only one we need. We note that in Item 6

$$V_\pi^n[a_n] = \pi^n[F^{-n}(a_n)],$$

i.e., $a_n' = a_n^{q^{-n}}$. It can easily be seen that $W_{\mathcal{O}_E,\pi}(A)$, the Teichüller lift and the Frobenius do not depend (up a canonical isomorphism) on π (but clearly, V_π does as $FV_\pi = \pi$). For details see [10, Section 1.2.1.]. From now on we will therefore supress π and simply write $W_{\mathcal{O}_E}$ instead of $W_{\mathcal{O}_E,\pi}$.

In the perfect case it is possible to reduce the construction of the ramified Witt vectors to the classical one (where $E = \mathbb{Q}_p, \pi = p$) because of the following lemma.

Lemma 3.8. *Let A be a perfect $\mathbb{F}_q$-algebra and let $E_0 \subseteq E$ be the maximal unramified subextension of E. Then*

$$W(A) \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E \cong W_{\mathcal{O}_E,\pi}(A)$$

as $\mathcal{O}_E$-algebras.

Note that $\mathcal{O}_{E_0} \cong W(\mathbb{F}_q)$, thus $W(A)$ is naturally a $W(\mathbb{F}_q)$-algebra. We leave the construction of a concrete isomorphism as an exercise, cf. [10, Lemma 1.2.3].

Proof. Both rings are π-adically complete and π-torsion free with perfect quotient A. This implies that they must be isomorphic as can either be seen by the concrete arguments we presented before Lemma 3.4 or by vanishing of the cotangent complex $L_{A/\mathbb{F}_q}$ (which implies that A deforms uniquely along any nilpotent thickening, cf. [2, Example 3.1.7]). □

In the perfect case, tilting, cf. 3.2, and Witt vectors are related by an adjunction, cf. [10, Proposition 2.1.7.].

Proposition 3.9. *The functor*

$$(-)^\flat\colon \{\pi - \textit{complete } \mathcal{O}_E - \textit{algebras}\} \to \{\textit{perfect } \mathbb{F}_q - \textit{algebras}\}$$

is right adjoint with left adjoint given by the functor $W_{\mathcal{O}_E}(-)$.

Before proving Proposition 3.9 we make some remarks.

Remark 3.10.
- The unit

$$R \to W_{\mathcal{O}_E}(R)^\flat = \varprojlim_{x\mapsto x^q} (W_{\mathcal{O}_E}(R)/\pi) = \varprojlim_{x\mapsto x^q} R,$$

which sends r to $(r, r^{1/q}, r^{1/q^2}, \ldots)$ is an isomorphism. In particular, the functor $W_{\mathcal{O}_E}(-)$ is fully faithful. Its essential image is given by π-complete, π-torsion free $\mathcal{O}_E$-algebras A, s.t., A/π is perfect.
- The counit $\theta\colon W_{\mathcal{O}_E}(A^\flat) \to A$ is called Fontaine's map θ.

Proof. (of Proposition 3.9) We only give the construction of the counit θ and leave the necessary verifications as an exercise. Fix $n \geq 0$. By definition of the Witt vectors the morphism

$$\mathcal{W}_n\colon W_{\mathcal{O}_E}(A) \to A/\pi^{n+1},\ (a_0, a_1, \ldots) \mapsto \sum_{i=0}^n a_i^{q^{n-i}} \pi^i$$

is a morphism of rings. If all $a_i \equiv 0 \bmod \pi$, then by the "key lemma", cf. Lemma 3.1, $a_i^{q^{n-i}} \equiv 0 \bmod \pi^{n-i+1}$ for each $0 \leq i \leq n$. This implies

$$\sum_{i=0}^{n} a_i^{q^{n-i}} \pi^i \equiv 0 \bmod (\pi^{n+1}),$$

i.e., $\mathcal{W}_n$ factors over $W_{\mathcal{O}_E}(A/\pi)$. Call the resulting map

$$\theta_n \colon W_{\mathcal{O}_E,n}(A/\pi) \to A/\pi^{n+1}.$$

One checks that the diagram

$$\begin{array}{ccc} W_{\mathcal{O}_E,n+1}(A/\pi) & \xrightarrow{\theta_{n+1}} & A/\pi^{n+2} \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \mathrm{can}} \\ W_{\mathcal{O}_E,n}(A/\pi) & \xrightarrow{\theta_n} & A/\pi^{n+1} \end{array}$$

where F denotes the Witt vector Frobenius (which is induced by the q-power Frobenius of A/π), and "can" the canonical projection. Passing to the limit yields therefore the map

$$\theta \colon W_{\mathcal{O}_E}(A^\flat) \cong \varprojlim_{n,F} W_{\mathcal{O}_E,n}(A/\pi) \to A \cong \varprojlim_{n} A/\pi^{n+1}$$

which serves as the counit. □

Using the $(-)^\sharp$-map from 3.3 we can give a more concrete description of θ-map.

Lemma 3.11. *For a π-complete $\mathcal{O}_E$-algebra the map*

$$\theta \colon W_{\mathcal{O}_E}(A^\flat) \to A$$

is given by $\sum\limits_{i=0}^{\infty} [a_i]\pi^i \mapsto \sum\limits_{i=0}^{\infty} a_i^\sharp \pi^i$.

Proof. This is a good exercise in unravelling the constructions. □

We now will introduce (perfect) prisms and perfectoid rings. For this lecture the following definition is convenient. See [3, Theorem 3.9] for its relation to former definitions of perfectoid rings, e.g., in [1].

Definition 3.12.

(1) A perfect prism over $\mathcal{O}_E$ is a pair $(W_{\mathcal{O}_E}(R), I)$ with R a perfect $\mathbb{F}_q$-algebra, $I \subseteq W_{\mathcal{O}_E}(R)$ an ideal generated by some $d \in W_{\mathcal{O}_E}(R)$, s.t., $\frac{F(d)-d^q}{\pi} \in W_{\mathcal{O}_E}(R)^\times$ (i.e., d is "distinguished") and $W_{\mathcal{O}_E}(R)$ is I-adically complete.
(2) An $\mathcal{O}_E$-algebra A is perfectoid if $A \cong W_{\mathcal{O}_E}(R)/I$ for some perfect prism $(W_{\mathcal{O}_E}(R), I)$ over $\mathcal{O}_E$.

Remark 3.13.

- An element
$$d = \sum_{i=0}^{\infty} [r_i]\pi^i \in W_{\mathcal{O}_E}(R)$$
is distinguished if and only if $r_1 \in R^\times$ as
$$\frac{F(d) - d^q}{\pi} \equiv r_1 \bmod \pi.$$

If d is distinguished, then $W_{\mathcal{O}_E}(R)$ is (d)-adically complete if and only if R is r_0-adically complete.

- Perfect $\mathbb{F}_q$-algebras are perfectoid by taking $d = \pi$.
- If $A \cong W_{\mathcal{O}_E}(R)/I$ is perfectoid, then

$$A^\flat \cong (W_{\mathcal{O}_E}(R)/I)^\flat \cong (W_{\mathcal{O}_E}(R)/(\pi, I))^\flat \cong R$$

by Proposition 3.3.

The last remark motivates the following definition of an "untilt".

Definition 3.14. Let R be a perfect $\mathbb{F}_q$-algebra. An untilt of R over $\mathcal{O}_E$ is a pair (A, ι) of a perfectoid $\mathcal{O}_E$-algebra A and an isomorphism $\iota\colon A^\flat \cong R$.

With this definition one checks that for any perfect $\mathbb{F}_q$-algebra R one obtains an equivalence[21]

$$\{\text{ untilts } (A,\iota) \text{ of } R \text{ over } \mathcal{O}_E\} \cong \{I \subseteq W_{\mathcal{O}_E}(R), \text{ s.t. } (W_{\mathcal{O}_E}(R), I) \text{ is a prism}\}.$$

With this terminology the tilting equivalence from [20] becomes an easy exercise.

Exercise 3.15. Let A be a perfectoid $\mathcal{O}_E$-algebra. Then the functors

$$\begin{array}{ccc} \{\text{ perfectoid } A-\text{algebras }\} & \cong & \{\text{ perfectoid } A^\flat-\text{algebras }\} \\ B & \mapsto & B^\flat \\ W_{\mathcal{O}_E}(S) \otimes_{W_{\mathcal{O}_E}(A^\flat)} A & \leftarrow\!\shortmid & S \end{array}$$

are mutually inverse equivalences.[22]

We now give the most important example of a perfectoid ring for this course.

Proposition 3.16. *Let C be an algebraically closed, non-archimedean extension of E with valuation $\nu\colon C \to \mathbb{R} \cup \{\infty\}$. Then the ring of integers*

$$\mathcal{O}_C := \{x \in C \mid \nu(x) \geq 0\}$$

is a perfectoid $\mathcal{O}_E$-algebra.

Proof. Let $\pi^{1/q^n} \in \mathcal{O}_C$, $n \geq 1$, be a compatible system of q^n-th roots of π. This yields the element

$$\pi^\flat := (\pi, \pi^{1/q}, \pi^{1/q^2}, \ldots) \in \varprojlim_{x \mapsto x^q} \mathcal{O}_C \cong \mathcal{O}_C^\flat$$

in the tilt of $\mathcal{O}_C$. We claim that

$$\xi := \pi - [\pi^\flat]$$

generates $\ker(\theta\colon W_{\mathcal{O}_E}(\mathcal{O}_C^\flat) \to \mathcal{O}_C)$. As θ is also surjective (because C is algebraically closed), this implies that $\mathcal{O}_C$ is perfectoid. We first show that

$$\mathcal{O}_C/\pi \cong \mathcal{O}_C^\flat/\pi^\flat.$$

Namely, let

$$y = (y_0, y_1, \ldots) \in \varprojlim_{x \mapsto x^q} \mathcal{O}_C \cong \mathcal{O}_C^\flat.$$

Then $\pi^\flat$ divides y if and only if for all $n \geq 0$ the element π^{1/q^n} divides y_n. Because $\mathcal{O}_C$ is a valuation ring this happens if and only if $\nu(y_n) \geq q^{-n}\nu(\pi)$ for all $n \geq 0$.

[21] for the obvious notions of morphisms

[22] As a hint, prove that if $W_{\mathcal{O}_E}(R, I) \to (W_{\mathcal{O}_E}(S), J)$ is a morphism of prisms, i.e., I is send to J, then necessary $J = IW_{\mathcal{O}_E}(S)$.

This occurs if and only if $\nu(y_0) \geq \nu(\pi)$ as $\nu(y_n) = q^{-1}\nu(y_0)$, i.e., if and only if $y_0 \equiv 0 \bmod \pi$. We have therefore proven that

$$\ker(\mathcal{O}_C^\flat \xrightarrow{(-)^\sharp} \mathcal{O}_C \to \mathcal{O}_C/(\pi)) = (\pi^\flat).$$

Because the $\sharp$-map is surjective, we can conclude that

$$\mathcal{O}_C^\flat/(\pi^\flat) \cong \mathcal{O}_C/(\pi).$$

Now we can prove that $\ker(\theta\colon \mathbb{A}_{\mathrm{inf}} \to \mathcal{O}_C)$ is generated by $\xi = \pi - [\pi^\flat]$. First note that

$$\theta(\pi - [\pi^\flat]) = \pi - (\pi^\flat)^\sharp = \pi - \pi = 0,$$

that is $\xi \in \ker(\theta)$. Let $x = \sum\limits_{i=0}^{\infty} [x_i]\pi^i \in \ker(\theta)$. Then

$$0 = \theta(x) = \sum_{i=0}^{\infty} x_i^\sharp \pi^i$$

which implies

$$x_0^\sharp \equiv 0 \bmod \pi.$$

As $\mathcal{O}_C^\flat/(\pi^\flat) \cong \mathcal{O}_C/(\pi)$ this implies

$$\pi^\flat | x_0$$

and thus that we can write

$$x = [\pi^\flat]x_1 + (\pi - [\pi^\flat])z_0$$

for some $x_1, z_0 \in \mathbb{A}_{\mathrm{inf}}$. Now,

$$0 = \theta(x) = \theta([\pi^\flat]x_1) = \pi\theta(x_1)$$

which implies that $x_1 \in \ker(\theta)$ as π is a non-zero divisor in $\mathcal{O}_C$. Continuing the argument with x_1 instead of x, we see that we can write

$$x = \xi(z_0 + [\pi^\flat](z_1 + \ldots)) \in (\xi)$$

where the infinite sum converges as $\nu([\pi^\flat]) > 0$. This finishes the proof. □

We leave the following proposition as an exercise.

Proposition 3.17. *Let S be a ring and let $\varpi \in S$ be a non-zero divisor such that $\varpi^p | p$, S is ϖ-adically complete and the Frobenius $\varphi\colon S/\varpi S \to S/\varpi^p S$ is an isomorphism. Then S is perfectoid.*

Proof. Exercise. □

Let C/E be a non-archimedean algebraically closed field with valuation

$$\nu\colon C \to \mathbb{R} \cup \{\infty\}.$$

Recall that by 3.3

$$\mathcal{O}_C^\flat = \varprojlim_{\mathrm{Frob_q}} \mathcal{O}_C/\pi \cong \varprojlim_{x \mapsto x^q} \mathcal{O}_C$$

via the map

$$x \mapsto (x^\sharp, (x^{1/q})^\sharp, \ldots).$$

We want to analyze $\mathcal{O}_C^\flat$. The following lemma is [10, Section 2.1.3.].

Lemma 3.18. *The ring $\mathcal{O}_C^\flat$ is a valuation ring with associated valuation given by*

$$\nu^\flat\colon \mathcal{O}_C^\flat \to \mathbb{R}\cup\{\infty\},\ x\mapsto \nu(x^\sharp).$$

Moreover, $\mathcal{O}_C^\flat$ is complete for its valuation topology and its fraction field $C^\flat := \mathrm{Frac}(\mathcal{O}_C^\flat)$ is algebraically closed.

In particular, $C^\flat$ is a non-archimedean, algebraically closed field. One can check that (as multiplicative monoids)

$$C^\flat \cong \varprojlim_{x\mapsto x^q} C.$$

Proof. It is clear that $\nu^\flat(xy) = \nu^\flat(x) + \nu^\flat(y)$ for $x, y \in \mathcal{O}_C^\flat$ because the $(-)^\sharp$-map is multiplicative. Moreover, $\nu^\flat(x) = \infty$ if and only if $x = 0$. We have to show that $\nu^\flat(-)$ satisfies the non-archimedean triangle inequality. Let $x, y \in \mathcal{O}_C^\flat$. Then

$$\begin{aligned}
\nu^\flat(x+y) &= \nu((x+y)^\sharp)\\
&= \nu(\lim_{n\to\infty}((x^{1/q^n})^\sharp + (y^{1/q^n})^\sharp)^{q^n})\\
&= \lim_{n\to\infty} q^n\nu((x^{1/q^n})^\sharp + (y^{1/q^n})^\sharp)\\
&\geq \lim_{n\to\infty} q^n\inf(\nu((x^{1/q^n})^\sharp), \nu((y^{1/q^n})^\sharp))\\
&= \inf(\nu^\flat(x), \nu^\flat(y)).
\end{aligned}$$

Next we will prove completeness of $\mathcal{O}_C^\flat$. For this it suffices to show that the valuation topology induced by $\nu^\flat(-)$ agrees with the inverse limit topology on

$$\mathcal{O}_C^\flat \cong \lim_{x\mapsto x^q} \mathcal{O}_C$$

as $\mathcal{O}_C$ is complete for its valuation topology. But a basis of neighborhoods of 0 for the valuation topology for $\nu^\flat(-)$ is given by the subsets

$$\{x\in \mathcal{O}_C^\flat \mid \nu^\flat(x) \geq m\}$$

for $m \geq 0$, while the system of subsets, $n, m \geq 0$,

$$\{x\in\mathcal{O}_C^\flat \mid \nu((x^{1/q^n})^\sharp) \geq m\}$$

is a basis of neighborhoods of 0 for the inverse limit topology. As

$$\nu((x^{1/q^n})^\sharp) = 1/q^n\nu^\flat(x)$$

these two systems of basis agree, which implies that the two topologies are the same. This finishes the proof of completeness. Let us now show that $C^\flat = \mathrm{Frac}(\mathcal{O}_C^\flat)$ is algebraically closed. Let

$$f(T)\in\mathcal{O}_C^\flat[T],\ f(T) = T^d + a_{d-1}T^{d-1} + \ldots + a_0,$$

be a monic polynomial.[23] It suffices to show that $f(T)$ has a zero in $\mathcal{O}_C^\flat$. Set

$$f_n(T) := T^d + (a_{d-1}^{1/q^n})^\sharp T^{d-1} + \ldots + (a_0^{1/q^n})^\sharp \in \mathcal{O}_C[T].$$

Then

$$f_{n+1}(T)^q \equiv f_n(T^q) \bmod \pi.$$

Fix some $n \geq 0$ and let $x\in\mathcal{O}_C$ be a zero of f_n. Choose moreover some $y\in\mathcal{O}_C$ such that $y^q = x$. Then

$$\nu(f_{n+1}(y)) \geq \frac{1}{q}\nu(\pi)$$

[23] As $\mathcal{O}_C^\flat$ is complete it suffices to consider monic polynomials with coefficients in $\mathcal{O}_C^\flat$.

by the above congruence. Let $z_1, \ldots, z_d \in \mathcal{O}_C$ be the zeros of f_{n+1}. Then

$$\nu(f_{n+1}(y)) = \sum_{i=1}^{d} \nu(y - z_i) \geq \frac{1}{q}\nu(\pi),$$

which implies that there exists some i such that

$$\nu(y - z_i) \geq \frac{1}{dq}\nu(\pi).$$

Then

$$\nu(x - z_i^q) \geq \frac{1}{d}\nu(\pi).$$

By induction we therefore obtain a sequence $(x_n)_{n\geq 0}$ such that $x_n \in \mathcal{O}_C$, $f_n(x_n) = 0$ and

$$\nu(x_{n+1}^q - x_n) \geq \frac{1}{d}\nu(\pi).$$

Set

$$\mathfrak{a} := \{y \in \mathcal{O}_C \mid \nu(y) \geq \frac{1}{d}\nu(\pi)\}.$$

Then

$$x := (x_n)_{n\geq 0} \in \varprojlim_{\mathrm{Frob}} \mathcal{O}_C/\mathfrak{a} \cong \varprojlim_{\mathrm{Frob}} \mathcal{O}_C/\pi = \mathcal{O}_C^\flat,$$

where we used Proposition 3.3 to identify the two limits. Clearly, $f(x) = 0$ as desired. □

4. Lecture of 30.10.2019: The ring $\mathbb{A}_{\mathrm{inf}}$

According to Colmez, cf. [6], the ring $\mathbb{A}_{\mathrm{inf}}$ is the "one ring to rule them all", namely all other rings like B_{dR}, B_{cris},... are derived from $\mathbb{A}_{\mathrm{inf}}$.

We need the notation introduced in Section 1, i.e., p is a prime, $E/\mathbb{Q}_p$ a finite extension, $\mathcal{O}_E$ its ring of integers, $\pi \in \mathcal{O}_E$ a uniformizer, $\mathbb{F}_q = \mathcal{O}_E/(\pi)$, $F/\mathbb{F}_q$ a non-archimedean algebraically closed field with valuation $\nu\colon F \to \mathbb{R}\cup\{\infty\}$ and ring of integers $\mathcal{O}_F := \{x \in F \mid \nu(x) \geq 0\}$.

In this setup we can define the ring $\mathbb{A}_{\mathrm{inf}}$, Fontaine's first period ring.

Definition 4.1. We define

$$\mathbb{A}_{\mathrm{inf}} := \mathbb{A}_{\mathrm{inf}\,E,F} := W_{\mathcal{O}_E}(\mathcal{O}_F)$$

as the ring of ramified Witt vectors of the perfect $\mathbb{F}_q$-algebra $\mathcal{O}_F$, cf. Lemma 3.4

As $\mathcal{O}_F$ is a perfect $\mathbb{F}_q$-algebra we know by Proposition 3.7 that

$$\mathbb{A}_{\mathrm{inf}} = \{\sum_{n=0}^{\infty} [x_n]\pi^n \mid x_n \in \mathcal{O}_F\},$$

that is, $\mathbb{A}_{\mathrm{inf}}$ is a ring of "power series in π with coefficients in $\mathcal{O}_F$". However, the addition and multiplication are much more complicated than their counterparts for the ring

$$\mathcal{O}_F[[u]]$$

of usual power series with coefficients in $\mathcal{O}_F$ (as can be seen from 3.6). We let

$$\varphi := F\colon \mathbb{A}_{\mathrm{inf}} \to \mathbb{A}_{\mathrm{inf}}$$

be the Witt vector Frobenius, or equivalently the morphism induced by the Frobenius on $\mathcal{O}_F$. Thus,

$$\varphi(\sum_{n=0}^{\infty}[x_n]\pi^n) = \sum_{n=0}^{\infty}[x_n^q]\pi^n.$$

On $\mathcal{O}_F[[u]]$ the analogous morphism would be the arithmetic Frobenius

$$\varphi(\sum_{n=0}^{\infty}x_n u^n) = \sum_{n=0}^{\infty}x_n^q u^n$$

which leaves u fixed. The ring $\mathcal{O}_F[[z]]$ is a non-noetherian local integral domain which looks as it could be of Krull dimension two. But it isn't, the Krull dimension of $\mathcal{O}_F[[z]]$ is infinite. The intuition that $\mathcal{O}_F[[u]]$ "is" 2-dimensional is supported by the fact that $\mathcal{O}_F[[u]]$ can naturally be interpreted as the ring of bounded functions on 1-dimensional the rigid-analytic open unit disc

$$\mathbb{D}_F := \{x \mid \nu(x) > 0\}$$

over F, which is one-dimensional. We will not introduce rigid-analytic or adic spaces and contend ourselves with the statement that for each $a \in \mathfrak{m}_F$, there is the natural evaluation morphism

$$\mathrm{ev}_a \colon \mathcal{O}_F[[u]] \to F,\ f(u) \mapsto f(a)$$

with kernel $(u-a)$. The exotic prime ideals on $\mathcal{O}_F[[u]]$, which imply that $\mathcal{O}_F[[u]]$ is of infinite Krull dimension, are all contained in the prime ideal $\mathfrak{m}_F[[u]]$. Apart from these $\mathrm{Spec}(\mathcal{O}_F[[u]])$ can be described completely.

Theorem 4.2. *The spectrum of $\mathcal{O}_F[[u]]$ is given by*

$$\mathrm{Spec}(\mathcal{O}_F[[u]]) = \{(0), (\mathfrak{m}_F, u)\} \cup \{(u-a) \mid a \in \mathfrak{m}_F\} \cup \mathrm{Spec}(\mathcal{O}_F[[u]]_{\mathfrak{m}_F[[u]]})$$

where $\mathcal{O}_F[[u]]_{\mathfrak{m}_F[[u]]}$ denotes the localization of $\mathcal{O}_F[[u]]$ at the prime ideal $\mathfrak{m}_F[[u]]$.

Note that for $a \neq b \in \mathfrak{m}_F$ the prime ideals $(u-a)$ and $(u-b)$ are distinct.

Proof. Clearly, the mentioned ideals are prime. Assume that $\mathfrak{q} \subset \mathcal{O}_F[[u]]$ is any prime ideal, which is not contained in $\mathfrak{m}_F[[u]]$. Then there exists an element

$$f(u) = \sum_{i=0}^{\infty} x_i u^i \in \mathfrak{q}$$

for which some $x_i \in \mathcal{O}_F^{\times}$ is a unit. Set

$$d := \min\{i \mid x_i \in \mathcal{O}_F^{\times}\}$$

According to Weierstraß factorization, cf. [4] or [17], the element f can be written as a monic polynomial $g(u) \in \mathcal{O}_F[u]$ of degree d times some unit in $\mathcal{O}_F[[u]]$. In particular, $g(u) \in \mathfrak{q}$ as $\mathfrak{q}$ is prime. But then by our assumption that F is algebraically closed

$$g(u) = \prod_{i=1}^{d}(u-a_i)$$

for some $a_i \in \mathcal{O}_F$ and some $(u-a_i)$ must lie in $\mathfrak{q}$, which finishes the proof. ∎

The first part of the course will be devoted to prove analogous statements, in particular the factorisation occuring in the proof of Theorem 4.2, for $\mathbb{A}_{\mathrm{inf}}$. As a start, it is clear that the ring $\mathbb{A}_{\mathrm{inf}}$ is a non-noetherian, local integral domain, which is $(\pi, [\varpi])$-adically complete for all $\varpi \in \mathfrak{m}_F \setminus \{0\}$, where $\mathfrak{m}_F := \{x \in F \mid \nu(x) > 0\}$.

The chain of prime ideals

$$0 \subsetneq \bigcup_{\varpi \in \mathfrak{m}_F} [\varpi]\mathbb{A}_{\mathrm{inf}} \subsetneq W_{\mathcal{O}_E}(\mathfrak{m}_F) \subsetneq (\pi, W_{\mathcal{O}_E}(\mathcal{O}_E))$$

shows that $\mathbb{A}_{\mathrm{inf}}$ is at least of Krull dimension ≥ 3. However, similarly to the case of $\mathcal{O}_F[[u]]$ the Krull dimension of $\mathbb{A}_{\mathrm{inf}}$ is in fact infinite.

Theorem 4.3 (Lang–Ludwig [16])**.** $\mathrm{Spec}(\mathbb{A}_{\mathrm{inf}})$ *has infinite Krull dimension.*

Again, all the "exotic" prime ideals predicted by Theorem 4.3 are contained in $W_{\mathcal{O}_E}(\mathfrak{m}_F)$.

Contrary to the case of $\mathcal{O}_F[[u]]$ it is much less clear how to interpret $\mathbb{A}_{\mathrm{inf}}$ as some ring of functions on a geometric object. In fact, for the sake of simplicity we will only introduce a weak substitute for $\mathbb{D}_F$.[24]

Definition 4.4. We define

$$|Y|_{[0,\infty)} := \{I \subseteq \mathbb{A}_{\mathrm{inf}} \mid I \text{ generated by a distinguished element}\},$$

i.e.,

$$|Y|_{[0,\infty)} = \{(u\pi - [a]) \subseteq \mathbb{A}_{\mathrm{inf}} \mid u \in \mathbb{A}_{\mathrm{inf}}^{\times}, a \in \mathfrak{m}_F\}.$$

Moreover, we set

$$|Y| := |Y|_{[0,\infty)} \setminus \{(\pi)\}.$$

In the case of $\mathcal{O}_F[[u]]$ the analogous definitions would exactly recover the sets $\mathfrak{m}_F$ and $\mathfrak{m}_F \setminus \{0\}$. Note that by Exercise 3.15 the set $|Y|_{[0,\infty)}$ is in bijection with the set of isomorphism classes of untilts of $\mathcal{O}_F$ and $|Y|$ with among these with the π-torsion free untilts of $\mathcal{O}_F$. Proposition 3.16 supplies us with a lot of these untilts.[25] Moreover, note that in the notation of Proposition 3.16

$$\mathbb{A}_{\mathrm{inf}}/(\pi - [\pi^\flat]) \cong \mathcal{O}_C$$

for *every* choice of $\pi^\flat = (\pi, \pi^{1/q}, \ldots)$. In particular, the map

$$\mathfrak{m}_F \to |Y|_{[0,\infty)},\ a \mapsto (\pi - [a])$$

is *not* bijective (but we will show that it is surjective, cf. Theorem 5.4).

The picture 1 of $\mathbb{A}_{\mathrm{inf}}$ is helpful (see also [23, Page 83, Figure 5]).

We will analyze the ring $\mathbb{A}_{\mathrm{inf}}$ further. Recall that for C/E non-archimedean and algebraically closed, there is an isomorphism

$$\mathcal{O}_C \cong \mathbb{A}_{\mathrm{inf}}/(\xi)$$

with $\xi := \pi - [\pi^\flat]$, cf. Proposition 3.16.

Definition 4.5. We define

$$B_{\mathrm{dR}}^+ := B_{\mathrm{dR},C}^+ := \mathbb{A}_{\mathrm{inf}}[1/\pi]_\xi^\wedge$$

as the ξ-adic completion of $\mathbb{A}_{\mathrm{inf}}[1/\pi]$.

---

[24]Although there exists a reasonable geometric replacement, cf. [22, Section 11.2].

[25]We will see in 5.4 that this example covers all untilts of $\mathcal{O}_F$.

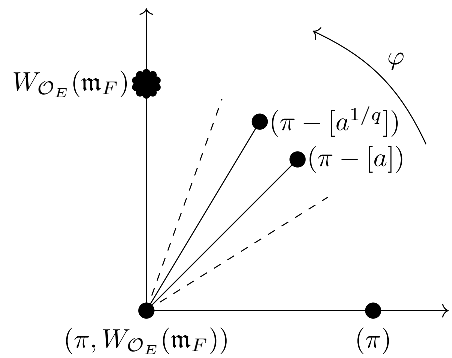

FIGURE 1. A picture of $\mathrm{Spec}(\mathbb{A}_{\mathrm{inf}})$. The arrow indicates the action of φ. All mysterious prime ideals are "close" to $W_{\mathcal{O}_E}(\mathfrak{m}_F)$.

By [27, Tag 05GG]

$$B_{\mathrm{dR}}^+ \cong \varprojlim_n \mathbb{A}_{\mathrm{inf}}[1/\pi]/\xi^n$$

and

$$B_{\mathrm{dR}}^+/\xi \cong C.$$

Lemma 4.6. *The morphism $\mathbb{A}_{\mathrm{inf}} \to B_{\mathrm{dR}}^+$ is injective and the two local rings B_{dR}^+, $\mathbb{A}_{\mathrm{inf},(\xi)}$ are discrete valuation rings.*

We remark that this statement can be interpreted as giving $|Y|$ at the point $y := (\pi - [\pi^\flat])$ a bit more geometric structure, namely the complete DVR[26]

$$B_{\mathrm{dR},y}^+ = \text{"}\widehat{\mathcal{O}}_{|Y|,y}\text{"}$$

and, in particular, the residue field $C_y := \mathbb{A}_{\mathrm{inf}}/y[1/\pi]$ of $|Y|$ at y.

The field of fractions B_{dR} of B_{dR}^+ is called Fontaine's field of p-adic periods (for C).

Proof. As $\xi \in \mathbb{A}_{\mathrm{inf}}$ is a non-zero divisor and $\mathcal{O}_C$ is π-torsion free, the ring

$$\mathbb{A}_{\mathrm{inf}}/\xi^n$$

is π-torsion free for each $n \geq 0$, i.e., $\mathbb{A}_{\mathrm{inf}}/\xi^n \hookrightarrow \mathbb{A}_{\mathrm{inf}}/\xi^n[1/\pi]$ for each $n \geq 0$. By left exactness of $\varprojlim$ one can conclude[27] that

$$\mathbb{A}_{\mathrm{inf}} \cong \varprojlim_n \mathbb{A}_{\mathrm{inf}}/\xi^n \hookrightarrow B_{\mathrm{dR}}^+.$$

To see that B_{dR}^+ is a discrete valuation ring we use [27, Tag 05GH], which implies that by completeness B_{dR}^+ is noetherian. Moreover, B_{dR}^+ is local, with non-zero maximal ideal generated by one element, of Krull dimension at least 1[28] and thus a DVR. We can conclude that the localization $\mathbb{A}_{\mathrm{inf},(\xi)}$ is a DVR, too. Namely, pick

[26] The plain localization $\mathbb{A}_{\mathrm{inf},(\xi)}$ is not so useful and only mentioned for completeness.
[27] We used that $\mathbb{A}_{\mathrm{inf}}$ is $(\pi, [\pi^\flat])$-adically complete and [27, Tag 090T] to conclude that $\mathbb{A}_{\mathrm{inf}}$ is also ξ-adically complete.
[28] as $\mathbb{A}_{\mathrm{inf}} \hookrightarrow B_{\mathrm{dR}}^+$.

$\mathfrak{p} \subseteq \mathbb{A}_{\mathrm{inf},(\xi)}$ a prime ideal such that $\xi \notin \mathfrak{p}$. Then $\mathfrak{p} \subseteq (\xi)$, which implies that for $a \in \mathfrak{p}$ also $a/\xi \in \mathfrak{p}$. In other words,

$$\xi\mathfrak{p} = \mathfrak{p}.$$

But, using that $\mathbb{A}_{\mathrm{inf},(\xi)} B_{\mathrm{dR}}^+$ and that B_{dR}^+ is a DVR, this implies that $\mathfrak{p} = 0$. Thus,

$$\mathrm{Spec}(\mathbb{A}_{\mathrm{inf},(\xi)}) = \{(0), (\xi)\}$$

which implies that $\mathbb{A}_{\mathrm{inf},(\xi)}$ is noetherian by [12, Chapitre 0, Proposition (6.4.7.)] and thus a DVR. □

The ring

$$B_{\mathrm{dR}}^+ = \varprojlim_n \mathbb{A}_{\mathrm{inf}}[1/p]/(\xi)^n$$

has two topologies. On the one hand, its topology as a valuation ring, i.e., the inverse limit topology with each $\mathbb{A}_{\mathrm{inf}}[1/p]/(\xi)^n$ given the discrete topology. On the other hand the inverse limit topology for the topology on $\mathbb{A}_{\mathrm{inf}}[1/p]/(\xi^n)$ such that $\mathbb{A}_{\mathrm{inf}}/(\xi)^n$ is open with the p-adic topology.[29] The second topology is called the canonical topology on B_{dR}^+. For both topologies the ring B_{dR}^+ is complete.

We make a short digression to explain the name "$\mathbb{A}_{\mathrm{inf}}$",cf. [11].

Definition 4.7. Let R be a π-complete $\mathcal{O}_E$-algebra. A surjection $D \to R$ of $\mathcal{O}_E$-algebras with kernel I, such that D is $I + (\pi)$-adically complete is called a π-adic pro-infinitesimal thickening of R.

For example, if $R = \mathcal{O}_C$ or $R = \mathcal{O}_C/\pi$, then Fontaine's map

$$\mathbb{A}_{\mathrm{inf}} \to R$$

is a pro-infinitesimal thickening.

The following lemma explains the terminology "$\mathbb{A}_{\mathrm{inf}}$".

Lemma 4.8. *Let $R \in \{\mathcal{O}_C, \mathcal{O}_C/p\}$. Then $\mathbb{A}_{\mathrm{inf}}$ is the universal π-adic pro-infinitesimal thickening of R, i.e., for each π-adic pro-infinitesimal thickening $D \to R$ exists a unique morphism $\mathbb{A}_{\mathrm{inf}} \to D$ over R.*

Proof. Proposition 3.3 implies that $D^\flat \cong R^\flat$. By Proposition 3.9 there exists therefore a morphism $\mathbb{A}_{\mathrm{inf}} \to D$ reducing to the canonical isomorphism $D^\flat \cong R^\flat$ on tilts. One checks that this morphism is unique. □

Now, assume that $E = \mathbb{Q}_p$. In characteristic p (or mixed characteristics) infinitesimal thickenings with a PD-structure are usually more interesting.

Definition 4.9. Let R be p-adically complete. A p-adic PD-thickening of R is a triple $(D, D \twoheadrightarrow R, (\gamma_n)_{n\geq 0})$ where D is p-adically complete, $D \twoheadrightarrow R$ is a surjection and $(\gamma_n)_{n\geq 0}$ is a PD-structure on $J := \ker(D \twoheadrightarrow R)$ compatible with the canonical PD-structure on (p).

For all facts related to PD-structures or crystalline cohomology we refer to [27, Tag 07GI].

Remark 4.10. (1) If D is p-torsion free, then necessarily $\gamma_n(x) = \frac{x^n}{n!}$ for $x \in J$.

[29] If R is any ring and $f \in R$ a non-zero divisor, then there exists a unique topology on $R[1/f]$ making $R[1/f]$ a topological ring such that R is open and the subspace topology on R is the f-adic one.

(2) For $x \in \mathcal{O}_C$

$$\nu(x^n/n!) \geq 0 \text{ for all } n \geq 0 \Leftrightarrow \nu(x) \geq \frac{1}{p-1}\nu(p).$$

This implies that $\{x \in \mathcal{O}_C \mid \nu(x) \geq \frac{1}{p-1}\nu(p)\} \subseteq \mathcal{O}_C$ is the largest ideal admitting divided powers.

Looking at the universal divided power envelope of $\mathcal{O}_C$ (or equivalently $\mathcal{O}_C/p$) yields the important crystalline period ring $\mathbb{A}_{\mathrm{crys}}$ of Fontaine.

Definition 4.11. We define

$$\mathbb{A}_{\mathrm{crys}}$$

as the universal divided power envelope of $\ker(\theta\colon \mathbb{A}_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_C) = (\xi)$ and

$$B^+_{\mathrm{crys}} := \mathbb{A}_{\mathrm{crys}}[1/p].$$

By the definition of the crystalline site

$$\mathbb{A}_{\mathrm{crys}} = H^0_{\mathrm{crys}}(\mathcal{O}_C/\mathbb{Z}_p) \cong H^0_{\mathrm{crys}}((\mathcal{O}_C/p)/\mathbb{Z}_p).$$

This explains the name of $\mathbb{A}_{\mathrm{crys}}$. More concretely,

$$\mathbb{A}_{\mathrm{crys}} = \mathbb{A}_{\mathrm{inf}}[\frac{\xi^n}{n!} \mid n \geq 0]^\wedge_p \cong \mathbb{A}_{\mathrm{inf}} \hat{\otimes}_{\mathbb{Z}_p[x]} D_{\mathbb{Z}_p[x]}((x))^\wedge_p$$

where $\mathbb{Z}_p[x] \to \mathbb{A}_{\mathrm{inf}},\ x \mapsto \xi$ and

$$D_{\mathbb{Z}_p[x]}((x))^\wedge_p = \hat{\bigoplus_{n\geq 0}} \mathbb{Z}_p \cdot \frac{x^n}{n!} \cong (\mathbb{Z}_p[y_0, y_1, y_2, \ldots]/(y_0 - x, y_1^p - py_0, y_2^p - py_1, \ldots))^\wedge_p$$

is the free p-complete PD-algebra on one generator. In particular,

$$\mathbb{A}_{\mathrm{crys}}/p \cong \mathcal{O}_C/p \otimes_{\mathbb{F}_p} \mathbb{F}_p[y_1, y_2, \ldots]/(y_1^p, y_2^p, \ldots).$$

is a rather horrible non-noetherian, non-perfect ring. Every element in $\mathbb{A}_{\mathrm{crys}}$ can (non-uniquely) be written as

$$\sum_{n\geq 0} a_n \frac{\xi^n}{n!}$$

with $a_n \in \mathbb{A}_{\mathrm{inf}}$ converging to 0 for the p-adic topology.

Lemma 4.12. *The natural morphism $\mathbb{A}_{\mathrm{inf}} \to B^+_{\mathrm{dR}}$ extends to an injection*[30]

$$B^+_{\mathrm{crys}} \to B^+_{\mathrm{dR}}.$$

Proof. We claim that the natural inclusion

$$\mathbb{A}_{\mathrm{inf}}[\frac{\xi^n}{n!} \mid n \geq 0] \to B^+_{\mathrm{dR}}$$

is continuous for the p-adic topology on the left and the canonical topology on the right. But for each $m \geq 0$ the image of

$$\mathbb{A}_{\mathrm{inf}}[\frac{\xi^n}{n!} | n \geq 0] \to \mathbb{A}_{\mathrm{inf}}[1/p]/(\xi^m)$$

is contained in $1/(m-1)!\mathbb{A}_{\mathrm{inf}}/(\xi)^m$ because each $\frac{\xi^n}{n!}$ with $n \geq m$ maps to 0. This implies continuity. By completeness of B^+_{dR} for its canonical topology we obtain the extension

$$\mathbb{A}_{\mathrm{crys}} \to B^+_{\mathrm{dR}}.$$

[30]Even $\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p^{\mathrm{un}}} B^+_{\mathrm{crys}} \to B^+_{\mathrm{dR}}$ is injective, cf. [5].

Each element $x \in \mathbb{A}_{\mathrm{crys}}$ can be written in the form

$$x = \sum_{n\geq 0} a_n \frac{\xi^n}{n!}$$

with $a_n \in \mathbb{A}_{\mathrm{inf}}$ converging p-adically to 0. Assume that $x \neq 0$. As

$$\ker(\theta\colon \mathbb{A}_{\mathrm{inf}} \to \mathcal{O}_C) = (\xi)$$

we can assume that $\theta(a_n) \neq 0$ for some $n \geq 0$. But then x cannot map to 0 in B_{dR}^+ as its (ξ)-adic valuation is

$$\inf_n\{a_n \neq 0\} < \infty.$$

This finishes the proof. □

We note that $\mathbb{A}_{\mathrm{crys}}$ depends on C, but $\mathbb{A}_{\mathrm{inf}}$ only on the tilt $C^\flat$.

5. Lecture of 06.11.2019: More on $\mathbb{A}_{\mathrm{inf}}$

For general notations see Section 1. Let us make a side remark about the occurrence of the two fields F, E. For this, let $K/\mathbb{Q}_p$ be a discretely valued non-archimedean field with perfect residue field and let $X \to \mathrm{Spec}(K)$ be a proper and smooth morphism. Of interest in p-adic Hodge theory is the p-adic étale cohomology

$$H^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$$

of X. Thus, there are implicitly *two* non-archimedean fields involved, namely,

$$\mathbb{Q}_p$$

as the field of coefficients and

$$C := \widehat{\overline{K}},$$

the completion of an algebraic closure $\overline{K}$ of K. In the setup for the Fargues-Fontaine curve the field E replaces the field $\mathbb{Q}_p$ and F plays the role of C (note that one can for example set F as the fraction field $C^\flat$ of $\mathcal{O}_C^\flat$, cf. Lemma 3.18).

We now introduce primitive elements in $\mathbb{A}_{\mathrm{inf}}$.

Definition 5.1 (cf. [10, Section 2.2.1.])**.** An element

$$x = \sum_{i=0}^{\infty} [x_i]\pi^i \in \mathbb{A}_{\mathrm{inf}}$$

is called primitive if $x_0 \neq 0$ and there exists $d \geq 0$, such that $x_d \in \mathcal{O}_F^\times$. The degree of a primitive element x is defined as

$$\min\{d \mid x_d \in \mathcal{O}_F^\times\}.$$

Furthermore, we denote by Prim_d the set of degree d primitive elements.

For example, Prim_0 is precisely the set of units in $\mathbb{A}_{\mathrm{inf}}$ and if $x \in \mathrm{Prim}_1$, then x is distinguished in the sense of Definition 3.12. Clearly, each distinguished element which is not a multiple of π is primitive of degree 1. Thus, with Definition 4.4,

$$|Y| \cong \mathrm{Prim}_1/\mathbb{A}_{\mathrm{inf}}^\times.$$

From Proposition 3.16 we obtain an injective map

$$\{(C, \iota) \mid C/E \text{ non-archimedean, algebraically closed}, \iota\colon \mathcal{O}_C^\flat \cong \mathcal{O}_F\} \hookrightarrow |Y|$$

and we wish to show this map is surjective. Thus let $u \in \mathbb{A}_{\mathrm{inf}}^\times$ and $a_0 \in \mathfrak{m}_F \setminus \{0\}$ and set

$$a := u\pi - [a_0] \in \mathrm{Prim}_1,\ D := \mathbb{A}_{\mathrm{inf}}/(a),\ \theta\colon \mathbb{A}_{\mathrm{inf}} \twoheadrightarrow D.$$

We have to prove that D is isomorphic to the ring of integers $\mathcal{O}_C$ in a non-archimedean, algebraically closed extension C/E. We follow [10, Section 2.2.2.].

Proposition 5.2. *(1) D is π-complete and π-torsion free.*
(2) $D^\flat \cong \mathcal{O}_F$.
(3) The map $D \to D,\ x \mapsto x^p$ is surjective.

Proof. The sequence (π, a) is regular. As $\mathbb{A}_{\mathrm{inf}}$ is π-complete this implies that the sequence (a, π) is regular, too.[31] This proves 1). By Proposition 3.3

$$D^\flat \cong {\mathbb{A}_{\mathrm{inf}}}^\flat \cong \mathcal{O}_F,$$

which shows 2). For 3): Let $E_0 \subseteq E$ be the maximal unramified extension. There exists a norm morphism

$$N_{E/E_0}\colon \mathbb{A}_{\mathrm{inf}\,E,F} = W_{\mathcal{O}_E}(\mathcal{O}_F) \to \mathbb{A}_{\mathrm{inf}\,E_0,F} = W(\mathcal{O}_F)$$

which sends primitive elements of degree 1 to primitive elements of degree 1 (this can be checked on $W_{\mathcal{O}_E}(k) \to W(k)$ and uses that E/E_0 is totally ramified). One checks that the resulting morphism

$$D' := \mathbb{A}_{\mathrm{inf}}/(N_{E/E_0}(a)) \to \mathbb{A}_{\mathrm{inf}}/(a) =: D$$

is surjective, which reduces us to the case that $E = E_0$, and then to $E = \mathbb{Q}_p$. Let

$$\theta\colon \mathbb{A}_{\mathrm{inf}} \to D,\ \sum_{n=0}^{\infty}[x_n]\pi^n \mapsto \sum_{n=0}^{\infty}\theta([x_n])\pi^n$$

be the natural projection. It is clear that every element

$$\theta([z])$$

with $z \in \mathcal{O}_F$ has a p-th root. We can write each $x \in D$ in the form

$$x = \sum_{n=0}^{\infty}\theta([x_n])\theta([a_0])^n$$

with $\nu(x_n) < \nu(a_0)$, $n \geq 0$, because D is $\theta([a_0]) = \theta(u)\pi$-adically complete. Multiplying with

$$\theta([x_{n_0}a_0^{n_0}])^{-1}$$

where n_0 is the least integer with $x_n \neq 0$ we may assume that

$$x \in 1 + (p, \mathfrak{m}_F),$$

i.e., that $x_0 \in \mathfrak{O}_F^\times$. We claim that there exists $z \in \mathfrak{O}_F^\times$, such that

$$x \equiv \theta([z]) \bmod p^2.$$

(resp. $x \equiv \theta([z]) \bmod p^3$, if $p = 2$). This is sufficient because $\theta([z])$ has a p-th root and if $p \neq 2$ each element in $1 + (p^2)$ (resp. if $p = 2$ each element in $1 + (p^3)$) has a p-th root. Write

$$x \equiv \theta([x_0] + p[y_1])$$

[31] Let R be a ring, (r, s) some regular sequence such that R is r-adically complete. Passing to the limit of the injections $R/r^n \xrightarrow{s} R/r^n$ implies that $s \in R$ is a non-zero divisor. The snake lemma implies then that (s, r) is regular because (r, s) is regular.

with $y_1 \in \mathcal{O}_F^\times$. After multiplying a with some Teichmüller lift we may assume

$$a = [a_0] + p \bmod p^2.$$

For $\lambda \in \mathcal{O}_F$ we obtain

$$[x_0] + p[y_1] + [\lambda]a \equiv [x_0 + \lambda a_0] + p[y_1^p + \lambda^p + S_1(x_0, \lambda a_0)]^{1/p} \bmod p^2$$

with

$$S_1(X,Y) = \frac{1}{p}((X+Y)^p - X^p - Y^p)$$

(cf. Example 3.6 and Item 6). As F is algebraically closed we find $\lambda \in F$ such that

$$[x_0] + p[y_1] + [\lambda]a \equiv [z] \bmod p^2$$

with $z = x_0 + \lambda a_0$. Necessarily, $\lambda \in \mathcal{O}_F$ and $z \in \mathcal{O}_F^\times$. This finishes the proof if $p \neq 2$. We leave the case $p = 2$ as an exercise. □

We can now finish our discussion of D.

Corollary 5.3. *D is a complete valuation ring with algebraically closed field of fractions whose valuation is given by $\nu_D \colon D \to \mathbb{R} \cup \{\infty\}, d = \theta([x]) \mapsto \nu(x)$.*

Proof. By Proposition 5.2 we know that the map

$$(-)^\sharp \colon \mathcal{O}_F \to \mathcal{D}$$

is surjective. This multiplicative map extends to a surjective multiplicative map

$$\mathcal{O}_F[1/a_0] \to \mathcal{D}[1/\pi]$$

This implies that $D[1/\pi]$ is a field as each non-zero element is invertible. As D is π-torsion free, we can conclude that D is a domain and $D[1/\pi] = \mathrm{Frac}(D)$. Moreover, D is a valuation ring because an integral domain R is a valuation ring if and only if for all $r \in \mathrm{Frac}(R) \setminus \{0\}$ either $r \in R$ or $r^{-1} \in R$. We leave as an exercise to check that the valuation on D has the desired shape.[32] We use finally the argument from [20, Proposition 3.8] to show that $\mathrm{Frac}(D)$ is irreducible. Let

$$P(T) = T^d + b_{d-1}T^{d-1} + \ldots + b_0 \in D[T]$$

be irreducible, $d > 0$.[33] Let $Q(T) \in \mathcal{O}_F[T]$ such that $Q(T) \equiv P(T)$ in $D/\pi[T] \cong \mathcal{O}_F/a_0[T]$, and let $y \in \mathcal{O}_F$ be a zero of Q. Then $P(T + y^\sharp)$ has constant term divisible by π and is again irreducible. Consider

$$P_1(T) := c^{-d}P(cT + y^\sharp)$$

where $d\nu_D(c) = \nu_D(P(y^\sharp)) \geq \nu_D(\pi)$. Then $P_1(T)$ has again coefficients in $D[T]$ and there exists $y_1 \in \mathcal{O}_F$ such that

$$\nu_D(P_1(y_1^\sharp)) \geq \nu_D(\pi),$$

i.e.,

$$\nu_D(P(cy_1^\sharp + y^\sharp) \geq d\nu_D(c) + \nu_D(\pi).$$

Iterating this process yields a zero of $P(T)$. □

We have thus finished the proof of the following theorem.

[32] Hint: Use that $D/\pi \cong \mathcal{O}_F/a_0$.

[33] By completeness of D this case is sufficient to see that $\mathrm{Frac}(D)$ is algebraically closed.

Theorem 5.4 (cf. [10, Corollaire 2.2.22.])**.** *The map*

$$\{C/E \text{ algebraically closed, non-archimedean, } \iota\colon \mathcal{O}_C^\flat \cong \mathcal{O}_F\} \to |Y|$$

defined by

$$(C,\iota) \mapsto \ker(\mathbb{A}_{\mathrm{inf}} \xrightarrow{\iota^{-1}} W_{\mathcal{O}_E}(\mathcal{O}_C^\flat) \xrightarrow{\theta} \mathcal{O}_C)$$

is bijective.

Continuing the discussion after Lemma 4.6 Theorem 5.4 can be seen as giving "a non-archimedean geometric structure" to $|Y|$. Namely, we can make the following definitions.

Definition 5.5. Let $y \in |Y|$. Then we set

- $\mathfrak{p}_y \subseteq \mathbb{A}_{\mathrm{inf}}$ the corresponding prime ideal
- $\xi_y \in \mathfrak{p}_y$ some generator
- $C_y := \mathbb{A}_{\mathrm{inf}}/\mathfrak{p}_y[1/\pi]$ the "residue field of y" (an algebraically closed, non-archimedean extension of E)
- $\theta_y\colon \mathbb{A}_{\mathrm{inf}} \to C_y$ the canonical projection
- $\nu_y\colon C_y \to \mathbb{R}\cup\{\infty\}$ the valuation[34]

$$\nu_y(\theta_y([x])) := \nu(x).$$

- $B^+_{\mathrm{dR},y}$ the ξ_y-adic completion of $\mathbb{A}_{\mathrm{inf}}[1/\pi]$, a complete discrete valuation ring (cf. Lemma 4.6) with residue field C_y.
- For $f \in \mathbb{A}_{\mathrm{inf}}$, we set $f(y) := \theta_y(f) \in C_y$ and $\nu(f(y)) = \nu_y(f(y))$.

For $f \in \mathbb{A}_{\mathrm{inf}}$ the map $y \mapsto f(y)$ allows us to think about elements of $\mathbb{A}_{\mathrm{inf}}$ as "functions on $|Y|$". This will be a useful viewpoint in Section 7.

6. Lecture of 13.11.2019: Newton polygons

We will now introduce the Newton polygons of elements in $\mathbb{A}_{\mathrm{inf}}$. These will be a powerful tool. We will however introduce them in greater generality. For this, let K be a non-archimedean field and $\nu\colon K \to \mathbb{R}\cup\{\infty\}$ its valuation. Let $f(T) = \sum\limits_{i=1}^{n} a_i T^i \in K[T]$ be a polynomial.

Definition 6.1. We define

$$\mathcal{N}ewt_{\mathrm{poly}}(f)$$

as the largest convex polygon below the set $\{(i,\nu(a_i))\}_{i=0}^n$.

The usefulness of the Newton polygon is Proposition 6.2. For us the slopes of a polygon are the usual slopes of its segments, and not as in [10, Section 1.5.1] the inverses.[35] For a polygon with integral breakpoints with call the length of the projection of a segment to the first coordinate the multiplicity of the slope of that segment.

Proposition 6.2. *Let $x_0,\ldots,x_n \in \overline{K}$ be the zeros of f, then*

$$-\nu(x_0),\ldots,-\nu(x_n)$$

are exactly the slopes of $\mathcal{N}ewt_{\mathrm{poly}}(f)$ with correct multiplicity.

[34] By Corollary 5.3 this is well-defined.

[35] We hope that by this convention the confusion within this lecture is reduced (although the confusion when comparing with our main source [10] is augmented).

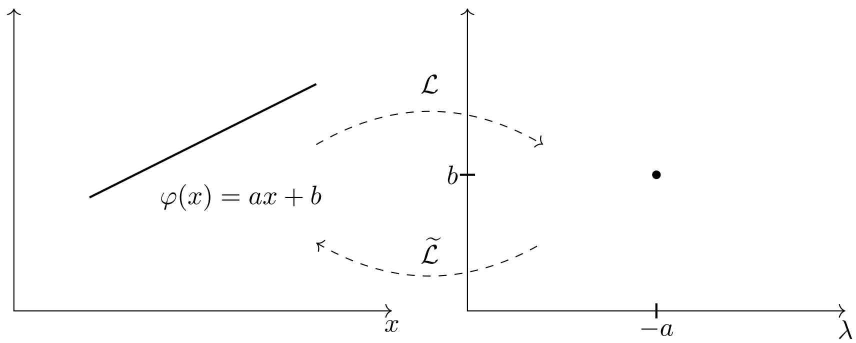

FIGURE 2. The Legendre transform and its inverse for $\varphi(x) = ax + b$.

We will present a proof of this proposition as a consequence of our discussions of the Legendre transform, cf. Example 6.17.

Definition 6.3. We set

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$$

and

$$\mathcal{F} := \{\varphi\colon \mathbb{R} \to \overline{\mathbb{R}}\}.$$

The set $\mathcal{F}$ is not an $\mathbb{R}$-vector space, not even an abelian group, in any reasonable sense.

The Legendre transform and its "inverse" are defined in the following way. For references on the Legendre transform we recommend [17], [10] and [29].

Definition 6.4. We define the "Legendre transform"

$$\mathcal{L}\colon \mathcal{F} \to \mathcal{F},\ \varphi \mapsto (\lambda \mapsto \inf_{x\in\mathbb{R}} \{\varphi(x) + \lambda x\})$$

and the "inverse Legendre transform"

$$\widetilde{\mathcal{L}}\colon \mathcal{F} \to \mathcal{F},\ \psi \mapsto (x \mapsto \sup_{\lambda\in\mathbb{R}} \{\psi(\lambda) - \lambda x\}).$$

Remark 6.5. We note that

$$\widetilde{\mathcal{L}}(\varphi) = -\mathcal{L}(-\varphi).$$

The Legendre transform interchanges x-coordinates and slopes as the following example shows.

Example 6.6. Assume $\varphi(x) = ax + b$ for some $a, b \in \mathbb{R}$. Then, see Figure 2,

$$\mathcal{L}(\varphi)(\lambda) = \begin{cases} b, & \text{if } \lambda = -a \\ -\infty, & \text{otherwise } \lambda \neq -a \end{cases}$$

and

$$\widetilde{\mathcal{L}}\mathcal{L}(\varphi) = \varphi.$$

To understand the behaviour of the Legendre transform it is useful to define the concept of a supporting resp. capping line.

Definition 6.7. Let $\varphi \in \mathcal{F}$ and $x \in \mathbb{R}$. We say that φ admits a supporting line at x of slope $\lambda \in \mathbb{R}$ if $\varphi(x) \neq \pm\infty$ and

$$\varphi(y) \geq \varphi(x) + \lambda(y - x)$$

for all $y \in \mathbb{R}$. Dually, we say that φ admits a capping line at x of slope λ if $\varphi(x) \neq \pm \pm \infty$ and

$$\varphi(y) \leq \varphi(x) + \lambda(y - x)$$

for all $y \in \mathbb{R}$.

If $\varphi(x) = \infty$ (resp. $\varphi(x) = -\infty$), then we call each linear function

$$c + \lambda(y - x)$$

with $c, \lambda \in \mathbb{R}$ a supporting line (resp. a capping line) of φ at x of slope λ if

$$\varphi(y) \geq c + \lambda(y - x)$$

(resp.

$$\varphi(y) \leq c + \lambda(y - x))$$

for each $y \in \mathbb{R}$. The Legendre transform induces a bijection between (non-extendable) convex resp. concave functions as we will see in Proposition 6.11.

Definition 6.8. A function $\varphi\colon \mathbb{R} \to \overline{\mathbb{R}}$ is called convex resp. concave if for all $x, y \in \mathbb{R}$ and all $a, b \geq 0$ such that $a + b = 1$

$$\varphi(ax + by)) \leq a\varphi(x) + b\varphi(y)$$

resp.

$$\varphi(ax + by)) \geq a\varphi(x) + b\varphi(y).$$

Thus a function $\varphi \in \mathcal{F}$ is convex (resp. concave) if and only if it admits a supporting (resp. capping line) at each $x \in \mathbb{R}$.

We need one more definition to exclude some pathological behaviour.

Definition 6.9. We call a convex (resp. concave) function $\varphi \in \mathcal{F}$ non-extendable if φ is the infimum over all its supporting lines (resp. the supremum over all its capping lines).

For example, the function

$$\varphi(x) = \begin{cases} \infty, & x \leq 0 \\ 0, & x > 0 \end{cases}$$

is convex and extendable with extension

$$\tilde{\varphi}(x) = \begin{cases} \infty, & x < 0 \\ 0, & x \geq 0. \end{cases}$$

Note that $\mathcal{L}(\varphi) = \mathcal{L}(\tilde{\varphi})$ is the function

$$\lambda \mapsto \begin{cases} -\infty, & \lambda < 0 \\ 0, & \lambda \geq 0. \end{cases}$$

Definition 6.10. Let $\varphi \in \mathcal{F}$. Then the non-extendable convex (resp. non-extendable concave) hull below (resp. above) of φ is defined as the infimum over all its supporting lines (resp. the supremum over all its capping lines).

We can now summarize the properties of the Legendre transform $\mathcal{L}$ and its "inverse" $\widetilde{\mathcal{L}}$.

Proposition 6.11. *Let $\varphi, \psi \in \mathcal{F}$.*

1. *$\mathcal{L}(\varphi)$ is non-extendable concave, and $\widetilde{\mathcal{L}}(\varphi)$ is non-extendable convex.*
2. *If $\varphi \leq \psi$, then $\mathcal{L}(\varphi) \leq \mathcal{L}(\psi)$ and $\widetilde{\mathcal{L}}(\varphi) \leq \widetilde{\mathcal{L}}(\psi)$.*
3. *$\widetilde{\mathcal{L}}\mathcal{L}(\varphi) \leq \varphi$ and $\varphi \leq \mathcal{L}\widetilde{\mathcal{L}}(\varphi)$.*
4. *If $\varphi \neq \infty$ admits a supporting line at x of slope λ, then $\mathcal{L}(\varphi)$ admits a capping line at $-\lambda$ of slope x.*
5. *$\widetilde{\mathcal{L}}\mathcal{L}(\varphi)$ is the non-extendable convex function below φ.*
6. *$\mathcal{L}, \widetilde{\mathcal{L}}$ define inverse bijections between non-extendable convex resp. non-extendable concave functions.*

Proof. Point (1) for $\mathcal{L}$ is clear as an infimum of linear functions is non-extendable concave. This implies the statement for $\widetilde{L}$ using the formula $\widetilde{L}(\varphi) = -\mathcal{L}(-\varphi)$.

Point (2), (3) follow directly from the definitions.

Let us prove (4). The condition $\varphi \neq \infty$ implies $\mathcal{L}(\varphi)(\mu) \neq \infty$ for all $\mu \in \mathbb{R}$, in particular for $\mu = -\lambda$. By assumption

$$\varphi(y) \geq c + \lambda(y - x)$$

for all $c \leq \varphi(x)$ and all $y \in \mathbb{R}$. We calculate for $\mu \in \mathbb{R}$:

$$\begin{aligned} & \mathcal{L}(\varphi)(-\lambda) + x(\lambda + \mu) \\ = \; & \inf_{y \in \mathbb{R}} \{\varphi(y) - \lambda y\} + x(\lambda + \mu) \\ = \; & \inf_{y \in \mathbb{R}} \{\varphi(y) - \lambda(y - x) + x\mu\} \\ \geq \; & \inf_{y \in \mathbb{R}} \{c + x\mu\} = c + x\mu \end{aligned}$$

If $\varphi(x) \neq +\infty$, then we can take $c = \varphi(x)$ and

$$c + x\mu \geq \mathcal{L}(\varphi)(\mu)$$

as desired. If $\varphi(x) = \infty$ (note that $\varphi(x) = -\infty$ is excluded by the existence of a supporting line), then trivially

$$\mathcal{L}(\varphi)(-\lambda) + x(\lambda + \mu) = \infty \geq \mathcal{L}(\varphi)(\mu).$$

In point (5) it is clear by (1), that $\widetilde{\mathcal{L}}\mathcal{L}(\varphi)$ is non-extendable, convex and below φ. If ℓ is any supporting line of φ, then by Example 6.6

$$\ell = \widetilde{\mathcal{L}}\mathcal{L}(\ell) \leq \widetilde{\mathcal{L}}\mathcal{L}(\varphi).$$

This implies the claim.

Point (6) is a formal consequence of the other statements. Namely, (2),(3) imply that $\mathcal{L}, \widetilde{\mathcal{L}}$ induce adjoint functors $(\mathcal{F}, \leq) \to (\mathcal{F}, \leq)$. But any adjunction induces an equivalence on fixed points and (1), (5) imply that functions φ satisfying $\widetilde{\mathcal{L}}\mathcal{L}(\varphi) = \varphi$ resp. $\mathcal{L}\widetilde{\mathcal{L}}(\psi) = \psi$ are precisely the non-extendable convex resp. concave functions. □

Proposition 6.11, point (4) is particularly useful for working out the shape of the Legendre transform without to much calculation. The most important example of the Legendre transform for us is the case of piecewise linear functions (such as Newton polygons), cf. Figure 3.

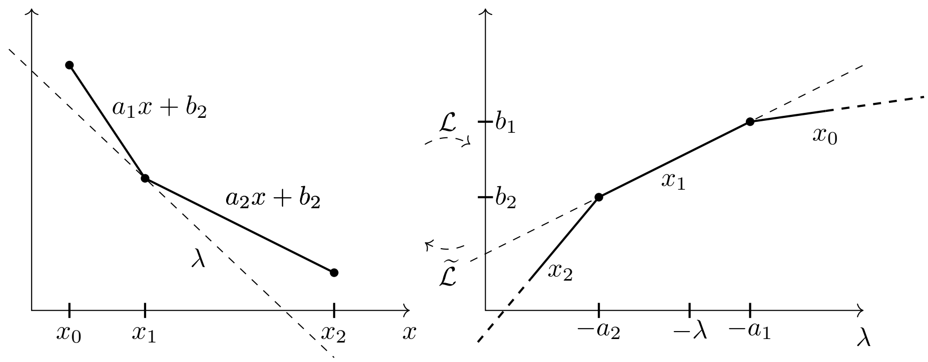

FIGURE 3. The Legendre transform of a piecewise linear function. Note how the Legendre transform exchanges slopes and abcisses. The picture also shows a supporting line at x_1 of λ and a capping line at $-\lambda$ of slope x_1.

Lemma 6.12. *The transform $\mathcal{L}$ sends a convex piecewise linear function to a concave piecewise linear function.*

Proof. This follows from point (4) in Proposition 6.11. □

In the future we will drop the adjective non-extendable and assume implicitly that all convex resp. concave functions are non-extendable.

Let us come back to a non-archimedean field K with valuation $\nu\colon K \to \mathbb{R} \cup \{\infty\}$ and pick a polynomial

$$f(T) = \sum_{i=0}^{n} a_i T^i \in K[T].$$

By Definition 6.1 $\mathcal{N}ewt_{\mathrm{poly}}(f)$ is the largest convex function below the set $\{(i, \nu(a_i)\}_{i\in\mathbb{Z}}$ (where $a_i = 0$ if $i \notin \{0, \ldots, n\}$). By Proposition 6.11 this implies

$$\mathcal{L}(\mathcal{N}ewt_{\mathrm{poly}}(f))(r) = \inf_{i\in\mathbb{Z}}\{\nu(a_i) + ri\} =: \nu_r(f).$$

for $r \in \mathbb{R}$. If $r \in \nu(\overline{K}^{\times})$, the function ν_r has the more geometric interpretation

$$\nu_r(f) = \inf\{\nu(f(x)) \mid x \in \overline{K}, \nu(x) = r\},$$

cf. [4, 6.1.5.Proposition 5].

The functions ν_r are, and this is important, not only norms, but valuations, i.e., multiplicative norms.

Lemma 6.13. *For $r \in \mathbb{R}$ and $f, g \in K[T]$*

$$\nu_r(f \cdot g) = \nu_r(f) + \nu_r(g).$$

Proof. After extending K and factoring f, we may assume that $f = T - a$ for some $a \in K$. We know

$$\nu_r(Tg) = r + \nu_r(g),\ \nu_r(-ag) = \nu(a) + \nu_r(g).$$

Let us first assume that $r \neq \nu(a)$. Then by the strong triangle inequality

$$\nu_r((T-a)g) = \inf\{\nu_r(Tg), \nu_r(-ag)\} = \nu_r(T-a) + \nu_r(g)$$

as desired. The case $r = \nu(a)$ can be reduced to this. Namely, the functions

$$r \mapsto \nu_r(f \cdot g),\ r \mapsto \nu_r(f) + \nu_r(g)$$

are continuous in r and they agree on $\mathbb{R} \setminus \{\nu(a)\}$, hence they must agree also for $r = \nu(a)$. □

This implies that

$$\mathcal{L}(\varphi_{f\cdot g}) = \mathcal{L}(\varphi_f) + \mathcal{L}(\varphi_g)$$

where φ_h is the piecewise linear function connecting the points $\{(i, \nu(a_i))\}_{i\in\mathbb{Z}}$ for

$$h = \sum_{i\in\mathbb{Z}} a_i T^i \in K[T].$$

To analyze how the Newton polygon for a product $f \cdot g$ can be described via $\mathcal{N}ewt_{\mathrm{poly}}(f)$ and $\mathcal{N}ewt_{\mathrm{poly}}(g)$ we need the convolution product of functions.

Definition 6.14. Let $\varphi, \psi \in \mathcal{F}$ such that $-\infty \notin \mathrm{Im}(\varphi) \cup \mathrm{Im}(\psi)$. The the convolution of φ and ψ is defined to be the function

$$\varphi * \psi \colon \mathbb{R} \to \widehat{\mathbb{R}},\ x \mapsto \inf_{a+b=x} \{\varphi(a) + \psi(b)\}.$$

The Legendre transform behaves well with convolution.

Lemma 6.15. *Let $\varphi, \psi \in \mathcal{F}$ such that $-\infty \notin \mathrm{Im}(\varphi) \cup \mathrm{Im}(\psi)$.*

(1) *If φ, ψ are convex, then $\varphi * \psi$ is convex.*
(2) *$\mathcal{L}(\varphi * \psi) = \mathcal{L}(\varphi) + \mathcal{L}(\psi)$.*

Proof. We leave this as an exercise. □

Item 2 implies that the convolution of two piecewise linear convex functions φ, ψ is obtained by concatenating the slopes of φ, ψ to a new convex function.

Moreover, Lemma 6.15 has the following important corollary.

Corollary 6.16. *For $f, g \in K[T]$*

$$\mathcal{N}ewt_{\mathrm{poly}}(f \cdot g) = \mathcal{N}ewt_{\mathrm{poly}}(f) * \mathcal{N}ewt_{\mathrm{poly}}(g).$$

Proof. Both sides are convex functions and

$$\begin{aligned} \mathcal{L}(\mathcal{N}ewt_{\mathrm{poly}}(f) * \mathcal{N}ewt_{\mathrm{poly}}(g)) &= \mathcal{L}(\mathcal{N}ewt_{\mathrm{poly}}(f)) + \mathcal{L}(\mathcal{N}ewt_{\mathrm{poly}}(g)) \\ &= \mathcal{L}(\varphi_f) + \mathcal{L}(\varphi_g) \\ &= \mathcal{L}(\varphi_{f\cdot g}) \\ &= \mathcal{L}(\mathcal{N}ewt_{\mathrm{poly}}(f \cdot)) \end{aligned}$$

using Lemma 6.15 and Lemma 6.13. □

Example 6.17. If $f = T - \alpha$, $g = T - \beta$, then the slopes of $\mathcal{N}ewt_{\mathrm{poly}}(fg)$ are the concatenation of the slopes of $\mathcal{N}ewt_{\mathrm{poly}}(f)$ and $\mathcal{N}ewt_{\mathrm{poly}}(f)$, cf. Figure 4. This yields a quick proof of Proposition 6.2.

The theory of Newton polygons can be done for power series. Let

$$f \in \mathcal{O}_K[[T]]^{36}$$

Then f defines a function on the open rigid-analytic unit disc

$$\mathbb{D}_K := \{|x| < 1\}$$

[36] We assume that the coefficients of f lie in $\mathcal{O}_C$, as opposed to K, in order to avoid convergence issues on the open rigid-analytic unit disc $\mathbb{D}_K$ over K.

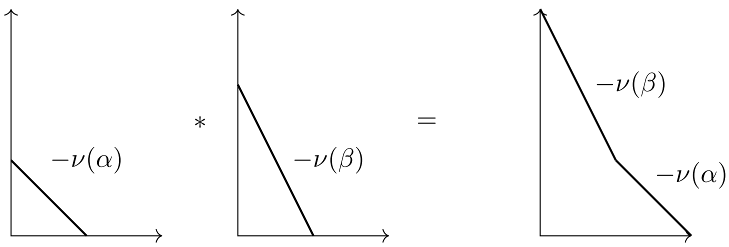

FIGURE 4. The convolution of $\mathcal{N}ewt_{\text{poly}}(T-\alpha)$ and $\mathcal{N}ewt_{\text{poly}}(T-\beta)$.

and the theory of Newton polygons for f will yield informations on the slopes of zeros of f in $\mathbb{D}_K$. In particular, even if f is a polynomial we will not be interested in zeros of f outside of $\mathbb{D}_K$, i.e., in those line segments of $\mathcal{N}ewt(f)$ of slopes > 0 (as these correspond to zeros with negative valuations). This explains the condition that the polygon is non-decreasing in Definition 6.18.

Definition 6.18. Let $f \in \mathcal{O}_K[[T]]$. Then $\mathcal{N}ewt(f)$ is defined as the largest, decreasing convex function below $\{(i, \nu(a_i))\}_{i\in\mathbb{Z}}$, i.e.,

$$\mathcal{L}(\mathcal{N}ewt(f))(r) = \begin{cases} \nu_r(f), \ r \geq 0 \\ -\infty, r < 0 \end{cases}$$

with

$$\nu_r(f) := \inf_{i\in\mathbb{Z}}\{\nu(a_i) + ri\}$$

for $r \geq 0$.

The condition that $\mathcal{L}(\mathcal{N}ewt(f))(r) = -\infty$ for $r < 0$ ensures precisely that $\mathcal{N}ewt(f)$ is decreasing. The functions ν_r are again valuations (this follows from Lemma 6.13).

Again the slopes of the Newton polygon $\mathcal{N}ewt(f)$ of $f \in \mathcal{O}_K[[T]]$ captures the valuation of the zeros of f.

Theorem 6.19 (Lazard [17])**.** *Let $f \in \mathcal{O}_K[[T]]$ and $\lambda \neq 0$ a slope of $\mathcal{N}ewt(f)$. Then there exists some $\alpha \in \widehat{\overline{K}}$ with $f(\alpha) = 0$ and $\nu(\alpha) = -\lambda$.*

The condition $f(\alpha) = 0$ is equivalent to the condition that there exists some $g \in \mathcal{O}_{\widehat{\overline{K}}}[[T]]$ such that $f = (T-\alpha)g$.

In the next lecture we will present the proof of Fargues and Fontaine about the analogue in mixed characteristic, i.e., with $\mathcal{O}_K[[T]]$ replaced by $\mathbb{A}_{\text{inf}}$.

The valuations ν_r (cf. Lemma 6.13) have obvious analogues for $\mathbb{A}_{\text{inf}}$.

Definition 6.20. For $r \geq 0$ and $f = \sum\limits_{i=0}^{\infty}[a_i]\pi^i \in \mathbb{A}_{\text{inf}}$ we set

$$\nu_r(f) := \inf_{i\in\mathbb{Z}}\{\nu(a_i) + ri\}.$$

With the ν_r at hand we can define the Newton polygon for elements in $\mathbb{A}_{\text{inf}}$.

Definition 6.21. Let $f \in \mathbb{A}_{\text{inf}}$. The Newton polygon $\mathcal{N}ewt(f)$ of f is the convex, decreasing, piecewise linear function with Legendre transform

$$\mathcal{L}(\mathcal{N}ewt(f)) := \begin{cases} \nu_r(f), \ r \geq 0 \\ -\infty, \ r < 0 \end{cases}.$$

Crucially, the functions ν_r are again multiplicative.

Lemma 6.22 (cf. [10, Proposition 1.4.9.]). *For $r \geq 0$ the function ν_r is a valuation. In particular,*

$$\mathcal{N}ewt(f \cdot g) = \mathcal{N}ewt(f) * \mathcal{N}ewt(g)$$

for $f, g \in \mathbb{A}_{\mathrm{inf}}$.

A reduction to an analogue of Lemma 6.13 like for power series is not possible as there is no replacement of the ring of polynomials in $\mathbb{A}_{\mathrm{inf}}$.[37]

Proof. We have to show $\nu_r(f \cdot g) = \nu_r(f) + \nu_r(g)$ for $f, g \in \mathbb{A}_{\mathrm{inf}}$. The other statements are then clear. The inequality

$$\nu_r(f \cdot g) \geq \nu_r(f) + \nu_r(g)$$

is easy. Moreover, we leave the case $r = 0$ as an exercise. Thus assume $r > 0$. Write

$$f = \sum_{i=0}^{\infty} [a_i]\pi^i, \ g = \sum_{i=0}^{\infty} [b_i]\pi^i.$$

Then there exist natural numbers $n, m \geq 0$ which are the least such that

$$\nu_r(f) = \nu_r([a_n]\pi^n), \ \nu_r(g) = \nu_r([b_m]\pi^m)$$

(this uses $r > 0$). Write

$$f = x' + [a_n]\pi^n + \pi^{n+1}x''$$

with $\nu_r(x') > \nu_r(f)$, $\nu_r(\pi^{n+1}x'') \geq \nu_r(f)$ and

$$g = y' + [b_m]\pi^m + \pi^{m+1}y''$$

with $\nu_r(y') > \nu_r(g)$, $\nu_r(\pi^{m+1}y'') \geq \nu_r(g)$. Then

$$f \cdot g = z + [a_n \cdot b_m]\pi^{n+m} + \pi^{n+m+1}w$$

with $\nu_r(z) > \nu_r(f) + \nu_r(g)$. We introduce the auxiliary function

$$\tilde{\nu} \colon \mathbb{A}_{\mathrm{inf}} \colon \mathbb{R} \cup \{\infty\}, \ h = \sum_{i=0}^{\infty} [c_i]\pi^i \mapsto \inf_{0 \leq i \leq n+m} \{\nu(c_i)\} + r(n+m).$$

Then $\tilde{\nu}$ satisfies, as is easily checked, the properties

(1) $\tilde{\nu}(0) = \infty$
(2) $\tilde{\nu}(h_1 + h_2) \geq \inf\{\tilde{\nu}(h_1), \tilde{\nu}(h_2)\}$ with equality if $\tilde{\nu}(h_1)$ and $\tilde{\nu}(h_2)$ differ.
(3) $\tilde{\nu}(h) \geq \nu_r(h)$.

We can conclude that

$$\begin{aligned} \nu_r(f \cdot g) \leq \tilde{\nu}(f) &= \tilde{\nu}(z + [a_n \cdot b_m]\pi^{n+m}) \\ &= \tilde{\nu}([a_n \cdot b_m]\pi^{n+m}) = \nu_r(f) + \nu_r(g) \end{aligned}$$

where we used that $\nu_r(z) > \nu_r([a_n \cdot b_m]\pi^{n+m})$ in the second to last step. ☐

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[37] The subset $\{\sum_{i=0}^{n} [x_i]\pi^i \in \mathbb{A}_{\mathrm{inf}} \mid n \in \mathbb{N}\}$ of $\mathbb{A}_{\mathrm{inf}}$ is not stable under addition and multiplication.

7. Lecture of 20.11.2019: The metric space $|Y|$ and factorizations

We continue with the notations from Section 1.

In this lecture we want to discuss the following theorem of Fargues/Fontaine, which is analogous to Theorem 6.19.

Theorem 7.1 (Fargues-Fontaine, cf. [10, Théorème 2.4.5.]). *Let $f \in \mathbb{A}_{\mathrm{inf}}$ and let $\lambda \neq 0$ be a slope of $\mathcal{N}ewt(f)$. Then there exists some $a \in \mathcal{O}_F$, such that $\nu(a) = -\lambda$ and $f = (\pi - [a])g$ for some $g \in \mathbb{A}_{\mathrm{inf}}$.*

For the proof of it is important to interpret $\mathbb{A}_{\mathrm{inf}}$ as "functions on a punctured open unit disc".

Recall the space

$$|Y| = \mathrm{Prim}^1/\mathbb{A}_{\mathrm{inf}}^\times$$

of ideals in $\mathbb{A}_{\mathrm{inf}}$ generated by primitive elements of degree 1, which was introduced in Definition 4.4. We saw in Theorem 5.4 that $|Y|$ is in bijection the set of isomorphism classes of algebraically closed non-archimedean extension C/E equipped with an isomorphism $\mathcal{O}_C^\flat \cong \mathcal{O}_F$ and we will use the notations introduced in Definition 5.5. As an additional "geometric structure" on $|Y|$ we introduce a metric on it, cf. [10, Section 2.3.1.].

Definition 7.2. For $y_1, y_2 \in |Y|$ we set

$$d(y_1, y_2) := \nu_{y_1}(\theta_{y_1}(\xi_{y_2}))$$

and

$$d(y_1, 0) := \nu(\pi(y_1)).$$

We will see that $d(y_1, y_2)$ is a metric on $|Y|$. For the moment, it is not even clear that d is symmetric.

Remark 7.3. Define the adic space

$$\mathcal{Y} := \mathrm{Spa}(\mathbb{A}_{\mathrm{inf}}) \setminus \{\pi[\varpi]\}.$$

Then one can embed $|Y| \subseteq \mathcal{Y}$ as the set of "classical points". Rigorously, one has $\mathbb{A}_{\mathrm{inf}} \subseteq \mathcal{O}(\mathcal{Y})$. On $\mathcal{Y}$ the Frobenius φ acts properly discontinuous (as $d(\varphi(y), 0) = 1/q \cdot d(y, 0)$). The "adic Fargues-Fontaine" curve is defined as the quotient (in adic spaces)

$$\mathcal{X}^{\mathrm{ad}} := \mathcal{Y}/\varphi^{\mathbb{Z}}.$$

For more informations on this viewpoint we refer to [22, Section 11.2.].

Using the notations introduced in Definition 5.5Theorem 7.1 has the more geometric reformulation: For $f \in \mathbb{A}_{\mathrm{inf}}$ and $\lambda \neq 0$ a slope of $\mathcal{N}ewt(f)$, there exists $y \in |Y|$ such that $d(y, 0) = \nu(\pi(y)) = -\lambda$ and $f(y) = 0$.

Let us first check that $d(-,-)\colon |Y| \times |Y| \to \mathbb{R} \cup \{\infty\}$ is a metric. For $r \geq 0$ set

$$\mathfrak{a}_r := \{x = \sum_{i=0}^{\infty} [x_i]\pi^i \in \mathbb{A}_{\mathrm{inf}} \mid \nu_0(x) = \inf\{\nu(x_i)\} \geq r\}.$$

Lemma 7.4. *Let $y_1, y_2, y_3 \in |Y|$. Then*

$$d(y_1, y_2) = \sup_{r \geq 0}\{\mathfrak{p}_{y_1} + \mathfrak{a}_r = \mathfrak{p}_{y_2} + \mathfrak{a}_r\}.$$

In particular, $d(-,-)$ is an ultrametric, i.e.,

(1) $d(y_1, y_2) = d(y_2, y_1)$
(2) $d(y_1, y_3) \geq \inf\{d(y_1, y_2), d(y_2, y_3)\}$
(3) $d(y_1, y_2) = \infty \Leftrightarrow y_1 = y_2$.

Proof. Let $\mathfrak{p}_{y_i} = (\xi_{y_i})$, $i = 1, 2$, and write

$$\xi_{y_1} = \sum_{n=0}^{\infty} [x_n] \xi_{y_2}^n$$

(this is possible as $\mathcal{O}_F \twoheadrightarrow \mathcal{O}_{C_{y_2}}$ via the map $x \mapsto [x] \mathrm{mod}(\xi_{y_2})$). Then

$$d(y_2, y_1) = \nu_{y_2}(\theta_{y_2}(\xi_{y_1})) = \nu(x_0).$$

Applying θ_{y_1} yields

$$0 = \sum_{n=0}^{\infty} \theta_{y_1}([x_n]) \theta_{y_1}(\xi_{y_2})^n,$$

i.e.,

$$-\theta_{y_1}([x_0]) = \theta_{y_1}(\xi_{y_2}) (\sum_{n=1}^{\infty} \theta_{y_1}([x_n]) \theta_{y_1}(\xi_{y_2})^{n-1}).$$

This implies that

$$d(y_1, y_2) = \nu(x_0) = \nu_{y_1}(\theta_{y_1}([x_0])) \geq \nu_{y_1}(\theta_{y_1}(\xi_{y_2})) = d(y_1, y_2)$$

with equality if and only if $x_1 \in \mathcal{O}_F^\times$ (as $\nu_{y_1}(\theta_{y_1}(\xi_{y_2}))$). By symmetry this implies

$$d(y_2, y_1) = d(y_1, y_2).$$

Moreover,

$$\mathfrak{p}_{y_1} + \mathfrak{a}_{\nu(x_0)} = \mathfrak{p}_{y_2} + \mathfrak{a}_{\nu(x_0)}.$$

If $r \geq 0$, such that

$$\mathfrak{p}_{y_1} + \mathfrak{a}_r = \mathfrak{p}_{y_2} + \mathfrak{a}_r,$$

then

$$\langle \theta_{y_1}(\xi_{y_2}) \rangle \subseteq \{x \in \mathcal{O}_{C_{y_1}} \mid \nu_{y_1}(x) \geq r\}.$$

But

$$\langle \theta_{y_1}(\xi_{y_2}) \rangle = \langle \theta_{y_1}([x_0]) \rangle.$$

This implies

$$d(y_1, y_2) = \nu(x_0) \geq r$$

as desired. The properties of an ultrametric are then clear, except perhaps that

$$d(y_1, y_2) = \infty$$

implies $y_1 = y_2$. But the ideals $\mathfrak{p}_{y_1}, \mathfrak{p}_{y_2}$ are closed and

$$\mathbb{A}_{\mathrm{inf}} = \varprojlim_{r \geq 0} \mathbb{A}_{\mathrm{inf}} / \mathfrak{a}_r,$$

which implies this. □

For $r \in (0, \infty)$ set

$$|Y_r| := \{y \in |Y| \mid d(y, 0) = r\}.$$

The following proposition is important.

Proposition 7.5. *For any $r \in (0, \infty)$ the metric space $(|Y_r|, d)$ is complete.*

The statement for each single $r \in (0, \infty)$ implies that also for each $I \subseteq (0, \infty)$ the metric space

$$|Y_I| := \{y \in |Y| \mid d(y, 0) \in I\}$$

is complete.

Proof. Let $\{y_n\}_{n \geq 0}$ be a Cauchy sequence in $|Y_r|$. We claim that for all $r' > 0$ the sequence of ideals

$$\{\mathfrak{p}_{y_n} + \mathfrak{a}_r\}_{n \geq 0}$$

is constant for $n \gg 0$. Indeed, there exists some $n_0 \geq 0$, such that $d(y_n, y_m) \geq r'$ for all $n, m \geq n_0$. Then by Lemma 7.4

$$\mathfrak{p}_{y_n} + \mathfrak{a}_{r'} = \mathfrak{p}_{y_m} + \mathfrak{a}_{r'}.$$

Set

$$I_{r'} = \mathfrak{p}_{y_n} + \mathfrak{a}_{r'} / \mathfrak{a}_{r'},$$

$n \gg 0$ as the eventually attained ideal, and

$$I := \varprojlim_{r \geq 0} I_{r'} \subseteq \mathbb{A}_{\mathrm{inf}}.$$

Note

$$I_{r'} = I + \mathfrak{a}_{r'} / \mathfrak{a}_{r'}$$

The ideal I is generated by a primitive element of degree 1, and $\mathfrak{p}_{y_n} \to I,\ n \to \infty$. To see this, fix $r' > r$ and n such that

$$\mathfrak{p}_{y_n} + \mathfrak{a}_{r'} = I + \mathfrak{a}_{r'}.$$

Write $\mathfrak{p}_{y_n} = (\xi_{y_n})$. Then there exist an $x \in \mathfrak{a}_{r'}$ such that

$$a := \xi_{y_n} + x \in I.$$

Then a is primitive of degree 1 as $r' > r = \nu(\pi(y_n))$. Note that $(a) \in |Y_r|$. Clearly, $(a) \subseteq I$. Let us prove that $I \subseteq (a)$. The ring

$$\mathbb{A}_{\mathrm{inf}}/(a)$$

is a valuation ring (cf. Theorem 5.4) and thus if $(a) \neq I$ there exists an $r_0 \geq 0$, such that

$$(a) + \mathfrak{a}_{r_0} \subseteq I.$$

Let $r'' > \sup\{r_0, r\}$ and $m \gg 0$ such that

$$I + \mathfrak{a}_{r''} = \mathfrak{p}_{y_m} + \mathfrak{a}_{r''}.$$

Then

$$\mathfrak{a}_{r_0} \subseteq I \subseteq \mathfrak{p}_{y_m} + \mathfrak{a}_{r''}.$$

Applying θ_{y_m} we can conclude the contradiction $r_0 \geq r''$. Thus, I is generated by a. Clearly,

$$\mathfrak{p}_{y_n} \to I,\ n \to \infty$$

as

$$\mathfrak{p}_{y_n} + \mathfrak{a}_{r'} = I + \mathfrak{a}_{r'}$$

for $r' > 0, n > 0$. □

Now, we can give a sketch of proof for Theorem 7.1. More details can be found in [10, Section 2.4.].

Theorem 7.1. 1) First, one reduces to the case that $f \in \mathbb{A}_{\mathrm{inf}}$ is primitive of some degree $d \geq 0$. For this, write

$$f = \sum_{n=0}^{\infty} [x_n]\pi^n = \lim_{d\to\infty} f_d$$

with

$$f_d := \sum_{n=0}^{\infty} [x_n]\pi^n,$$

up to multiplying by some Teichmüller lift, primitive of some degree $\leq d$. For some $D \geq 0$, λ appears in $\mathcal{N}ewt(f_d)$ for $d \geq D$ (with multiplicity bounded independent of d). Set

$$X_d := \{y \in |Y| \mid f_d(y) = 0, \nu(\pi(y)) = -\lambda\}.$$

Then one checks that one can construct a Cauchy sequence $\{y_d\}_{d\geq 0}, y_d \in X_d$. By Proposition 7.5 this Cauchy sequence converges to some $y \in |Y|$ with $\nu(\pi(y)) = -\lambda$ such that $f(y) = 0$.

2) Thus, we may assume that $f \in \mathbb{A}_{\mathrm{inf}}$ is primitive of some degree $d \geq 1$. In this case we may assume that $\lambda < 0$ is the maximal slope of f (by iterating the resulting factorization). We claim that there exists a sequence $y_n \in |Y|$ such that

- $\nu(f(y_n)) \geq -(d+n)\lambda$
- $d(y_n, y_{n+1}) \geq -\frac{d+n}{d}\lambda$
- $\nu(\pi(y_n)) = -\lambda$.

The existence of such a series finishes the proof as it will converge to the desired zero. We give the construction of y_1. Write

$$f = \sum_{n=0}^{\infty} [x_n]\pi^n.$$

Then $x_d \in \mathcal{O}_F^\times$. Let $z \in \mathcal{O}_F$ be a zero of the polynomial

$$\sum_{n=0}^{d} x_n T^n$$

with $\nu(z) = -\lambda$. Such a zero exists by Example 6.17. Set $y_1 = (\pi - [z])$. As λ was of maximal slope,

$$\nu(x_i) \geq \lambda(d-i)$$

for $0 \leq i \leq d$. This implies

$$x_i z^i = w_i z^d$$

with $w_i \in \mathcal{O}_F$. Then

$$\begin{aligned} f(y_1) = \theta_{y_1}(f) = \theta_{y_1}(\sum_{n=0}^{\infty}[x_n]\pi^n) \\ = \sum_{n=0}^{d} \theta_{y_1}([x_i z^i]) + \pi^{d+1} \sum_{i=d+1} \theta_{y_1}([x_i])\pi^{i-d-1} \end{aligned}$$

where we can write

$$\sum_{n=0}^{d} \theta_{y_1}([x_i z^i]) = \pi^d \sum_{i=0}^{d} \theta_{y_1}([w_i]).$$

But $\sum_{i=0}^{d} w_i = 0$, i.e.,

$$\sum_{i=0}^{d} [w_i] \in \pi \mathbb{A}_{\text{inf}}.$$

Thus

$$f(y_1) \in \pi^{d+1} \mathcal{O}_{C_{y_1}},$$

i.e.,

$$\nu(f(y_1)) \geq (d+1)\lambda.$$

This finishes the construction of y_1. Assume y_n is constructed and write

$$f = \sum_{i=0}^{\infty} [a_i] \xi^i y_n.$$

Let $z \in F$ be a zero of

$$\sum_{i=0}^{d} a_i T^i$$

of maximal valuation. Then $z \in \mathcal{O}_F$ (check $a_d \in \mathcal{O}_F^\times$ using the projection

$$\mathbb{A}_{\text{inf}} \to W_{\mathcal{O}_E}(k))$$

and one checks that $y_{n+1} = (\xi_{y_n} - [z])$ works. □

We know that the map

$$\mathfrak{m}_F \to |Y|, \ a \mapsto (\pi - [a])$$

is surjective (but not injective!). In the case $E = \mathbb{Q}_p$ one give the following better description of $|Y|$, cf. [10, Proposition 2.3.10.], which includes a discussion for general E.

Exercise 7.6. (1) Let $C/\mathbb{Q}_p$ be an algebraically closed, non-archimedean extension with an isomorphism $\mathcal{O}_C^\flat \cong \mathcal{O}_F$. Let $\varepsilon = (1, \zeta_p, \ldots) \in \mathcal{O}_C^\flat$ with $\zeta_p \in \mathcal{O}_C$ a non-trivial p-th root of unity. Then

$$u_\varepsilon := 1 + [\varepsilon^{1/p}] + \ldots + [\varepsilon^{p-1/p}] = \frac{[\varepsilon] - 1}{[\varepsilon^{1/p}] - 1}$$

is a distinguished element and $u_\varepsilon \in \ker(\theta)$.

(2) Let $a, b \in \mathbb{A}_{\text{inf}}$ be distinguished elements, such that $(a) \subseteq (b)$. Show $(a) = (b)$. In particular, $(u_\varepsilon) = \ker(\theta)$.

(3) For $\varepsilon \in 1 + \mathfrak{m}_F \setminus \{1\}$ define u_ε as in 1). Then u_ε is primitive of degree 1 and $(u_\varepsilon) = (u_{\varepsilon'})$ if and only if there exists an $a \in \mathbb{Z}_p^\times$ such that

$$\varepsilon' = \varepsilon^a := \sum_{n=0}^{\infty} \binom{a}{i} (\varepsilon - 1)^i.$$

Conclude, $|Y| \cong 1 + \mathfrak{m}_F \setminus \{1\}/\mathbb{Z}_p^\times$.
Hint: Use that for $C/\mathbb{Q}_p$-algebraically closed, non-archimedean the Tate module $T_p C^\times \subseteq \mathcal{O}_C^\flat$ is free of rank 1 over $\mathbb{Z}_p$.

8. Lecture of 27.11.2019: The ring B

We follow the notation introduced in Section 1. We recall that for $r \geq 0$ the map

$$\nu_r \colon \mathbb{A}_{\mathrm{inf}} \to \mathrm{R} \cup \{\infty\},\ f = \sum_{i=0}^{\infty} [x_i]\pi^i \mapsto \inf_{i\in\mathbb{Z}} \{\nu(\xi) + ir\}$$

is a valuation, cf. Definition 6.20 and that the Newton polygon of f is the convex, decreasing, piecewise linear function with Legendre transform

$$\mathcal{L}(\mathcal{N}ewt(f)) := \begin{cases} \nu_r(f),\ r \geq 0 \\ -\infty,\ r < 0 \end{cases},$$

cf. Definition 6.21. Clearly, $\nu_r, \mathcal{N}ewt(f)$ can be extend to the ring

$$B^b := \mathbb{A}_{\mathrm{inf}}[\frac{1}{\pi}, \frac{1}{[\varpi]}] = \{ \sum_{n \gg -\infty}^{\infty} [x_n]\pi^n \mid x_i = 0, j \ll 0,\ \inf_{i\in\mathbb{Z}} \{\nu(x_i)\} > -\infty \}.$$

Definition 8.1 (cf. [10, Section 1.6.]). Let $I \subseteq (0, \infty)$ be an interval. We set

$$B_I$$

as the completion of B^b for the family of valuations $(\nu_r)_{r\in I}$.

The intuition is that "$B_I = \mathcal{O}(|Y_I|)$", where

$$|Y_I| := \{y \in |Y| \mid d(y, 0) \in I\}.$$

Remark 8.2.

- Let R be a topological ring such that 0 has fundamental system $\mathcal{F}$ of neighborhoods which are subgroups. Then

  $$\widehat{R} := \varprojlim_{U\in\mathcal{F}} R/U$$

  is the completion of R. By continuity of multiplication $\widehat{R}$ is again a ring.
- In B^b consider

  $$\mathcal{F} := \{\bigcap_{i=1}^{n} \nu_{r_i}^{-1}([m, \infty)) \mid n, m \in \mathbb{N}, r_i \in I\}.$$

  According to the previous remark

  $$B_I = \varprojlim_{U\in\mathcal{F}} B^b/U.$$
- For any inclusion $I \subseteq I'$ there is a natural injection (cf. [10, Proposition 1.6.15.])

  $$B_{I'} \to B_I,$$

  which is the identity on B^b.
- We can often reduce to the case that I is compact using that

  $$B_I \cong \varprojlim_{J\subseteq I \text{ compact}} B_J.$$

The most important case is $I = (0, \infty)$,i.e., of "$\mathcal{O}(|Y|)$".

Definition 8.3. We set

$$B := B_{(0,\infty)}.$$

We denote by $\varphi \colon B \to B$ the extension by continuity of $\varphi \colon B^b \to B^b$.

More generally, for any interval $I \subseteq (0,\infty)$ the morphism $\varphi\colon B^b \to B^b$ induces an isomorphism

$$\varphi\colon B_I \cong B_{qI}.$$

With B at hand we can define the "schematic Fargues-Fontaine curve".

Definition 8.4 (cf. [10, Definition 6.5.1.])**.** The schematic Fargues-Fontaine curve is defined as

$$X := X_{E,F} := \mathrm{Proj}(\bigoplus_{d\geq 0} B^{\varphi=\pi^d})$$

Thus, in some sense,

$$X = \text{"}|Y|/\varphi^{\mathbb{Z}}\text{"}.$$

Remark 8.5. Recall that we shortly discussed the adic Fargues-Fontaine curve

$$\mathcal{X} = \mathcal{Y}/\varphi^{\mathbb{Z}}$$

in Remark 7.3. The isomorphism

$$\pi^{-1}\varphi\colon \mathcal{O}_{\mathcal{Y}} \cong \mathcal{O}_{\mathcal{Y}}$$

defines by descent a line bundle $\mathcal{O}(1)$ on $\mathcal{X}$. More or less by definition,

$$H^0(\mathcal{X}, \mathcal{O}(1)^{\otimes d}) \cong B^{(\pi^{-1}\varphi)^d=1} = B^{\varphi=\pi^d}.$$

Thus, in some sense, we are declaring $\mathcal{O}(1)$ to be "ample" in Definition 8.4.

Let us give a simple explanation why completing B^b to B is necessary.

Lemma 8.6. *We have*

$$(B^b)^{\varphi=\pi^d} = \begin{cases} E, & d=0 \\ 0, & d\neq 0 \end{cases}$$

Thus, in B^b the φ-eigenspaces for π^d, $d\in\mathbb{Z}$, are too small to define something reasonable. We will see in Remark 8.8 that $B^{\varphi=\pi^d}$ for $d>0$ is *huge*, more precisely infinite dimensional over $\mathbb{Q}_p$. The argument is typical for the use of Newton polygons and taken from [10, Proposition 4.1.2.] if $d<0$.

Proof. Assume $d=0$ and pick

$$f = \sum_{i\gg-\infty}^{\infty} [x_i]\pi^i \in (B^b)^{\varphi=1}.$$

Then for all $i\in\mathbb{Z}$ we get $\varphi(x_i) = x_i$, i.e., $x_i\in\mathbb{F}_q$. This implies $f\in E$. Now, assume that $d\neq 0$ and $0\neq f\in B^{\varphi=\pi^d}$. Then for any $x\in\mathbb{R}$

$$q\mathcal{N}ewt(f)(x) = \mathcal{N}ewt(\varphi(f))(x) = \mathcal{N}ewt(f)(x-d)$$

and thus for $n\in\mathbb{N}$

$$q^n\mathcal{N}ewt(f)(x) = \mathcal{N}ewt(\varphi^n(f))(x) = \mathcal{N}ewt(f)(x-nd).$$

Assume $d>0$. Then we know that

$$\mathcal{N}ewt(f)(x-nd) = \infty$$

for $n\gg 0$. But this implies $q^n\mathcal{N}ewt(f)(x)=\infty$ for all $x\in\mathbb{R}$ and thus $f=0$.

Assume now $d<0$ and let $x\in\mathbb{R}$. Then there exists $x_0\ll 0$ with

$$\mathcal{N}ewt(f)(x_0) \geq \mathcal{N}ewt(f)(x).$$

As $\mathcal{N}ewt(f)$ is non-decreasing, for $n \gg 0$ we get

$$\mathcal{N}ewt(f)(x) \geq \mathcal{N}ewt(f)(x_0 - nd) = q^n \mathcal{N}ewt(f)(x_0) \geq q^n \mathcal{N}ewt(f)(x).$$

But this implies that $\mathcal{N}ewt(f)(x) = \infty$ as desired. □

Under some assumption elements in B can be constructed via two both side infinite power series.

Lemma 8.7. *Let $(x_n)_{n\in\mathbb{Z}}$ be a sequence of elements in F such that*

$$\lim_{|n|\to\infty} \nu(x_n) + nr = \infty$$

for all $r \in (0, \infty)$. Then

$$\sum_{n\in\mathbb{Z}} [x_n]\pi^n$$

converges in B.

Proof. It is sufficient to show

$$\nu_r([x_n]\pi^n) \to 0,\ |n| \to \infty$$

for all $r \in (0, \infty)$. But $\nu_r([x_n]\pi^n) = \nu(x_n) + rn$ so this is precisely our assumption. □

Remark 8.8.

- Let $a \in \mathfrak{m}_F$. Then
$$f_a := \sum_{i\in\mathbb{Z}} [a^{q^{-i}}]\pi^i$$
converges in B by Lemma 8.7[38]. Moreover,
$$\varphi(f_a) = \sum_{i\in\mathbb{Z}} [a^{q^{-(i-1)}}]\pi^i = \pi f_a.$$
Thus, we have constructed a map $\mathfrak{m}_F \to B^{\varphi=\pi}$. We will in fact prove that $\mathfrak{m}_F \cong B^{\varphi=\pi}$.[39]
- In general, it is unknown whether elements in B can be written in the form
$$\sum_{n\in\mathbb{Z}} [x_n]\pi^n,\ x_n \in F,$$
and, if they can, if the $x_n,\ n \in \mathbb{Z}$, are unique.

To analyze elements in B (or in B_I) we need to extend the theory of Newton polygons, cf. [10, Section 1.6.3.]. First note that for $r \in I$ the valuation $\nu_r \colon B^b \to \mathbb{R} \cup \{\infty\}$ extends to $\nu_r \colon B_I \to \mathbb{R} \cup \{\infty\}$ by continuity.

Definition 8.9. Assume $I \subseteq (0, \infty)$ is an open interval and $f \in B_I$. We define

$$\mathcal{N}ewt_I^0(f)$$

as the decreasing convex function whose Legendre transform is

$$r \mapsto \begin{cases} \nu_r(f),\ r \in I \\ -\infty,\ r \notin I \end{cases}$$

[38] $\nu(a^{q^{-i}}) + ir = q^{-i}\nu(a) + ir \to \infty,\ |i| \to \infty$ as $\nu(a) > 0$.

[39] And the given map $\mathfrak{m}_F \to B^{\varphi=\pi}$ is bijective. Strictly speaking, we will not prove this. A proof can be found in [10, Proposition 4.2.1.].

and

$$\mathcal{N}ewt_I(f) \subseteq \mathbb{R}^2$$

as the subset of the graph of $\mathcal{N}ewt_I(f)$ with slopes contained in $-I$.

Remark 8.10. If $K \subseteq I$ is compact, $f_n \in B^b$ converging to f, $f \neq 0$, then there exists and N such that for all $n \geq N$, $\nu_r(f_n) = \nu_r(f)$ for all $r \in K$. This implies that $\mathcal{L}(\mathcal{N}ewt_I^0(f))$ is a concave function with integral slopes and thus that $\mathcal{N}ewt_I^0(f)$ is a decreasing convex polygon with integral break points.
If $f \in B$, λ_i the slope of $\mathcal{N}ewt_{(0,\infty)}(f)$ on $[i, i+1]$, then

(1) $\lambda_i < 0$
(2) $\lim\limits_{i\to\infty} \lambda_i = 0$
(3) $\lim\limits_{i\to-\infty} \lambda_i = \infty$.

In the case that I is compact, we have to modify the definition of the Newton polygon.

Definition 8.11. Let $I = [a, b] \subseteq (0, \infty)$ be a compact interval, $f \in B_I, f \neq 0$. Set $\mathcal{N}ewt_I^0(f)$ as the decreasing convex function whose Legendre transform is

$$r \mapsto \begin{cases} \nu_r(f), & r \in I \\ \nu_a(f) + (r-a)\partial_g \nu_a(f), & r < a \\ \nu_b(f) + (r-a)\partial_d \nu_b(f), & r \geq b \end{cases}$$

and $\mathcal{N}ewt_I(f) \subseteq \mathbb{R}^2$ as the subset of the graph of $\mathcal{N}ewt_I^0(f)$ with slopes in $-I$.

Remark 8.12.

- If $f_n \to f,\ n \to \infty,\ f_n \in B^b$, then

$$\partial_g \nu_r(f) = \lim_{n\to\infty} \partial_g \nu_r(f_n),$$

where ∂_g denotes the left derivative of the function $r \mapsto \nu_r(f_n)$. This definition is independent of the choice of converging sequence $f_n \in B^b$. Similarly,$\partial_d \nu_r(f)$ is defined using the right derivative, cf. [10, Lemme 1.6.10].
- For $f \in B^b$, λ a slope of $\mathcal{N}ewt(f)$, the number

$$\partial_g \nu_{-\lambda}(f) - \partial_d \nu_{-\lambda}(f)$$

is the multiplicity of λ in $\mathcal{N}ewt(f)$. This explains why in the compact case, i.e., in Definition 8.11 the definition is different than in the open case, i.e., in Definition 8.9. Only Definition 8.11 yields the correct multiplicities for the slopes $-a, -b$.
- It is clear that $\mathcal{N}ewt_I(f)$ is a decreasing convex polygon and all slopes have finite multiplicities.
- Similarly to Lemma 6.22, one can see

$$\mathcal{N}ewt_I^0(f \cdot g) = \mathcal{N}ewt_I^0(f) * \mathcal{N}ewt_I^0(g)$$

in both cases, I open or compact.
- For any interval $I \subseteq (0, \infty)$, e.g., half-open, and $f \in B_I$ one can set

$$\mathcal{N}ewt_I(f) := \bigcup_{J \subseteq I \text{ compact}} \mathcal{N}ewt_J(f)$$

(which agrees with Definition 8.9 in the open case, cf. [10, Section 1.6.3.2.]).

9. Lecture of 11.12.2019: The graded algebra P

We continue to use the notation from Section 1. Let $I \subseteq (0,\infty)$ be an interval. Last time we introduced the ring

$$B_I$$

as the completion of $B^b := \mathbb{A}_{\mathrm{inf}}[\frac{1}{\pi}, \frac{1}{[\varpi]}]$ for the family $(\nu_v)_{r\in I}$ of valuations, which, heuristically, can be seen as the "ring of functions on $|Y_I|$" (cf. 8.1).

Definition 9.1 (cf. [10, Definition 6.1.1.]). We set

$$P := \bigoplus_{d\geq 0} P^d = \bigoplus_{d\geq} B^{\varphi=\pi^d}.$$

Recall that the schematic Fargues-Fontaine curve is defined as

$$X = \mathrm{Proj}(P),$$

cf. Definition 8.4.

A first aim of this lecture is to analyze the multiplicative structure of P.

Theorem 9.2 (cf. [10, Théorème 6.2.1.]). *The algebra P is graded factorial with irreducible elements of degree 1, i.e., the multiplicative monoid*

$$\bigcup_{d\geq 0} (P_d \setminus \{0\})/E^\times$$

is free on $P_1 \setminus \{0\}$. In particular, if $d \geq 1$ and $x \in P_d = B^{\varphi=\pi}$, then there exist $t_1, \ldots, t_d \in P_1 = B^{\varphi=\pi}$ such that $x = t_1 \cdots t_d$.

This theorem does not imply that $P \cong \mathrm{Sym}^\bullet_{\mathbb{Q}_p}(P_1)$ is a symmetric algebra.[40] To prove Theorem 9.2 we will need the following theorem.

Theorem 9.3 (cf. [10, Théorème 2.5.1.]). *Assume $I \subseteq (0,\infty)$ is compact. Then the ring B_I is a principal ideal domain.*

As will be clear form the proof, the set of maximal ideals in B_I is in bijection with $|Y_I|$.

Lemma 9.4. *Let A be an integral domain. Then A is a principal ideal domain if and only if A is factorial and each (non-invertible) irreducible element generates a maximal ideal.*

Proof. The "only if" part is clear. For the converse, let $I \subseteq A$ be a non-zero ideal and let $a \in I$ be non-zero. As A is factorial we can write

$$a = a_1^{i_1} \cdots a_n^{i_n}$$

with the a_i pairwise prime irreducible elements. Then

$$A/a \cong \prod_{j=1}^n A/a_j^{i_j}$$

and because the ideals (a_j) are maximal we see that I/a is generated by the residue class of some divisor of a. This implies that as desired I is principal. □

[40] If $P \cong \mathrm{Sym}^\bullet(P_1)$, then $\mathrm{Proj}(P)$ would have points with residue field $\mathbb{Q}_p$, which will turn out not to be true.

Lemma 9.5. *Let $y \in |Y_I|$. Then the morphism $\theta_y \colon B^b \to C_y$ extends to*

$$\theta_y \colon B_I \to C_y.$$

Moreover, $\ker(\theta_y \colon B_I \to C_y) = \xi_y B_I$ *is principal, generated by* ξ_y.

Proof. Set $r = d(y, 0)$. It suffices to see that θ_y is continuous for the topology on B^b induced by ν_r. Let

$$x = \sum_{i \gg -\infty}^{\infty} [x_i]\pi^i \in B^b.$$

Then

$$\theta_y(x) = \sum_{i \gg -\infty}^{\infty} \theta_y([x_i])\pi^i \in B^b$$

and thus

$$\nu_y(\theta_y(x)) \geq \inf_{i\in\mathbb{Z}}\{\nu(x_i) + i\nu_y(\pi)\} = \inf_{i\in\mathbb{Z}}\{\nu(x_i) + ir\} = \nu_r(x)$$

as $\nu_y(\pi) = r$. This proves that θ_y extends as desired to B_I.

It follows from the description of the completion as an inverse limit, that

$$\ker(\theta_y \colon B_I \to C_y)$$

is the closure of $\xi_y B^b$ in B_I. Let $f \in B_I$ be in this closure and write

$$f = \varinjlim_{n\to\infty} f_n$$

with $f_n = g_n\xi_y \in \xi_y B^b$. Let $r \in I$. Then

$$\nu_r(g_m - g_n) = \nu_r(f_m - f_n) - \nu_r(\xi_y)$$

with $\nu_r(\xi_y) \neq \infty$. Thus, the sequence $(g_n)_n$ is again Cauchy which finishes the proof. □

The following lemma is [10, Proposition 1.6.25.].

Lemma 9.6. *Let $f \in B_I$ be non-zero with $\mathcal{N}ewt_I(f) = \emptyset$, then $f \in B_I^\times$.*

Thus, if $f \in B_I$ with $\mathcal{N}ewt_I(f) = \emptyset$, then $f = 0$ or f is invertible.

Proof. As $B_I = \varprojlim_{J\subseteq I \text{ compact}} B_J$ and

$$\mathcal{N}ewt_I(f) = \bigcup_{J\subseteq I \text{ compact}} \mathcal{N}ewt_J(f)$$

it suffices to treat the case that $I = [a, b]$ is compact. If $f_n \to f,\ n \to \infty$ with $f_n \in B^b$, then $\mathcal{N}ewt_I(f_n) = \emptyset$ for $n \gg 0$, which reduces us the the case that $f \in B^b$. Here one implicitly used that $f \neq 0$ to assure that the sequences f_n^{-1} for $n \gg 0$ is Cauchy. Write

$$f = \sum_{n\gg-\infty} [x_n]\pi^n = \sum_{n>N} [x_n]\pi^n + \sum_{n\leq N} [x_n]\pi^n$$

such that each slope of $\mathcal{N}ewt(f)$ on $(-\infty, N)$ is strictly less than $-b$ and each slope of $\mathcal{N}ewt(f)$ on (N, ∞) is strictly greater than $-a$ (this is possible as $\mathcal{N}ewt_I(f) = \emptyset$). Set

$$z := \sum_{n>N} [x_n]\pi^n,\ y := \sum_{n\leq N} [x_n]\pi^n.$$

Let λ_1 be the slope of $\mathcal{N}ewt(f)$ on $[N-1,N]$ (we allow $\lambda_1=\infty$) and λ_2 the slope of $\mathcal{N}ewt(f)$ on $[N,N+1]$. Then $-\lambda_1>b$ and $-\lambda_2<a$. Moreover,

$$\nu(x_n)\geq(n-N)\lambda_1+\nu(x_N)$$

for $n\leq N$ and

$$\nu(x_n)\geq(n-N)\lambda_2+\nu(x_N)$$

for $n\geq N$. Write

$$y=[x_N]\pi^N(1+\sum_{-\infty\ll n<N}[x_nx_N^{-1}]\pi^{n-N}).$$

We claim that

$$\tilde{y}:=\sum_{-\infty\ll n\leq N}[x_nx_N^{-1}]\pi^{n-N}$$

is topologically nilpotent in B_I. Let $r\in I$. Then

$\nu_r(\tilde{y})=\inf_{n<N}\{\nu(x_n)-\nu(x_N)+r(n-N)\}\geq\inf_{n<N}\{(\lambda_1+r)(n-N\}=-\lambda_1-r>0$

as $r\leq b<-\lambda_1$. Thus,

$$y=[x_N]\pi^N(1+\tilde{y})\in B_I^{\times}.$$

To finish the proof it suffices to show that $y^{-1}z$ is topologically nilpotent in B_I as

$$f=y(1+y^{-1}z).$$

For this it is sufficient (because $\nu_r(y)=\nu_r([x_N]\pi^N)$ for $r\in I$) to show that

$$\nu_r(z)>\nu_r([x_N]\pi^N)$$

for $r\in I$. But

$$\nu_r(z)=\inf_{n>N}\{\nu(x_n)+rn\}\geq\inf_{n>N}\{(n-N)\lambda_2+\nu(x_N)+rn\}$$
$$=\inf_{n>N}\{n(\lambda_2+r)-N\lambda_2+\nu(x_N)\}=\lambda_2+r+Nr+\nu(x_N)>\nu_r([x_N]\pi^N)$$

because $\lambda_2+r>0$. □

Now we can proof the following theorem.

Theorem 9.7 (cf. [10, Théorème 2.4.10.]). *Let $I\subseteq(0,\infty)$ be an intervall and let $f\in B_I$. If λ is a slope of $\mathcal{N}ewt(f)$, then there exists $a\in\mathfrak{m}_F$ with $\nu(a)=-\lambda$ and $f=(\pi-[a])g$ for some $g\in B_I$.*

Proof. By Lemma 9.5 it suffices to prove that there exists an element $y\in|Y|$ with $d(y,0)=-\lambda$ such that

$$f(y):=\theta_y(f)$$

is zero. This is again an approximation argument using Proposition 7.5. □

Now, the proof of Theorem 9.3 can be finished.

Proof. (of 9.3) We apply Lemma 9.4 to B_I. Let $f\in B_I$. As I is compact, $\mathcal{N}ewt_I(f)$ has only finitely many slopes. If $\mathcal{N}ewt_I(f)=\emptyset$, then $f=0$ or $f\in B_I^{\times}$ by Lemma 9.6. If λ is a slope of $\mathcal{N}ewt_I(f)$ we can factor $f=\xi_yg$ with $y\in|Y_{-\lambda}|$ and $g\in B_I$. The slopes of $\mathcal{N}ewt_I(g)$ are the slopes of $\mathcal{N}ewt_I(f)$ with multiplicity of λ one less. Iterating implies that we can write

$$f=u\xi_{y_1}\cdots\xi_{y_n}$$

for some unit $u\in B_I^{\times}$ and some $y_1,\ldots,y_n\in|Y_I|$. This finishes the proof as all the ξ_y for $y\in|Y_I|$ generate maximal ideals on B_I by Lemma 9.5. □

Lemma 9.8. *Let $y \in |Y_I|$, then the ξ_y-adic completion of B_I is $B^+_{\mathrm{dR},y}$.*

Proof. As $B^+_{\mathrm{dR},y}$ is the ξ_y-adic completion of B^b and $(\xi_y)B_I$ the kernel of

$$\theta_y \colon B_I \to C_y,$$

we obtain a canonical morphism

$$B^+_{\mathrm{dR},y} \to (B_I)^\wedge_{\xi_y}.$$

As both rings are complete discrete valuation rings, $\xi_y(B_I)^\wedge_{\xi_y}$ is the maximal ideal of $(B_I)^\vee_{\xi_y}$ and this morphism is an isomorphism on residue fields, the claim follows. □

This allows us to associate to any B_I a divisor on $|Y_I|$ as done in [10, Section 2.7.1.].

Definition 9.9. Let $I \subseteq (0, \infty)$ be any interval. We define $\mathrm{Div}^+(|Y_I|)$ as the monoid of formal sums

$$\sum_{y \in |Y_I|} n_y y,\ n_y \in \mathbb{N}$$

such that for each compact $J \subseteq I$ the set $\{y \in |Y_J| \mid n_y \neq 0\}$ is finite.

Thus,

$$\mathrm{Div}^+(|Y_I|) = \mathbb{N}[|Y_I|]$$

if I is compact and

$$\mathrm{Div}^+(|Y_I|) = \varprojlim_{J \subseteq I \text{ compact}} \mathrm{Div}^+(|Y_J|)$$

in general.

Definition 9.10. Let $I \subseteq (0, \infty)$ be any interval and $f \in B_I \setminus \{0\}$. Then

$$\mathrm{div}(f) := \sum_{y \in |Y_I|} \mathrm{ord}_y(f) y$$

where $\mathrm{ord}_y(f)$ is the valuation of the image of f in the valuation ring $B^+_{\mathrm{dR},y}$

Note that this is well-defined by 9.3. The monoid $\mathrm{Div}^+(|Y_I|)$ is naturally partially ordered by

$$\sum_{y \in |Y_I|} n_y y \geq \sum_{y \in |Y_I|} m_y y$$

if $n_y \geq m_y$ for all $y \in |Y_I|$.

Proposition 9.11. *The map*

$$\mathrm{div} \colon B_I \setminus \{0\}/B_I^\times \to \mathrm{Div}^+(|Y_I|)$$

is injective, and bijective if I is compact. Moreover,

$$\mathrm{div}(f) \geq \mathrm{div}(g)$$

if and only if $f \in gB_I$.

The morphism div need not be surjective if I is not compact, the problem are divisors with infinite support for $r \to 0$.

Proof. The case that I is compact is clear by 9.3. The general case follows as $B_I = \varprojlim_{J \subseteq I \text{ compact}} B_J$. □

Recall that

$$P := \bigoplus_{d\geq 0} P_d$$

with $P_d := B^{\varphi=\pi^d}$, cf. 9.1.

Lemma 9.12. *We have*

$$B^{\varphi=\pi^d} = \begin{cases} B^{\varphi=\pi^d}, & d>0 \\ E, & d=0 \\ 0, & d<0 \end{cases}$$

As presented in Remark 8.8 the space $B^{\varphi=\pi}$ is big.

Proof. Using that

$$\mathbb{A}_{\mathrm{inf}} = \{f \in B \mid \mathcal{N}ewt_{(0,\infty)}(f) \subseteq \mathbb{R}^2_{\geq 0}\}$$

the arguments for $d \leq 0$ are exactly the same as in the case of B^b, cf. Lemma 8.6. □

Note that φ acts on

$$\mathrm{Div}^+(|Y|)$$

By $\varphi^*(y) := (\varphi^{-1}(\xi_y))$.

Definition 9.13. Set

$$\mathrm{Div}^+(|Y|/\varphi^{\mathbb{Z}}) := \mathrm{Div}^+(|Y|)^{\varphi^{\mathbb{Z}}}.$$

As φ induces a bijection

$$\varphi\colon |Y_I| \cong |Y_{\frac{1}{q}I}|$$

for any interval $I \subseteq (0,\infty)$, we see that restriction of divisors induces an isomorphism

$$\mathrm{Div}^+(|Y|/\varphi^{\mathbb{Z}}) \cong \mathrm{Div}^+(|Y_I|)$$

for any interval of the form $I = [a, qa)$.

Theorem 9.2 is implied by the following more precise statement.

Theorem 9.14 (cf. [10, Théorème 6.2.7.])**.** *The morphism*

$$\mathrm{div}\colon \bigcup_{d\geq 0} (P_d \setminus \{0\})/E^\times \to \mathrm{Div}^+(|Y|/\varphi^{\mathbb{Z}})$$

is an isomorphism of monoids.

For $x \in P_d$, $d \geq 0$

$$\varphi^*(\mathrm{div}(x)) = \mathrm{div}(\varphi(x)) = \mathrm{div}(\pi^d x) = \mathrm{div}(x),$$

thus the morphism div in Definition 9.13 is well-defined.

Proof. Let $x \in P_d, y \in P_{d'}$ such that $\mathrm{div}(x) = \mathrm{div}(y)$. Without loss of generality we may assume

$$d' \geq d.$$

Then by Proposition 9.11

$$x = uy$$

for some unit $u \in B^\times$. Moreover,

$$\varphi(u) = \pi^{d-d'},$$

i.e.,

$$u \in B^{\varphi=\pi^{d-d'}} = \begin{cases} 0, & \text{if } d \neq d' \\ E, & \text{if } d = d' \end{cases}$$

(cf. Lemma 9.12). Thus, $d = d'$ and x, y are equivalent modulo the action of $E^\times$. Thus we are left with proving surjectivity of div. Let $y \in |Y|$ and write $\xi_y = \pi - [a]$ for some $a \in \mathfrak{m}_F$. It suffices to show that the divisor

$$\sum_{n\in\mathbb{Z}} \varphi^{-n}(y)$$

lies in the image of $\bigcup_{d\geq 0} (P_d \setminus \{0\})/E^\times$. Set

$$x := \prod_{n\geq 0} (1 - \frac{a^{q^n}}{\pi}) = \prod_{n\geq 0} \frac{\varphi^n(\xi_y)}{\pi}.$$

Then x converges in B[41] and satisfies

$$\mathrm{div}(x) = \sum_{n\geq 0} \varphi^{-n}(y).$$

By Lemma 9.15 there exists a non-zero element

$$z \in B$$

such that $\varphi(z) = \tilde{\xi} z$. Then $t := xy$ satisfies

$$\varphi(t) = \varphi(x)\varphi(z) = \pi\xi^{-1}x\xi z = \pi t$$

and

$$\mathrm{div}(t) = \sum_{n\geq 0} \varphi^*(y)$$

as desired. □

The element z in the proof is a substitute for the non-existent element

$$\text{"}\prod_{n<0} \frac{\varphi^n(\xi)}{\pi}\text{"}.$$

Lemma 9.15. *Let $b \in B^b \cap W_{\mathcal{O}_E}(F)^\times$ be any element. Then the E-vector space*

$$B^{b\varphi=b} := \{x \in B^b \mid \varphi(x) = bx\}$$

is one-dimensional.

The assumption that $b \in W_{\mathcal{O}_E}(F)^\times$ is essential. For $b = \pi$ the space

$$(B^b)^{\varphi=\pi}$$

is 0.

[41]An infinite product $\prod_{n\in\mathbb{N}} x_n$ of elements $x_n \in B$ converges if and only if $x_n \to 1,\ n \to \infty$. Now use that $a^{q^n} \to 0$ for $n \to \infty$.

Proof. As

$$W_{\mathcal{O}_E}(F)[1/\pi]^{\varphi=1} = E$$

one can conclude that $(B^b)^{\varphi=b}$ is at most one-dimensional. By assumption we can write

$$b = \sum_{n\geq 0} [a_n]\pi^n$$

with $a_n \in F$ such that $a_0 \neq 0$ and $\inf\{\nu(a_n) \mid n \geq 0\}$ bounded below. If $a \in F$, then multiplication by $[a]$ induces an isomorphism

$$B^{\varphi=b} \cong B^{\varphi=a^{q-1}b}.$$

Thus, after multiplying b by some Teichmüller lift we can assume that $b \in \mathbb{A}_{\mathrm{inf}} \setminus \pi\mathbb{A}_{\mathrm{inf}}$. We will define by induction a converging sequence $x_n \in \mathbb{A}_{\mathrm{inf}}$, $x_1 \notin \pi\mathbb{A}_{\mathrm{inf}}$, such that

$$\varphi(x_n) \equiv bx_n \bmod \pi^n$$

and $x_n \equiv x_{n+1} \bmod \pi^n$ for any $n \geq 1$. For $n = 1$ it suffices take the Teichmüller lift of a non-zero solution, necessarily in $\mathcal{O}_F$, of the equation

$$\varphi(X) \equiv X^q \equiv bX \bmod \pi$$

Note that a non-zero solution exists as F is algebraically closed. Assume x_n is constructed and write

$$\varphi(x_n) = bx_n + \pi^n[z] \bmod \pi^{n+1}$$

for some $z \in \mathcal{O}_F$. We need to find $u \in \mathcal{O}_F$ solving the equation

$$\varphi(x_n + \pi^n[u]) \equiv b(x_n + \pi^n[u]) \bmod \pi^{n+1}.$$

Expanding yields the equivalent equation

$$a_0u - z - u^q \equiv 0 \bmod \pi.$$

As F is algebraically closed we can solve this equation in $\mathcal{O}_F$. This finishes the proof. □

Assume that $E = \mathbb{Q}_p$. In this case one can choose

$$\xi_y = 1 + [\varepsilon^{1/p}] + \ldots + [\varepsilon^{p-1/p}] = \frac{[\varepsilon]-1}{\varphi^{-1}([\varepsilon]-1)}$$

and see that

$$\mu := [\varepsilon] - 1$$

satisfies

$$\varphi(z) = \varphi(\xi_y)z.$$

10. Lecture of 18.12.2019: The curve

The aim of this lecture is to prove that the schematic Fargues-Fontaine curve

$$X = \mathrm{Proj}(P)$$

with

$$P = \bigoplus_{d\geq 0} B^{\varphi=\pi^d}$$

is actually a "curve" or more precisely a Dedekind scheme. In some aspects the "curve" behaves like $\mathbb{P}^1_E$ as we will explain.

We start by proving the fundamental exact sequence of p-adic Hodge theory. Recall that in the proof of Theorem 9.14 we constructed for every $y \in |Y|$ an element $t \in B^{\varphi=\pi}$, unique up to multiplication by an element in $E^\times$, such that

$$\mathrm{div}(t) = \sum_{n\in\mathbb{Z}} (\varphi^n)^*(y) \in \mathrm{Div}^+(|Y|/\varphi^{\mathbb{Z}}).$$

Let us denote $\Pi(\xi_y) := t$.

Theorem 10.1 (cf. [10, Théorème 6.4.1.])**.** *Let $y \in |Y|$ and set $t := \Pi(\xi_y) \in B^{\varphi=\pi}$. Then for $d \geq 0$ the natural sequence*

$$0 \to Et^d \to B^{\varphi=\pi^d} \to B^+_{\mathrm{dR},y}/\xi_y^d B^+_{\mathrm{dR},y} \to 0$$

is exact.

Proof. If $x \in B^{\varphi=\pi^d}$ maps to 0 in $B^+_{\mathrm{dR},y}/\xi_y^d B^+_{\mathrm{dR},y}$, then

$$\mathrm{div}(x) \geq d \cdot y,$$

which implies that

$$\mathrm{div}(x) \geq d \cdot \mathrm{div}(t)$$

because $\mathrm{div}(x)$ is φ-invariant. By Theorem 9.14 this implies that

$$x \in Et^d.$$

By induction, the surjectivity of $B^{\varphi=\pi^d} \to B^+_{\mathrm{dR},y}/\xi_y^d B^+_{\mathrm{dR},y}$ can be reduced to the case that $d = 1$. Then it suffices to see that the map

$$\theta_y \colon B^{\varphi=\pi} \to C_y$$

is surjective. For simplicity, we only deal with the case $E = \mathbb{Q}_p$. But then we can apply the more precise Lemma 10.2. □

Lemma 10.2. *Assume $E = \mathbb{Q}_p$ and $y \in |Y|$. Then*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \varepsilon^{\mathbb{Q}_p} & \longrightarrow & 1+\mathfrak{m}_F & \longrightarrow & C_y & \longrightarrow & 1 \\
 & & \downarrow \simeq & & \simeq \downarrow \log([-]) & & \| & & \\
0 & \longrightarrow & \mathbb{Q}_p t & \longrightarrow & B^{\varphi=p} & \xrightarrow{\theta_y} & C_y & \longrightarrow & 1
\end{array}$$

is a commutative diagram with exact rows, where

$$\varepsilon := [1, \zeta_p, \zeta_{p^2}, \ldots] \in \mathcal{O}^\flat_{C_y} \cong \mathcal{O}_F$$

is a compatible system of primitive p^n-roots of unity and $t = \log([\varepsilon]) = \Pi(\xi_y)$.

If $E/\mathbb{Q}_p$ is arbitrary, one can use Lubin-Tate theory to construct an isomorphism

$$\mathfrak{m}_F \cong B^{\varphi=\pi},$$

cf. [10, Proposition 4.4.5.].

Proof. The

$$\log([-]) \colon 1+\mathfrak{m}_F \to B,\ x \mapsto \log([x]) = \sum_{n=1}^{\infty} (-1)^{n-1}([x]-1)^n/n$$

map is well-defined by general properties of the logarithm. The image lies in $B^{\varphi=p}$ as

$$\varphi(\log([x])) = \log([x^p]) = p\log([x]).$$

The composition

$$1+\mathfrak{m}_F \xrightarrow{\log([-])} B^{\varphi=p} \xrightarrow{\theta_y} C_y$$

factors as

$$1+\mathfrak{m}_F \xrightarrow{(-)^\sharp} 1+\mathfrak{m}_{C_y} \xrightarrow{\log} C_y.$$

Both of these morphisms are surjective. The first because C_y is algebraically closed and the second because the image of $\log\colon 1+\mathfrak{m}_{C_y} \to C_y$ is p-divisible (as C_y is algebraically closed) and open (because the exponential is a local inverse to log). From the proof of injectivity in Theorem 10.1 we have now established that the bottom row is exact. Thus it suffices to see exactness of the first row. But the first row is the inverse limit of the exact sequence

$$1 \to \mu_{p^\infty}(C_y) \to 1+\mathfrak{m}_{C_y} \xrightarrow{\log} C_y \to 1$$

along the transition maps which are rising to the p-th power resp. multiply by p. This finishes the proof. □

We can draw the following consequence.

Corollary 10.3. *In the notation of Lemma 10.2 there is a canonical isomorphism*

$$P/tP \cong S := \{f \in C_y[T] \mid f(0) \in E\}.$$

of graded E-algebras. In particular, $\mathrm{Proj}(P/tP) = (0)$.

Proof. Let $\theta_y\colon B \to C_y$ be the canonical quotient associated to y. Then we claim that the morphism

$$\alpha\colon P/tP \to S,\ \sum_{d\geq 0} x_d \mapsto \sum_{d\geq 0} \theta_y(x_d)T^d$$

is an isomorphism of graded E-algebras. It is trivially an isomorphism in degree 0 and in degrees ≥ 1 by Theorem 10.1. Indeed, surjectivity is clear as each element in C_y has arbitrary roots. To prove injectivity let $x \in P_d$, $d \geq 1$, with $\theta_y(t) = 0$. Then $x \equiv t \cdot t' \mod \xi_y^d B^+_{\mathrm{dR},y}$ for some $t' \in B^{\varphi=\pi^{d-1}}$ by Theorem 10.1. This implies, again by Theorem 10.1, that $x - t\cdot t' \in E \cdot t^d$ as desired. Let us prove that $\mathrm{Proj}(P/tP) = \{(0)\}$. For this pick a graded prime ideal $\mathfrak{p} \subseteq S$. If $cT^d \in \mathfrak{p}$ for some $d \geq 1$ and $c \in C_y^\times$ multiplying with $c^{-1}T$ yields $T^{d+1} \in \mathfrak{p}$, and then $\mathfrak{p} = (T)$, i.e., $\mathfrak{p}$ does not appear in $\mathrm{Proj}(S)$. □

As $X = \mathrm{Proj}(P)$ is defined as the Proj of an E-algebra which is generated by its elements in degree 1, X comes equipped with canonical line bundles

$$\mathcal{O}_X(n),$$

which are associated to the graded P-module $P[n]$ where $P[n]_d := P_{d+n}$, $d \in \mathbb{Z}$.

Lemma 10.4. *For each $n \in \mathbb{Z}$*

$$H^0(X, \mathcal{O}_X(n)) \cong B^{\varphi=\pi^n}.$$

Proof. By construction, there is a natural map

$$B^{\varphi=\pi^n} \to H^0(X, \mathcal{O}_X(n)),$$

which is injective as P is an integral domain. Using that P is graded factorial (cf. Theorem 9.14) one obtains that it is surjective. □

Definition 10.5. Let $t \in P_1 = B^{\varphi=\pi} = H^0(X, \mathcal{O}_X(1))$ non-zero. Then let

$$\infty_t \in X$$

be the unique closed point in the vanishing locus of t.

Now we can prove one main result of this course.

Theorem 10.6 (Fargues-Fontaine, cf. [10, Théorème 6.5.2.(7)] resp. [10, Théorème 5.2.7.]). *Let $t \in P_1 = B^{\varphi=\pi}$ be non-zero. Then*

$$B_t := P[1/t]_0 = B[1/t]^{\varphi=1}$$

is a principal ideal domain, and

$$\mathrm{Proj}(P) \cong \mathrm{Spec}(B_t) \cup \{\infty_t\}.$$

In particular, X is noetherian and regular of Krull dimension 1.

Classically, in the case $E = \mathbb{Q}_p$ the ring B_t for $t = \log[\varepsilon]$ is called B_e.

Proof. We apply Lemma 9.4. If $x \in B_t$ is non-zero, then for some $d \geq 0$

$$x = \frac{t'}{t^d}$$

with $t' \in B^{\varphi=\pi^d}$. Applying Theorem 9.14 we can factor t' into elements $t_1, \ldots, t_d \in B^{\varphi=\pi}$, i.e.,

$$x = \frac{t_1}{t} \cdots \frac{t_d}{t}.$$

By Theorem 10.1 and Corollary 10.3 each t_i/t is either a unit or generates a maximal ideal in B_t. This finishes the proof that B_t is a principal ideal domain. To see that X is noetherian and regular of Krull dimension 1, pick two non-colinear $t, t' \in B^{\varphi=\pi}$. Then

$$X = \mathrm{Spec}(B_t) \cup \mathrm{Spec}(B_{t'}),$$

which finishes the proof. □

Let $|X|$ denote the set of closed points of X.[42]

Lemma 10.7. *There are canonical bijections*

$$|X| = |Y|/\varphi^{\mathbb{Z}} = (P_1 \setminus \{0\})/E^\times.$$

Moreover, for $y \in |Y|$ with image $x \in |X|$ there is a canonical isomorphism

$$\mathcal{O}^\wedge_{X,x} \cong B^+_{\mathrm{dR},y}.$$

The bijection

$$|X| = (P_1 \setminus \{0\})/E^\times$$

exists similarly for $X = \mathbb{P}^1_E$.

[42]This notation can potentially be misleading as it commonly also denotes the underlying topological space of X. However, we already chose the similar notation $|Y|$.

Proof. The bijection
$$|X| = (P_1 \setminus \{0\})/E^\times$$
is clear by Theorem 10.6 and Theorem 9.14. The bijection
$$|Y|/\varphi^{\mathbb{Z}} = (P_1 \setminus \{0\})/E^\times$$
follows directly from Theorem 9.14. Let $y \in |Y|$ with image $x \in |X|$. Then
$$\{x\} = V^+(t)$$
with $t = \Pi(\xi_y)$. Pick some $t' \in B^{\varphi=\pi}$ such that $t' \notin Et$. Then the canonical morphism $B \to B^+_{\mathrm{dR},y} \cong B^\wedge_{\xi_y}$ (cf. Lemma 9.8) yields a homomorphism
$$B[1/t']^{\varphi=1} \to B^\wedge_{\xi_y}.$$
Moreover, t/t' is send to a uniformizer in $B^+_{\mathrm{dR},y}$ as t generates $\ker(\theta_y \colon B \to C_y)$. As the residue fields of $(B[1/t']^{\varphi=1})^\wedge_{t/t'}$ and $B^+_{\mathrm{dR},y}$ agree and both are discrete valuation rings, one can conclude that they are isomorphic. □

Thus, the following definition is sensible.

Definition 10.8. For $x \in |X|$ we define the complete discrete valuation ring
$$B^+_{\mathrm{dR},x} := \mathcal{O}^\wedge_{X,x}$$
and its field of fractions $B_{\mathrm{dR},x}$.

If $D = \sum\limits_{x \in X_0} n_x x \in \mathrm{Div}(X)$ is a divisor, then
$$\deg(D) := \sum_{x \in |X|} n_x \in \mathbb{Z}.$$
Let $k(X)$ be the function field of the Fargues-Fontaine curve. Then we can define
$$\mathrm{div}(f) := \sum_{x \in |X|} \mathrm{ord}_x(f) x,$$
where $\mathrm{ord}_x \colon \mathcal{O}_{X,x} \to \mathbb{Z} \cup \{\infty\}$ is the valuation on the DVR $\mathcal{O}_{X,x}$.

Proposition 10.9. *Let $f \in k(X)^\times$. Then*
$$\deg(\mathrm{div}(f)) = 0$$
and the resulting morphism
$$\deg \colon \mathrm{Pic}(X) \to \mathbb{Z}$$
is an isomorphism (with inverse $n \mapsto \mathcal{O}_X(n)$).

The statement that $\deg(\mathrm{div}(f)) = 0$ in the proposition can be interpreted as the heuristic that "X is proper".

Proof. The first assertion follows from Theorem 10.6 and Theorem 9.14 as these imply that one can reduce to the case $f = t'/t$ with $t', t \in B^{\varphi=\pi}$. Then
$$\mathrm{div}(f) = \infty_{t'} - \infty_t$$
has degree 0. The second assertion follows from Theorem 10.6 as there is an exact sequence
$$0 \to \mathbb{Z} \cdot [\mathcal{O}_X(1)] \to \mathrm{Pic}(X) \to \mathrm{Pic}(\mathrm{Spec}(B_t) \to 0$$
and $\mathrm{Pic}(\mathrm{Spec}(B_t)) = 0$. □

The fundamental exact sequence in Theorem 10.1 can be interpreted as the statement that the sequence

$$0 \to \mathcal{O}_X \xrightarrow{t} \mathcal{O}_X(1) \to k(\infty_t) \to 0$$

obtained by the choice of some non-zero section $t \in H^0(X, \mathcal{O}_X(n))$ remains exact after taking global sections.

More generally, we can calculate the higher cohomology of the line bundles $\mathcal{O}_X(n)$.

Proposition 10.10. *The cohomology of $\mathcal{O}_X(n)$ is*

$$H^i(X, \mathcal{O}_X(n)) = \begin{cases} B^{\varphi=\pi^n} & \text{if } i = 0 \\ 0, & \text{if } i \geq 2, or i = 1 and n \geq 0 \\ B^+_{\mathrm{dR},x}/\mathrm{Fil}^{-n} B^+_{\mathrm{dR},x} + E \neq 0, & \text{if } n \leq -1 \end{cases}$$

Here $x \in |X|$ is any point.

Note that the situation is a bit similar to the case that $X = \mathbb{P}^1_E$ with the important exception that $H^1(X, \mathcal{O}_X(-1)) \cong k(\infty_t)/E \neq 0$.

Proof. The case $i = 0$ was already proven in Lemma 10.4. The vanishing of the cohomology in degrees $i \geq 2$ follows from the fact that X can be covered by two affine open subschemes. Using the long exact sequences in cohomology associated to the short exact sequences

$$0 \to \mathcal{O}_X(n-1) \xrightarrow{t} \mathcal{O}_X(n) \to k(\infty_t) \to 0$$

for some $t \in B^{\varphi=\pi} \backslash \{0\}$ reduces the statement to the assertions that $H^1(X, \mathcal{O}_X) = 0$ and $H^1(X, \mathcal{O}_X(-1)) \neq 0$. Let us show that

$$H^1(X, \mathcal{O}_X) = 0.$$

Pick some non-zero $t \in B^{\varphi=\pi}$ and let $j\colon \mathrm{Spec}(B_t) \to X$ be the associated open immersion. Then there is an exact sequence

$$0 \to \mathcal{O}_X \to j_*(\mathcal{O}_{\mathrm{Spec}(B_t)}) \to \mathcal{O}_{X,\infty_t}[1/t]/\mathcal{O}_{X,\infty_t} \to 0.$$

Theorem 10.1 implies, by passing to the colimit[43], that the sequence stays exact after taking global sections. The morphism j is affine and hence

$$H^1(X, j_*(\mathcal{O}_{\mathrm{Spec}(B_t)}) = H^1(\mathrm{Spec}(B_t), \mathcal{O}_{\mathrm{Spec}(B_t)}) = 0$$

which implies as desired $H^1(X, \mathcal{O}_X) = 0$.

As $H^0(X, \mathcal{O}_X) = E$ we moreover obtain from the exact sequence

$$0 \to \mathcal{O}_X(-1) \xrightarrow{t} \mathcal{O}_X \to k(\infty_t) \to 0$$

an isomorphism

$$H^1(X, \mathcal{O}_X(-1)) \cong k(\infty_t)/E \neq 0$$

as desired. □

[43]note that $j_*(\mathcal{O}_X) \cong \varinjlim_d \mathcal{O}_X(d\infty_t)$

By the vanishing of $H^1(X, \mathcal{O}_X)$ taking cohomology of the short exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{t^d} \mathcal{O}_X(d) \to B^+_{\mathrm{dR},\infty_t}/\mathrm{Fil}^d B^+_{\mathrm{dR},\infty_t} \to 0$$

recovers the fundamental exact sequence Theorem 10.1. Thus we see that Theorem 10.1 is (basically) equivalent to the statement that $H^1(X, \mathcal{O}_X) = 0$.

We will close this lecture with relating the curve to the crystalline period ring

$$B^+_{\mathrm{crys}} = \mathbb{A}_{\mathrm{crys}}[1/p]$$

which was introduced in Definition 4.11. The following material is taken from [10, Section 1.10.]

Definition 10.11. We define the following rings.

(1) Set
$$B^{b,+} := \mathbb{A}_{\mathrm{inf}}[1/p].$$
(2) Let $I \subseteq (0, \infty)$ be an interval. Then define B^+_I as the completion of $B^{b,+}$ with respect to the family of norms $(\nu_r)_{r \in I}$.[44]
(3) If $I = \{r\}$, set $B^+_r := B^+_{\{r\}}$.
(4) Set
$$B^+ := B^+_{(0,\infty)}.$$

The elements in B^+ can be interpreted as functions on $|Y|$ which "extend to the crystalline point $W_{\mathrm{O}_E}(\mathfrak{m}_F)$".

The norms ν_r for $r \in (0, \infty)$ on $B^{b,+}$ enjoy the additional property that

$$\nu_{r'}(x) \geq \frac{r'}{r}\nu_r(x)$$

for $r' \leq r$.[45] This implies that

$$B^+_r \subseteq B^+_{r'}$$

and thus that the Frobenius $\varphi\colon B^b \to B^b$, which extends by continuity to an isomorphism $\varphi\colon B^+_r \cong B^+_{qr}$, yields a (non-invertible) endomorphism

$$\varphi\colon B^+_r \cong B^+_{qr} \subseteq B^+_r.$$

for each $r \in (0, \infty)$.

Lemma 10.12. *Let $r \in (0, \infty)$. Then there is an isomorphism*

$$B^+_r \cong \{x \in B_{(0,r]} \mid \mathcal{N}ewt_{(0,r]}(x) \geq 0\}$$

of topological E-vector spaces, and similarly

$$B^+ \cong \{x \in B \mid \mathcal{N}ewt_{(0,\infty)}(x) \geq 0\}.$$

The condition $\mathcal{N}ewt_{(0,r]}(x)$ is to be interpreted as $\mathcal{N}ewt_{(0,r]}(x) \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}$.

Proof. Cf. [10, Proposition 1.10.7.]. □

Lemma 10.13. *Let $a \in \mathfrak{m}_F \setminus \{0\}$ and set $r := \nu(a)$. Then*

$$B^+_r \cong (\mathbb{A}_{\mathrm{inf}}[\frac{[a]}{p}])^\wedge_p[1/p]$$

---

[44] Equivalently, B^+I is the closure of $B^{b,+}$ in B_I.

[45] Indeed, let $x = \sum\limits_{n \gg -\infty}^{\infty} [x_n]\pi^n \in B^{b,+}$. Then $\frac{r'}{r}\nu_r(x) = \inf_{i\in\mathbb{Z}}\{\nu(x_n) + ir\} \geq \nu_{r'}(x)$ as each $\nu(x_n) \geq 0$ and $r'/r \leq 1$.

Proof. By general properties of completions for norms

$$B_r^+ \cong U_p^\wedge[1/p]$$

where $U := \{x \in B^{b,+} \mid \nu_r(x) \geq 0\}$ is the "unit ball". But

$$U = \mathbb{A}_{\mathrm{inf}}[\frac{[a]}{p}]$$

as can easily be checked. □

We assume now that $E = \mathbb{Q}_p$ and fix a non-archimedean, algebraically closed extension $C/\mathbb{Q}_p$. Let

$$\nu_C \colon C \to \mathbb{R} \cup \{\infty\}$$

be the non-archimedean valuation on C and assume $F := C^\flat$. Moreover, fix an element

$$(p, p^{1/p}, \ldots) \in \mathcal{O}_C^\flat.$$

Recall (cf. Definition 4.11) that

$$\mathbb{A}_{\mathrm{crys}}$$

is the p-adic completion of the subring

$$\mathbb{A}_{\mathrm{inf}}[\frac{[p^\flat]^n}{n!} \ |n \geq 0] \subseteq \mathbb{A}_{\mathrm{inf}}[\frac{1}{p}].$$

Lemma 10.14. *There are natural inclusions*

$$B_{pr}^+ \subseteq B_{\mathrm{crys}}^+ \subseteq B_r^+,$$

where $r = \nu_C(p) = \nu(p^\flat)$. *Moreover,*

$$B^+ \cong \bigcap_{i=1}^{\infty} \varphi^n(B_{\mathrm{crys}}^+)$$

is the largest subring of B_{crys}^+ *such that* φ *is bijective.*

Proof. There are natural inclusions

$$\mathbb{A}_{\mathrm{inf}}[\frac{[p^\flat]^p}{p}] \subseteq \mathbb{A}_{\mathrm{inf}}[\frac{[p^\flat]^n}{n!} |n \geq 0] \subseteq \mathbb{A}_{\mathrm{inf}}[\frac{[p^\flat]}{p}].$$

Passing to p-adic completions yields the inclusions[46]

$$B_{pr}^+ \subseteq B_{\mathrm{crys}}^+ \subseteq B_r^+$$

(cf. Lemma 10.13). For the second statement it suffices to see that

$$B^+ = \bigcap_{i=1}^{\infty} \varphi^n(B_r^+)$$

for any $r \in (0, \infty)$. If $n \geq 0$, then the image of

$$\varphi^n \colon B_r^+ \to B_r^+$$

is B_{nr}^+ by construction of the Frobenius on B_r^+. As

$$B^+ = \varprojlim_{r \to \infty} B_r^+.$$

this implies the claim. □

[46]All of these rings embed compatibly into B_{dR}^+, cf. Lemma 4.12 for B_{crys}^+.

Proposition 10.15 (cf. [10, Proposition 4.1.3.]). *There is a canonical isomorphism of graded algebras*

$$P = \bigoplus_{d\geq 0} B^{\varphi=p^d} \cong \bigoplus_{d\geq 0} (B^+_{\mathrm{crys}})^{\varphi=p^d}$$

Proof. Clearly,

$$(B^+_{\mathrm{crys}})^{\varphi=p^d} \cong (B^+)^{\varphi=p^d}$$

for each $d \geq 0$ as B^+ is the largest subring on B^+_{crys} such that φ is bijective. Now the claim follows from Lemma 10.12 as for each $x \in B^{\varphi=p^d}$ for some $d \geq 0$ the Newton polygon $\mathcal{N}ewt(x)$ must lie in $\mathbb{R}^2_{\geq 0}$. Alternatively, one can use Theorem 9.14, Lemma 10.2 and that for $x \in 1+\mathfrak{m}_F$ the element $\log([x])$ lies in B^+. □

Let $y \in |Y|$ be the point determined by C, that is $(\xi_y) = \ker(\theta\colon B^b \to C)$, and fix a compatible system

$$\varepsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \in \mathcal{O}_C^\flat$$

of primitive p^n-roots of unity. Set

$$t := \log([x]) \in (B^+_{\mathrm{crys}})^{\varphi=1},$$
$$B_{\mathrm{crys}} := B^+_{\mathrm{crys}}[1/t]$$

and

$$B_e := B^{\varphi=1}_{\mathrm{crys}}.$$

Then

$$X = \text{"}\mathrm{Spec}(B_e) \cup \mathrm{Spec}(B^+_{\mathrm{dR}})\text{"}$$

glued "along $\mathrm{Spec}(B_{\mathrm{dR}})$". Thus we see how the Fargues-Fontaine curve geometrizes the various period rings in p-adic Hodge theory which were introduced by Fontaine.

11. Lecture of 08.01.2020: The vector bundles $\mathcal{O}_X(\lambda)$

For the classification of vector bundles on the Fargues-Fontaine curve the Harder-Narasimhan formalism is an indispensable tool. We will introduce it greater generality (cf. [10, Section 5.5.]). For this let $\mathcal{C}$ be an exact category[47] with two functions

$$\deg\colon \mathrm{Ob}(\mathcal{C}) \to \mathbb{Z},$$
$$\mathrm{rk}\colon \mathrm{Ob}(\mathcal{C}) \to \mathbb{N},$$

both additive in short exact sequences. Moreover, we assume that there exists an abelian category $\mathcal{A}$ and an exact faithful functor, the "generic fiber"

$$F\colon \mathcal{C} \to \mathcal{A}$$

such that F induces for every $\mathcal{E} \in \mathcal{C}$ a bijection

$$\{\text{strict subobjects of } \mathcal{E}\} \cong \{\text{subobjects of } F(\mathcal{E})\},$$

where a strict subobject is one which can be prolonged into a short exact sequence. Finally, we assume that the rank function $\mathrm{rk}\colon \mathcal{C} \to \mathbb{N}$ is the restriction of an additive function $\mathrm{rk}\colon \mathcal{A} \to \mathbb{N}$ which satisfies

$$\mathrm{rk}(\mathcal{E}) = 0 \Leftrightarrow \mathcal{E} \cong 0,$$

and, most importantly, that the following condition is satisfied: If $u\colon \mathcal{E} \to \mathcal{E}'$ is a morphism in $\mathcal{C}$ such that $F(u)$ is an isomorphism, then $\deg(\mathcal{E}) \leq \deg(\mathcal{E}')$ with

[47] roughly, an additive category with a notion of short exact sequences

equality if and only if u is an isomorphism. For example, one can take the category of vector bundles

$$\mathcal{C} := \mathrm{Bun}_C$$

for C a connected, smooth, proper curve over a field k, with $\mathcal{A} := \mathcal{C}oh(C)$ its category of coherent sheaves,

$$\deg(\mathcal{E}) := \deg(\Lambda^r \mathcal{E})$$

for $\mathcal{E}$ a vector bundle on C and $\mathrm{rk}\colon \mathcal{A} \to \mathbb{N}$ the generic rank of a coherent sheaf. Most importantly for us, using Proposition 10.9 for the degree function, we can take exactly the same definition with C replaced by the Fargues-Fontaine curve $X = X_{\mathrm{FF}} = X_{E,F}$ (associated to E, F).

Definition 11.1. In the general situation from above, we define the slope of $\mathcal{E} \in \mathcal{C}$ as

$$\mu(\mathcal{E}) := \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})} \in \mathbb{Q} \cup \{\infty\}$$

and we call $\mathcal{E}$ semistable if $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ for all non-zero strict subobjects $\mathcal{F} \subseteq \mathcal{E}$

$$\mu(\mathcal{F}) \leq \mu(\mathcal{E}).$$

The following lemma is a useful consequence of semistability.

Lemma 11.2. *Let $\mathcal{E}, \mathcal{E}' \in \mathcal{C}$. Assume $\mathcal{E}, \mathcal{E}'$ are semistable of slopes λ, λ'. If $\lambda > \lambda'$, then*

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{E}, \mathcal{E}') = 0.$$

For the cases we are interested in we leave the proof of Lemma 11.2 as an exercise.

Proposition 11.3. *With the above notations from above, each $\mathcal{E} \in \mathcal{C}$ has a unique, functorial filtration, the Harder-Narasimhan filtration,*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_r = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable for each $1 \leq i \leq r$ and the sequence of slopes $\mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ is strictly decreasing.

The Harder-Narasimhan polygon is the unique concave polygon in $\mathbb{R}^2$ with origin $(0,0)$ and slopes $\mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ with respective multiplicity $\mathrm{rk}(\mathcal{E}_i/\mathcal{E}_{i-1})$.

Proof. The statement can be proven using induction on $\mathrm{rk}(\mathcal{E})$. If $F(\mathcal{E})$ is simple in $\mathcal{A}$, then necessarily $\mathcal{E}$ is semistable and its own Harder-Narasimhan filtration. Thus assume that $\mathcal{E}$ has a strict subobject

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

with $\mathrm{rk}(\mathcal{F}), \mathrm{rk}(\mathcal{G}) < \mathrm{rk}(\mathcal{E})$. By induction $\mathcal{F}, \mathcal{G}$ admit Harder-Narasimhan filtrations. This implies, using Lemma 11.2, that the slopes of strict subobjects of $\mathcal{E}$ are bounded. Then take the strict subobject $\mathcal{F}' \subseteq \mathcal{E}$ of maximal slope and maximal rank. We claim that each non-zero strict subobject $\mathcal{G}'$ of $\mathcal{E}/\mathcal{F}'$ has slope $< \mu(\mathcal{F}')$. Indeed, if $\mu(\mathcal{G}') \geq \mu(\mathcal{F}')$, then the preimage $\mathcal{E}'$ of $\mathcal{G}'$ in $\mathcal{E}$ must have slope $\geq \mu(\mathcal{F}')$ and rank $\geq \mathrm{rk}(\mathcal{F}')$, which by construction of $\mathcal{F}'$ implies $\mathcal{E}' = \mathcal{F}'$. Thus, we can set

$$\mathcal{E}_1 := \mathcal{F}'$$

and continue with $\mathcal{E}/\mathcal{E}_1$. Uniqueness and naturality follow from Lemma 11.2. □

Proposition 11.4. *Let $\lambda \in \mathbb{Q} \cup \{\infty\}$, then the subcategory*

$$\mathcal{C}_\lambda^{\mathrm{sst}}$$

of semistable objects of $\mathcal{C}$ of slope λ (or ∞), is abelian and of finite length, i.e., each object has a finite filtration by simple objects.

Proof. Exercise. □

For example, if $\mathcal{C} = \mathrm{Bun}_{\mathbb{P}^1_k}$ for some field k, then each $\mathcal{E} \in \mathcal{C}$ is isomorphic to

$$\bigoplus_{d \in \mathbb{Z}} \mathcal{O}(d)^{n_d}$$

for some $n_d \in \mathbb{N}$ (only finitely many non-zero) and $\mathcal{E}$ is semistable if and only if $n_d = 0$ except for at most one $d \in \mathbb{Z}$. The Harder-Narasimhan filtration of $\mathcal{E}$ is, with repetitions, the one by the subbundles.

$$\mathcal{E}_i := \bigoplus_{d \geq i} \mathcal{O}(d)^{n_d}$$

We will now introduce the second example, next to Bun_X for X the Fargues-Fontaine curve, which will be most relevant to us.

Definition 11.5. We define

$$\breve{E} := W_{\mathcal{O}_E}(\overline{\mathbb{F}}_q)[1/\pi]$$

with $\overline{\mathbb{F}}_q$ the algebraic closure of $\mathbb{F}_q$ in F.

Definition 11.6. Let A be a ring with an endomorphism $\varphi\colon A \to A$. A φ-module over A is a finite projective A-module M together with an isomorphism

$$\varphi_M \colon \varphi^* M \cong M.$$

We denote by $\varphi - \mathrm{Mod}_A$ the category of φ-modules over A.

If M is free, then by choosing a basis $e_1, \ldots, e_n$ of M we can write

$$\varphi_M(e_i \otimes 1) = \sum_{j=1}^n a_{ij} e_j$$

for some uniquely determined matrix $(a_{ij}) \in \mathrm{GL}_n(A)$. Changing the basis $e_1, \ldots, e_n$ according to some $g \in \mathrm{GL}_n(A)$ changes the matrix $a := (a_{i,j})$ to the φ-conjugated matrix $g a \varphi(g)^{-1}$. Thus, isomorphism classes of free φ-modules of rank n are in bijection of φ-conjugacy classes of matrices in $\mathrm{GL}_n(A)$.

We will now assume $A = \breve{E}$ with φ the natural "Frobenius" on $\breve{E}$. In this case the category

$$\varphi - \mathrm{Mod}_{\breve{E}}$$

of φ-modules over $\breve{E}$[48] is abelian and fits in our previous formalism. Indeed, the valuation on $\breve{E}$ induces an isomorphism

$$\deg\colon \{\varphi - \text{modules of rank } 1\}/\mathrm{isom} \cong \mathrm{coker}(\breve{E}^\times \xrightarrow{\varphi(-)/(-)}) \cong \mathbb{Z}.$$

Now set

$$\mathcal{C} := \mathcal{A} := \varphi - \mathrm{Mod}_{\breve{E}},$$
$$\mathrm{rk}(M) := \dim_{\breve{E}} M$$

[48]If E is unramified over $\mathbb{Q}_p$, these are usually called isocrystals over $\overline{\mathbb{F}}_q$.

and

$$\deg(M) := \deg(\Lambda^{\mathrm{rk}(M)} M).$$

In particular, the Harder-Narasimhan filtration is available for the category $\varphi - \mathrm{Mod}_{\breve{E}}$.[49] Note, that in this example we could also consider $\mathcal{C} = \varphi - \mathrm{Mod}_{\breve{E}}$ with degree function $-\deg$ (as $\mathcal{C}$ is already abelian and the generic fiber functor is the identity). This implies that the Harder-Narasimhan filtration is *canonically* split and each morphism between semistable objects of different slopes is zero.

For example, for $\lambda \in \mathbb{Q}$ with $\lambda = d/r$ with $d \in \mathbb{Z}, r > 0$ coprime we define $(D(\lambda), \varphi_{D(\lambda)})$ as the φ-module $D(\lambda) := \breve{E}^r$ with associated matrix

$$\begin{pmatrix} 0 & \ldots & 0 & \pi^d \\ 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & 0 \end{pmatrix}$$

The category $\varphi - \mathrm{Mod}_{\breve{E}}$ can be described completely.

Theorem 11.7 ("Dieudonné-Manin classification")**.** *The category $\varphi - \mathrm{Mod}_{\breve{E}}$ is semisimple with simple objects given, up to isomorphism, by the $D(\lambda)$ for $\lambda \in \mathbb{Q}$. For $\lambda \in \mathbb{Q}$ the division algebra $\mathrm{End}_{\varphi-\mathrm{Mod}_{\breve{E}}}(D(\lambda))$ over E is central of invariant $-[\lambda] \in \mathrm{Br}(E) \cong \mathbb{Q}/\mathbb{Z}$.*

Proof. Using the Harder-Narasimhan formalism and passing to unramified coverings of E, the essential point is to see that each semistable φ-modules over $\breve{E}$ of slope 0 is a direct sum of the trivial isocrystal $D(0) = (\breve{E}, \varphi)$. By direct inspection, one can see that

$$\mathrm{Ext}^1_{\varphi-\mathrm{Mod}_{\breve{E}}}(D(0), D(0)) \cong \breve{E}/(\varphi - \mathrm{Id})(\breve{E})$$

and we will prove that this group is zero. We claim that even the sequence

$$0 \to \mathcal{O}_E \to \mathcal{O}_{\breve{E}} \xrightarrow{\varphi-\mathrm{Id}} \mathcal{O}_{\breve{E}} \to 0$$

is exact. The crucial point is surjectivity which, however, can be checked (by completeness) modulo π, where it follows from the fact that $\overline{\mathbb{F}}_q = \mathcal{O}_{\breve{E}}/\pi$ is algebraically closed. Thus, in order to finish the proof it suffices, using induction and semistability, to see that each semistable φ-module D of slope 0 admits a non-zero morphism $D(0) \to D$. For this, write (after the choice of some basis)

$$\varphi_D = a\varphi$$

for some matrix $a \in \mathrm{GL}_n(\breve{E})$. After row operations the matrix a is triangular with all diagonal entries entry $a_{i,i} \in \mathcal{O}_{\breve{E}}^\times$, because D is semistable of slope 0. As $\overline{\mathbb{F}}_q$ is algebraically closed, we can write

$$a_{1,1} = \varphi(x)/x$$

for some $x \in \mathcal{O}_{\breve{E}}^\times$. This implies that if e denotes the first basis vector of the (implicitly chosen basis) the φ-submodule $(\breve{E}, a_{1,1}\varphi)$ of D is isomorphic to $D(0)$. This finishes the proof. □

More generally, one can prove the following, sometimes useful, statement.

[49] For classical reasons, semistable isocrystals are called isoclinic.

Lemma 11.8. *Let R be a perfect ring. Then the functor*

$$\{\underline{\mathbb{Z}}_p - \text{ local systems on } \mathrm{Spec}(R)\} \to \{\varphi - \text{modules over } W(R)\},$$

which is defined by

$$\mathbb{L} \mapsto \Gamma(\mathrm{Spec}(R), \mathbb{L} \otimes_{\mathbb{Z}_p} W_R)^{\varphi=1}$$

is an equivalence of categories.

Here by a $\underline{\mathbb{Z}}_p$-local system we mean a pro-étale sheaf over $\mathrm{Spec}(R)$, which is locally equivalent to a finite direct sum of the sheaf $\underline{\mathbb{Z}}_p = \mathrm{Hom}_{\mathrm{cont}}(\pi_0(-), \mathbb{Z}_p)$, where $\mathbb{Z}_p$ is given the p-adic topology. By W_R we denote the pro-étale sheaf of Witt vectors over $\mathrm{Spec}(R)$.

Proof. We leave this as an elaborate exercise. □

Now, we will connect φ-modules over $\breve{E}$ to vector bundles on the Fargues-Fontaine curve. More details for the following discussion can be found in [10, Section 8.2.].

We note that $\breve{E} \subseteq B$.

Definition 11.9 (cf. [10, Section 8.2.3.])**.** We define the functor

$$\mathcal{E}(-)\colon \varphi - \mathrm{Mod}_{\breve{E}} \to \mathrm{Bun}_X,\ (D, \varphi_D) \mapsto \widetilde{\bigoplus_{d \geq 0} (B \otimes_{\breve{E}} D)^{\varphi \otimes \varphi_D = \pi^d}}$$

For example, for $d \in \mathbb{Z}$ the φ-module $D(-d)$, i.e., $\varphi_{D(d)} = \pi^{-d}\varphi$ is send to $\mathcal{O}_X(d)$.

We note that at this moment, it is not clear that the functor $\mathcal{E}(-)$ is well-defined, i.e., that $\mathcal{E}(D, \varphi_D)$ is a vector bundle. To see this we will introduce a different construction of the vector bundles $\mathcal{E}(D(\lambda))$.

For clarity, let us for the moment denote $\mathcal{E}(-)$ by $\mathcal{E}_E(-)$ to stress its dependence on E. Similarly, for $X = X_E, B = B_E, \ldots$.

Lemma 11.10. *Let E_h/E be the unramified extension of E of degree h and $(D, \varphi_D) \in \varphi - \mathrm{Mod}_{\breve{E}}$. Then for any $d \geq 0$ the canonical morphism*

$$E_h \otimes_E (B \otimes_{\breve{E}} D)^{\varphi \otimes \varphi_D = \pi^d} \to (B \otimes_{\breve{E}} D)^{\varphi^h \otimes \varphi_D^h = \pi^{hd}}$$

is an isomorphism. In particular, $X \otimes_E E_h \cong X_{E_h}$ and the diagram

$$\begin{array}{ccc} \varphi - \mathrm{Mod}_{\breve{E}} & \xrightarrow{\mathcal{E}_E(-)} & \mathcal{C}oh_{X_E} \\ \downarrow & & \downarrow {\scriptstyle - \otimes_E E_h} \\ \varphi^h - \mathrm{Mod}_{\breve{E}} & \xrightarrow{\mathcal{E}_{E_h}(-)} & \mathcal{C}oh_{X_{E_h}} \end{array}$$

commutes (up to natural isomorphism).

The functor $\varphi - \mathrm{Mod}_{\breve{E}} \to \varphi^h - \mathrm{Mod}_{\breve{E}}$ sends (D, φ_D) to (D, φ_D^h), where φ_D^h is the composition

$$(\varphi^h)^* D \xrightarrow{(\varphi^{h-1})^*(\varphi_D)} (\varphi^h)^* D \to \ldots \xrightarrow{\varphi^*(\varphi_D)} D.$$

An analogous base change holds for every finite extension E'/E. We only need the case that E'/E is unramified. We note that $B_E = B_{E_h}$, $\breve{E} = \breve{E}_h$ while $\varphi_{E_h} = \varphi_E^h$.

Proof. The group $\mathbb{Z}/h\mathbb{Z} \cong \mathrm{Gal}(E_h/E)$ acts on

$$(B \otimes_{\breve{E}} D)^{\varphi^h \otimes \varphi_D^h = \pi^{hd}}$$

E_h-semilinearly via $\pi^{-d}\varphi \otimes \varphi_D$ with invariants

$$(B \otimes_{\breve{E}} D)^{\varphi \otimes \varphi_D = \pi^d}.$$

Thus the first claim follows from Hilbert's theorem 90 (resp. Galois descent). By general properties of the Proj-construction

$$X_{E_h} = \mathrm{Proj}(\bigoplus_{d \geq 0} B^{\varphi^h = \pi^d}) \cong \mathrm{Proj}(\bigoplus_{d \geq 0} B^{\varphi^h = \pi^{hd}})$$

and thus the isomorphism $X \otimes_E E_h \cong X_{E_h}$ follows from the first statement in the case that $(D, \varphi) = (\breve{E}, \varphi)$. The commutativity of the diagram is again a consequence of the first statement. □

Lemma 11.10 implies that the functor $\mathcal{E}(-)$ has values in vector bundles, because for each φ-module this can be checked after pullback to some X_{E_h} and for $D(\lambda)$ with $\lambda = \frac{d}{r}$ the pullback $E_r \otimes_E \mathcal{E}(D(\lambda))$ is a direct sum of $\mathcal{O}_{X_{E_r}}(-d)$.

For each $h \in \mathbb{N}$ we denote by E_h the unramified extension of E of degree h. Moreover, we set

$$X_h := X_{E_h}.$$

Lemma 11.11. *Let $\lambda \in \mathbb{Q}$ and write $\lambda = \frac{d}{r}$ with $d \in \mathbb{Z}$, $r \geq 0$ and d, r coprime. Then*

$$\mathcal{E}(D(\lambda)) \cong (f_r)_*(\mathcal{O}_{X_h}(d)).$$

Proof. For both sides the pullback along f_r are isomorphic to

$$\mathcal{F} := \mathcal{O}_{X_h}(-d)^r.$$

For $\mathcal{E}(D(\lambda))$ this follows from Lemma 11.10 and for $(f_r)_*(\mathcal{O}_{X_h}(d))$ because $(f_r)_*\mathcal{O}_{X_h} \cong E_h \otimes_E \mathcal{O}_X$. Thus, both sides define elements in

$$H^1(\mathrm{Gal}(E_h/E), \mathrm{Aut}(\mathcal{F}))$$

with Galois action of $\mathrm{Gal}(E_h/E)$ induced by either isomorphism $\mathcal{F} \cong f_r^*(\mathcal{E}(D(\lambda)))$ or $\mathcal{F} \cong f_r^*((f_r)_*\mathcal{O}_{X_h}(d))$. But $\mathrm{Aut}(\mathcal{F}) \cong \mathrm{GL}_r(E_h)$, with Galois action in both cases given by the natural one on E_h. By Hilbert's theorem 90

$$H^1(\mathrm{Gal}(E_h/E), \mathrm{GL}_r(E_h)) = \{*\}$$

which finishes the proof. □

We make the following definition.

Definition 11.12. Let $\lambda = \frac{d}{r} \in \mathbb{Q}$ with $\lambda \in \mathbb{Z}, r \in \mathbb{N}$ and r minimal. Then

$$\mathcal{O}_X(\lambda) := \mathcal{E}(D(\lambda)) \cong f_{r,*}(\mathcal{O}_{X_r}(d)),$$

Lemma 11.13. *For $\lambda \in \mathbb{Q}$ the functor $\mathcal{E}(-)$ induces an isomorphism*

$$\mathrm{End}_{\varphi-\mathrm{Mod}_{\breve{E}}}(D(\lambda)) \cong \mathrm{End}(\mathcal{O}_X(-\lambda)).$$

Proof. This follows by descent from the statement that $\mathrm{End}_{X_h}(\mathcal{O}_{X_h}(d)) \cong E$ for any $d \in \mathbb{Z}$, $h \geq 0$. We leave the details as an exercise, cf. [10, Proposition 8.2.8.]. □

Because f_r is affine, Lemma 11.11 and our knowledge of the cohomology of the line bundles $\mathcal{O}_{X_h}(d)$ (cf. Proposition 10.10 implies that we know the cohomology of the $\mathcal{O}_X(\lambda)$, too.

The classification of vector bundles on the Fargues-Fontaine curve is the second main theorem of the course.

Theorem 11.14 (Fargues-Fontaine, cf. [10, Théorème 8.2.10.]). *The functor $\mathcal{E}(-)$ induces a bijection on isomorphism classes*

$$\varphi - \mathrm{Mod}_{\breve{E}}/\mathrm{isom.} \cong \mathrm{Bun}_X/\mathrm{isom.}.$$

By the Dieudonné-Manin classification of φ-modules, cf. Theorem 11.7, this means concretely that each vector bundle on X is a direct sum of the vector bundles $\mathcal{O}_X(\lambda)$. Moreover, the functor $\mathcal{E}(-)$ is compatible with the Harder-Narasimhan filtration, and the Harder-Narasimhan filtration of each vector bundle on X splits (non-canonically). Clearly, the functor $\mathcal{E}(-)$ is not equivalence (as the category $\varphi - \mathrm{Mod}_{\breve{E}}$ is abelian, but Bun_X not).

A sketch of proof of Theorem 11.14 will occupy us for two lectures. For the proof we will have to relate vector bundles on X to p-adic Hodge theory for p-divisible groups.

12. Lecture of 15.01.2020: p-divisible groups and $\mathbb{A}_{\mathrm{inf}}$-cohomology (by Ben Heuer)

Recall that last time we stated our second main theorem

Theorem 12.1. *The functor $\mathcal{E}$ defines a bijection of isomorphism classes*

$$\varphi\text{–}\mathrm{Mod}_{\breve{E}}/\sim \longrightarrow \mathrm{Bun}_{X_{\mathrm{FF}}}/\sim$$

A sketch of proof will be presented next time. Today, we discuss a few things we need for the proof. On the way, we will see some applications of X_{FF} to p-divisible groups and p-adic Hodge theory.

For this we will need some background that strictly speaking isn't in the prerequisites of this course (so don't worry if there are bits you don't understand, this lecture is a bit of a "survey" anyway). We therefore start with a crash course on p-divisible groups

Definition 12.2. Let R be a ring. A p-divisible group G over R of height h is a collection $(G_n, i_n)_{n\in\mathbb{N}}$ of finite flat group schemes[50] G_n of order p^{hn} over R together with closed immersions $i_n : G_n \to G_{n+1}$ such that the following sequence is left exact:

$$0 \to G_n \xrightarrow{i_n} G_{n+1} \xrightarrow{[p^n]} G_{n+1}$$

We also write $G[p^n] := G_n$. It follows from the axioms that we have a short exact sequence[51] for any n, m:

$$0 \to G[p^n] \xrightarrow{i_{n,m}} G[p^{n+m}] \xrightarrow{j_{n,m}} G[p^m] \to 0.$$

where j is induced from $[p^n] : G[p^{n+m}] \to G[p^{n+m}]$. Surjectivity on the right is the reason for the name "p-divisible".

[50] Always assumed to be commutative.

[51] As fppf-sheaves.

Example 12.3.
- Let $G_n = \frac{1}{p^n}\mathbb{Z}/\mathbb{Z}$ be the constant group scheme, and i_n the natural inclusion. This defines a p-divisible group called $\mathbb{Q}_p/\mathbb{Z}_p$ of height 1.
- Let $G_n = \mu_{p^n} := \mathrm{Spec}(R[X]/(X^{p^n}-1))$ be the group scheme of p^n-th unit roots. This defines a p-divisible group μ_{p^∞} of height 1.
- Let A be an abelian scheme over R of dimension d. Then the p^n-torsion $G_n := A[p^n]$ defines a p-divisible group $A[p^\infty]$ of height $2d$. The p-divisible group $A[p^\infty]$ is equipped with an action of $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{End}_R(A)$. In particular, each idempotent in this ring yields a decomposition of $A[p^\infty]$ into p-divisible groups, which usually cannot be obtained from the previous examples.

Definition 12.4. For any p-divisible group G, we obtain its dual p-divisible group $G^\vee$ by letting $(G^\vee)_n := (G_n)^\vee = \underline{\mathrm{Hom}}(G_n, \mathbb{G}_m)$ be the Cartier dual of G_n, and $i_n^\vee := j_{1,n}^\vee$.

The natural evaluation isomorphisms $G_n \to (G_n^\vee)^\vee$ are compatible and define an isomorphism of p-divisible groups

$$G \to (G^\vee)^\vee$$

. The functor $G \mapsto G^\vee$ is thus a (contravariant) auto-duality.

Example 12.5.
- $(\mathbb{Q}_p/\mathbb{Z}_p)^\vee = \mu_{p^\infty}$ and $(\mu_{p^\infty})^\vee = \mathbb{Q}_p/\mathbb{Z}_p$.
- $A[p^\infty]^\vee = A^\vee[p^\infty]$ where $A^\vee$ is the dual abelian variety.

We now specialise to the case of C a complete algebraically closed extension of $\mathbb{Q}_p$ and $R = \mathcal{O}_C$. This is the case we shall focus on today.

Definition 12.6. For a p-divisible group G over $\mathcal{O}_C$, we define its Tate module to be

$$T_pG := \varprojlim \Big(\dots \xrightarrow{[p]} G[p^2](C) \xrightarrow{[p]} G[p](C) \to 1 \Big).$$

Example 12.7.
- We have $T_p(\mathbb{Q}_p/\mathbb{Z}_p) = \mathbb{Z}_p$.
- We write $T_p\mu_{p^\infty} =: \mathbb{Z}_p(1)$ for the Tate module of μ_{p^∞} over $\mathcal{O}_C$. It is isomorphic to $\mathbb{Z}_p$, but the isomorphism depends on a choice of compatible p^n-th roots of unity ζ_{p^n}. For any $\mathbb{Z}_p$-module M and $n \in \mathbb{Z}$ we set $M(n) := M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)^{\otimes n}$ (interpreted as the module dual for $n < 0$)
- $T_p(A[p^\infty]) = T_pA$. This can be canonically identified with the dual of $H^1_{\text{ét}}(A, \mathbb{Z}_p)$ (and when we work over any field K, this identification is Galois equivariant).

Lemma 12.8.
(1) *There is a natural isomorphism of $\mathbb{Z}_p$-modules*

$$T_pG = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G).$$

(2) *The natural map $T_pG = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G) \xrightarrow{(-)^\vee} \mathrm{Hom}(G^\vee, \mu_{p^\infty})$ defines a perfect pairing*

$$T_pG \times T_pG^\vee \to T_p\mu_{p^\infty} = \mathbb{Z}_p(1).$$

Proof. Exercise. Use $\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G) \xrightarrow{T_p(-)} \mathrm{Hom}(\mathbb{Z}_p, T_pG) \xrightarrow{ev(1)} T_pG$ □

For $G = A[p^\infty]$, this duality pairing can be identified with the Weil pairing.

It is an important task in arithmetics to classify all p-divisible groups over a given ring, by (semi-)linear algebra. This was first known for perfect fields k of characteristic p:

Theorem 12.9 (Dieudonné, Cartier (60's))**.** *Let k be a perfect field of characteristic p. Then there is an exact equivalence of categories*

$$M : \{p\text{-divisible groups over } k\} \to \{\text{Dieudonné modules over } W(k)\}.$$

Here a Dieudonné module is a finite free $W(k)$-module together with a φ-linear action of an operator F and a φ^{-1}-linear action of an operator V such that $FV = p = VF$.

Actually, they prove this for finite flat group schemes, the case of p-divisible groups follows. In fact, p-divisible groups where historically introduced after this.

Remark 12.10. In this setting, one often instead uses a *contra*-variant version of Dieudonné modules, making the above an anti-equivalence. We will today use the *co*-variant version. The two can be translated into each other via the auto-duality $G \mapsto G^\vee$. Note: Going from contra to co does not affect the linearity properties of F and V, but it means that now F on M corresponds to V on G and vice versa.

Nowadays, one can classify p-divisible groups in many more cases e.g. over perfectoid bases. Grothendieck–Messing and Berthelot–Breen–Messing (70') extended the definition of Dieudonné modules to any ring R on which p is nilpotent, using the formalism of the crystalline site. Important for us is the following case:

Let C be as before and consider the semi-perfect[52] ring $\mathcal{O}_C/p$. Recall that we had defined in Definition 4.11 a ring $\mathbb{A}_{\mathrm{inf}} \to \mathbb{A}_{\mathrm{crys}}$.

Definition 12.11. A Dieudonné module over $\mathcal{O}_C/p$ is a finite free $\mathbb{A}_{\mathrm{crys}}$-module M together with linear operators

$$F : M \otimes_{A_{\mathrm{crys}},\varphi} A_{\mathrm{crys}} \to M$$
$$V : M \to M \otimes_{A_{\mathrm{crys}},\varphi} A_{\mathrm{crys}}$$

such that $FV = p = VF$.

Proposition 12.12 (Grothendieck–Messing, Scholze–Weinstein)**.** *There is a fully faithful covariant functor*

$$M_{\mathrm{crys}}(-) : \{p\text{-divisible groups over } \mathcal{O}_C/p\} \to \{\text{Dieudonné modules over } \mathbb{A}_{\mathrm{crys}}\}.$$

We have $\mathrm{rk}(M_{\mathrm{crys}}(G)) = \mathrm{ht}(G)$ *and* $M_{\mathrm{crys}}(G^\vee) = M_{\mathrm{crys}}(G)^\vee$*, the dual Dieudonné module.*

The definition of $M_{\mathrm{crys}}(-)$ is due to Grothendieck-Messing (cf. [18]), fully faithfulness was proven by Scholze-Weinstein (cf. [24]).

Example 12.13.
- $M(\mathbb{Q}_p/\mathbb{Z}_p) = \mathbb{A}_{\mathrm{crys}}$ with $F = p$, $V = 1$.
- $M(\mu_{p^\infty}) = \mathbb{A}_{\mathrm{crys}}$ with $F = 1$, $V = p$.
- If A is an abelian scheme over $\mathcal{O}_C/p$, then $M(A[p^\infty])^\vee = H^1_{\mathrm{crys}}(A)$ can be naturally identified with the crystalline cohomology, in a way that identifies F with the Frobenius φ.

To a p-divisible group over $\mathcal{O}_C$ we have therefore associated two very different invariants, T_pG and $M(G_{\mathcal{O}_C/p})$. How can one compare T_pG and $M(G_{\mathcal{O}_C/p})$, can one perhaps recover one from the other? In the case of $G = A[p^\infty]$ for an abelian variety A over $\mathcal{O}_C$, this is essentially asking how to compare $H^1_{\text{ét}}(A_C, \mathbb{Z}_p)$ and $H^1_{\mathrm{crys}}(A_{\mathcal{O}_C/p}|\mathbb{A}_{\mathrm{crys}})$.

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[52] An $\mathbb{F}_p$-algebra is semiperfect if its Frobenius is surjective.

The mathematical field studying such comparison isomorphisms between p-adic cohomology theories is p-adic Hodge theory as was explained in Section 2.

Theorem 12.14 ([1, Theorem 14.5.(i)], '16)**.** *Let X be a smooth proper formal scheme over $\mathcal{O}_C$. Then for any $i \geq 0$, we have an étale-crystalline comparison isomorphism*

$$H^i_{\text{ét}}(X_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{crys}} = H^i_{\text{crys}}(X_{\mathcal{O}_C/p}) \otimes_{\mathbb{A}_{\text{crys}}} B_{\text{crys}}.$$

Remark 12.15.

- Recall from Section 2 that we think of this as a p-adic analogue of the following complex comparison isomorphism: Let Y be a smooth variety over $\mathbb{C}$, then
$$H^i_{\text{sing}}(Y(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H^i_{\text{dR}}(Y).$$
In the p-adic version, the role of the ring of periods $\mathbb{C}$ (that one needs for the Poincaré-Lemma) is played by the much more complicated ring B_{crys} defined after Proposition 10.15.
- If X comes via base-change from $\mathcal{O}_K$ for $K/\mathbb{Q}_p$ finite, this is already due to Tsuji, after previous work by Fontaine–Messing, Bloch–Kato and Faltings). One can then also identify the Galois actions on both sides. In particular, one obtains that the Galois representation $V := H^*(X_C, \mathbb{Q}_p)$ is "crystalline", i.e., we have
$$\dim_{\mathbb{Q}_p} V = \dim_{\mathbb{Q}_p} (V \otimes_{\mathbb{Q}_p} B_{\text{crys}})^{G_K}$$
is an isomorphism.

Back to p-divisible groups. It turns out that such a comparison holds in this case, too.

Proposition 12.16. *Let G be a p-divisible group over $\mathcal{O}_C$. We write $M(G)$ for the Dieudonné module associated to $G_{\mathcal{O}_C/p}$. Then there is a natural φ-equivariant isomorphism*

$$\beta_G : T_pG \otimes_{\mathbb{Z}_p} B_{\text{crys}} \cong M(G) \otimes_{\mathbb{A}_{\text{crys}}} B_{\text{crys}}.$$

After tensoring up to B^+_{dR}, the respective B^+_{dR}-sublattices satisfy

$$T_pG \otimes_{\mathbb{Z}_p} B^+_{\text{dR}} \subseteq \Xi := M(G) \otimes_{\mathbb{A}_{\text{crys}}} B^+_{\text{dR}} \subseteq t^{-1}(M(G) \otimes_{\mathbb{A}_{\text{crys}}} B^+_{\text{dR}}).$$

This proposition implies Theorem 12.14 for abelian varieties with good reduction.

Sketch of proof. We'll construct β_G and an inverse: Recall that $T_pG = \operatorname{Hom}_{\mathcal{O}_C}(\mathbb{Q}_p/\mathbb{Z}_p, G)$. Given any $\alpha : \mathbb{Q}_p/\mathbb{Z}_p \to G$, we can base change it to $\mathcal{O}_C/p$ and apply $M(-)$ to get a φ-equivariant map

$$M(\alpha)\colon \mathbb{A}_{\text{crys}} \cong M(\mathbb{Q}_p/\mathbb{Z}_p) \to M(G_{\mathcal{O}_C/p}).$$

We define β_G by sending α to the image of 1. Extending $\mathbb{Z}_p$-linearly, this defines a map

$$T_pG \otimes_{\mathbb{Z}_p} \mathbb{A}_{\text{crys}} \to M(G).$$

To show that this is an isomorphism after inverting p and t, we construct a (generic) inverse mapping using the dual: Applying the discussion so far to $G^\vee$, we obtain a map

$$\beta_{G^\vee} : T_p(G^\vee) \otimes_{\mathbb{Z}_p} \mathbb{A}_{\text{crys}} \to M(G^\vee). \tag{3}$$

The inverse will be induced by its $\mathbb{A}_{\text{crys}}$-module dual $(\beta_{G^\vee})^\vee$ (this is a common trick).

Using $M(G^\vee) = M(G)^\vee$, we have a natural isomorphism

$$M(G) = M(G^\vee)^\vee$$

(this is an isomorphism by full faithfulness of M but we don't need this). On the other hand, we have by Lemma Lemma 12.8.2

$$T_p(G^\vee) = \mathrm{Hom}(T_pG, T_p\mu_{p^\infty}) = (T_pG)^\vee \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1).$$

Recall $\mathbb{A}_{\mathrm{crys}}(-1) := (\mathbb{A}_{\mathrm{crys}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1))^\vee$. Upon applying duals to (3), we then obtain a map

$$M(G) = M(G^\vee)^\vee \xrightarrow{(\beta_{G^\vee})^\vee} (T_p(G^\vee) \otimes_{\mathbb{Z}_p} \mathbb{A}_{\mathrm{crys}})^\vee = T_pG \otimes_{\mathbb{Z}_p} \mathbb{A}_{\mathrm{crys}}(-1).$$

Using the natural B^+_{crys}-linear isomorphism

$$\mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} B^+_{\mathrm{crys}} \to tB^+_{\mathrm{crys}}, \quad \epsilon^a \otimes x \mapsto \log(\epsilon^a) \cdot x = t \cdot a \cdot x,$$

we thus get a map

$$M(G) \otimes B^+_{\mathrm{crys}} \to T_pG \otimes_{\mathbb{Z}_p} B^+_{\mathrm{crys}}(-1) \to T_pG \otimes_{\mathbb{Z}_p} t^{-1}B^+_{\mathrm{crys}}.$$

One can check that after passing to $B_{\mathrm{crys}} = B^+_{\mathrm{crys}}[\frac{1}{t}]$, this defines an inverse to β_G. □

We now start to interpret this result via the modifications of vector bundles on the Fargues–Fontaine curve. Take $E = \mathbb{Q}_p$, $\pi = p$. Recall that

$$X_{\mathrm{FF}} = \mathrm{Proj}(P) = \mathrm{Proj}(\oplus_{d \geq 0} (B^+_{\mathrm{crys}})^{\varphi = p^d})$$

is the Fargues–Fontaine curve, a Dedekind scheme over $\mathbb{Q}_p$ (see Proposition 10.15, Theorem 10.6). Recall also that the natural morphism

$$\theta \colon \mathbb{A}_{\mathrm{inf}} \to \mathcal{O}_C$$

defines a point $\infty \in X_{\mathrm{FF}}$ with associated closed immersion

$$i_\infty : \mathrm{Spec}(C) \to X_{\mathrm{FF}}.$$

By definition, the completion of X_{FF} at this point is given by

$$\mathrm{Spec}({B_{\mathrm{dR}}}^+) \to X_{\mathrm{FF}}$$

where ${B_{\mathrm{dR}}}^+$ is a DVR with pseudo-uniformizer $t = \log([\epsilon])$, $\epsilon = (1, \zeta_p, \zeta_{p^2}, \dots)$ (see Definition 5.5, Section 10).

Here the element t is already defined in the smaller ring $\mathbb{A}_{\mathrm{inf}} \to \mathbb{A}_{\mathrm{crys}} \to B^+_{\mathrm{dR}}$. Moreover,

- $B^+_{\mathrm{crys}} = \mathbb{A}_{\mathrm{crys}}[\frac{1}{p}]$,
- $B_{\mathrm{crys}} = B^+_{\mathrm{crys}}[\frac{1}{t}]$,
- $B_{\mathrm{dR}} = B^+_{\mathrm{dR}}[\frac{1}{t}]$, a discretely valued field.

The Fargues–Fontaine curve organises all these p-adic period rings in a nice, geometric way.

Definition 12.17. Let G be a p-divisible group over $\mathcal{O}_C/p$. We can associate to G a quasi-coherent sheaf on X_{FF} by setting

$$\mathcal{E}(G) := \widetilde{\left(\bigoplus_{d \geq 0} M_{\mathrm{crys}}(G)[\frac{1}{p}]^{\varphi = p^d} \right)}.$$

Here, $\widetilde{-}$ denotes the sheaf associated to a graded module. We will soon see that this is in fact a vector bundle on X_{FF} of rank $\mathrm{ht}(G)$.

By Proposition Proposition 12.16, there is a natural morphism

$$\beta_G : T_pG \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{\mathrm{FF}}} \to \mathcal{E}(G)$$

that is an isomorphism over the locus $X_{\mathrm{FF}} \setminus \{\infty\}$ where t is invertible:

$$\mathrm{Spec}(C) \xhookrightarrow{\iota_\infty} X_{\mathrm{FF}} \hookleftarrow X_{\mathrm{FF}} \setminus \{\infty\}$$

Corollary 12.18. *Let $\mathcal{F} := T_p(G) \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{\mathrm{FF}}}$. Then there is a natural short exact sequence of sheaves on X_{FF}*

$$0 \to \mathcal{F} \xrightarrow{\beta_G} \mathcal{E}(G) \to i_{\infty *}W \to 0$$

where W is the fin. dim. C-vector space given by the image of $M(G) \otimes_{\mathbb{A}_{\mathrm{crys}}} B_{\mathrm{dR}}^+$ under

$$T_pG \otimes_{\mathbb{Z}_p} t^{-1}B_{\mathrm{dR}}^+ \xrightarrow{\mathrm{Id}\otimes\theta(-1)} T_pG \otimes_{\mathbb{Z}_p} C(-1).$$

Proof. It is clear that the cokernel of $\mathcal{F} \to \mathcal{E}(G)$ is supported at ∞. To calculate the stalk at ∞, we need to reduce mod C, i.e. tensor with $B_{\mathrm{crys}}^+ \to B_{\mathrm{dR}}^+ \to C$. We then use the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_pG \otimes {B_{\mathrm{dR}}}^+ & \longrightarrow & M(G) \otimes B_{\mathrm{dR}}^+ & \longrightarrow & \mathrm{coker} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow \wr & & \\
0 & \longrightarrow & T_pG \otimes B_{\mathrm{dR}}^+ & \longrightarrow & T_pG \otimes t^{-1}B_{\mathrm{dR}}^+ & \xrightarrow{\mathrm{Id}\otimes\theta(-1)} & T_pG \otimes C(-1) & \longrightarrow & 0.
\end{array}$$

□

Definition 12.19. A short exact sequence on X_{FF} of the form

$$0 \to \mathcal{F} \xrightarrow{\beta} \mathcal{F}' \to i_{\infty *}W \to 0$$

where $\mathcal{F}, \mathcal{F}'$ are vector bundles and W is a finite dimensional C-vector space is called a **minuscule modification** (at ∞). We have thus defined a functor (important for next time)

$$\{p\text{-divisible groups over } \mathcal{O}_C\} \to \{\text{minuscule modifications on } X_{\mathrm{FF}}\}.$$

Sending a minuscule modification to its cokernel defines a forgetful morphism

$$\{\text{minuscule modifications on } X_{\mathrm{FF}}\} \to \{W \subseteq T_pG \otimes_{\mathbb{Z}_p} C(-1)\}.$$

The following amazing Theorem says that the data of T_pG together with W is equivalent to the datum of G:

Theorem 12.20 ([25], Scholze–Weinstein '12)**.** *The functor defined above*

$$\{p\text{-divisible groups over } \mathcal{O}_C\} \to \left\{ \begin{array}{l} \text{pairs } (T, W) \text{ consisting of} \\ \bullet\ T \text{ finite free } \mathbb{Z}_p\text{-module,} \\ \bullet\ W \subseteq T \otimes_{\mathbb{Z}_p} C(-1) \end{array} \right\}$$

is an equivalence of categories.

Remark 12.21.

- This is in stark contrast to the usual classifications of p-divisible groups in terms of *semi*-liner algebra data.

- We think of this as an analogue to Riemann's Theorem: complex abelian varieties A are equivalent to pairs (Λ, W) of a finite free $\mathbb{Z}$-module Λ and $W \subseteq \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ such that (Λ, W) is a polarisable Hodge structure of weight -1.
- There is a second, equivalent, way to characterise W, namely as the Hodge–Tate filtration $\mathrm{Lie}(G) \subseteq T_pG \otimes_{\mathbb{Z}_p} C(-1)$ of G (this is the definition in [25]). In particular,
  - For $G = \mathbb{Q}_p/\mathbb{Z}_p$, we have $(T, W) = (\mathbb{Z}_p, C(-1) \subseteq C(-1))$.
  - For $G = \mu_{p^\infty}$, we have $(T, W) = (\mathbb{Z}_p(1), 0 \subset C)$.
  - For $G = A[p^\infty]$, the Hodge–Tate filtration has $\dim W = \dim A$ and is of the form

$$0 \to W = \mathrm{Lie}(A) \to T_pA \otimes C(-1) \to \omega_A(-1) \to 0.$$

We discuss in more details modifications of vector bundles on X_{FF}. Our next goal is to see that one can in fact reconstruct $\mathcal{E}(G)$ from the trivial vector bundle $T_pG \otimes \mathcal{O}_{X_{\mathrm{FF}}}$ when given the data of the B_{dR}^+-lattice $\Xi \subseteq T_pG \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$ from Prop. Proposition 12.16. By the second part of the proposition, for this to work, the lattice must satisfy

$$T_pG \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}^+ \subseteq \Xi \subseteq t^{-1}(T_pG \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}^+).$$

Definition 12.22. Such a lattice is called a **minuscule lattice**.

We note that taking the image under θ defines an equivalence

$$\{\text{minuscule lattices in } T_pG \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}^+\} \to \{\text{sub-}C\text{-vector spaces of } T_pG \otimes_{\mathbb{Z}_p} C(-1)\}$$

(and in particular, one can also state Theorem Theorem 12.20 in terms of minuscule B_{dR}^+-lattices).

The idea is that this gives "infinitesimal information at ∞" that one use to extend $\mathcal{F}$ to a new vector bundle over all of X_{FF}. This hinges on the following algebra fact (see [27, Tag 0BNI] for a discussion in greater generality):

Lemma 12.23 (Beauville–Laszlo)**.** *Let A be a Noetherian ring and let $f \in A$. Consider the completion $\widehat{A} := \varprojlim_{n \in \mathbb{N}} A/f^n$. Then the natural functor*

$$\begin{aligned} A\text{−Mod} \to & A[\tfrac{1}{f}]\text{−Mod} \times_{\widehat{A}[\frac{1}{f}]\text{−Mod}} \widehat{A}\text{−Mod} \\ M \mapsto \quad & (M[\tfrac{1}{f}], \widehat{M}, \mathrm{can}) \end{aligned}$$

is an equivalence of categories. Moreover, it restricts to an equivalence of categories on finite projective modules.

Here the category on the right is given by triples (N_1, N_2, α) where N_1 is an $A[\frac{1}{f}]$-module, N_2 is an $\widehat{A}$-module, and α is an $\widehat{A}[\frac{1}{f}]$-linear isomorphism

$$\alpha : N_1 \otimes_{A[\frac{1}{f}]} \widehat{A}[\tfrac{1}{f}] \to N_2 \otimes_{\widehat{A}} \widehat{A}[\tfrac{1}{f}].$$

The functor is given by sending $M \mapsto (M[\frac{1}{f}], \widehat{M}, \mathrm{can})$.

Proof. We explain how the inverse is constructed: Let (N_1, N_2, α) be an object in the RHS. We define M as the kernel of the natural map

$$0 \to M \to N_1 \oplus N_2 \xrightarrow{\alpha, -\mathrm{Id}} N_2 \otimes_{\widehat{A}} \widehat{A}[\tfrac{1}{f}].$$

Exercise in commutative algebra: This defines an inverse. Hint: Use the exact sequence

$$0 \to A \to A[\tfrac{1}{t}] \times \widehat{A} \to \widehat{A}[\tfrac{1}{f}] \to 0$$

and the fact that $A \to A[\frac{1}{t}] \times \widehat{A}$ is faithfully flat (flatness uses A Noetherian).

The statement about finite proj. modules follows by fpqc-descent along $A \to A[\frac{1}{f}] \times \hat{A}$. □

The following Corollary explains what the Lemma means geometrically

Corollary 12.24. *Let X be a Dedekind scheme and let $x \in X$ be a closed point. Let $\widehat{X} := \operatorname{Spec}(\hat{\mathcal{O}}_{X,x}) \to X$ be the completion at x. Then the natural functor*

$$\mathrm{Bun}_X \to \mathrm{Bun}_{X\setminus\{x\}} \times_{\mathrm{Bun}_{\widehat{X}\setminus\{x\}}} \mathrm{Bun}_{\widehat{X}}$$

is an equivalence of categories.

Proof. By passing to an open neighbourhood of x, we can without loss of generality assume that X is affine. After shrinking $X = \operatorname{Spec}(A)$ if necessary, x is cut out by a single $f \in A$ since X is Dedekind, and we can apply Beauville–Laszlo. □

Definition 12.25. A (finite free) Breuil–Kisin–Fargues module (BKF-module) is a finite free $\mathbb{A}_{\mathrm{inf}}$-module M together with an $\mathbb{A}_{\mathrm{inf}}$-linear isomorphism

$$\varphi_M : \varphi^* M[\tfrac{1}{\varphi(\xi)}] \simeq M[\tfrac{1}{\varphi(\xi)}]$$

A Breuil-Kisin-Fargues module can be thought of as a, not necessarily minuscule, Dieudonné module in mixed-characteristic.

Theorem 12.26 (Fargues, [22, Thm 14.1.1])**.** *The following categories are equivalent:*

1. *Breuil–Kisin–Fargues modules,*
2. *Quadruples $(\mathcal{F}, \mathcal{F}', \beta, T)$ consisting of vector bundles $\mathcal{F}, \mathcal{F}'$ on X_{FF} such that $\mathcal{F}$ is trivial, $\beta : \mathcal{F}_{|X_{FF}\setminus\{\infty\}} \to \mathcal{F}'_{|X_{FF}\setminus\{\infty\}}$, and $T \subseteq H^0(X_{\mathrm{FF}}, \mathcal{F})$ is a $\mathbb{Z}_p$-lattice,*
3. *Pairs (T, Ξ) where T is a finite free $\mathbb{Z}_p$-module and $\Xi \subseteq T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$ is a B_{dR}^+-lattice.*

This restricts to an equivalence of categories

1. *BKF-modules such that $M \subseteq \varphi_M(M) \subseteq \frac{1}{\varphi(\xi)} M$,*
2. *Quadruples for which β extends to a minuscule modification,*
3. *Pairs where Ξ is minuscule,*
4. *p-divisible groups over $\mathcal{O}_C$.*

Proof. We first show that 2 and 3 are equivalent:

Recall that $H^0(X_{FF}, \mathcal{O}_{X_{\mathrm{FF}}}) = \mathbb{Q}_p$. Consequently, the category of trivial vector bundles on X_{FF} is equivalent to the category of finite dimensional $\mathbb{Q}_p$ vector space via the functors

$$\mathcal{F} \mapsto H^0(X_{\mathrm{FF}}, \mathcal{F}) \tag{4}$$

$$V \otimes_{\mathbb{Q}_p} \mathcal{O}_{X_{\mathrm{FF}}} \leftarrow\!\shortmid V. \tag{5}$$

Under this equivalence, the datum of $T \subseteq H^0(X_{\mathrm{FF}}, \mathcal{F})$ corresponds to a $\mathbb{Z}_p$-lattice $T \subseteq V$.

We now apply Beauville–Laszlo glueing to the diagram

$$\begin{array}{ccc} \mathrm{Spec}(B_{\mathrm{dR}}^+) & \xrightarrow{\iota_\infty} & X_{\mathrm{FF}} \\ \uparrow & & \uparrow \\ \mathrm{Spec}(B_{\mathrm{dR}}) & \xrightarrow{\iota_{\infty,\eta}} & X_{\mathrm{FF}} \setminus \{\infty\} \end{array}$$

Starting with $(\mathcal{F}, \mathcal{F}', \beta, T)$, we now have an isomorphism

$$\iota_{\infty,\eta}^*\beta : T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}} = i_{\infty,\eta}^*(T \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{\mathrm{FF}}\setminus\{\infty\}}) = \iota_{\infty,\eta}^*\mathcal{F}_{|X_{\mathrm{FF}}\setminus\{\infty\}} \xrightarrow{\beta} i_{\infty,\eta}^*\mathcal{F}'_{|X_{\mathrm{FF}}\setminus\{\infty\}}$$

and we can define Ξ as the preimage of $H^0(\mathrm{Spec}(B_{\mathrm{dR}}^+), \iota_\infty^*\mathcal{F}')$.

Conversely, given (T, Ξ), we obtain a vector bundle $\mathcal{F}'$ on X_{FF} by extending

$$\iota_{\infty,\eta}^*\mathcal{F}_{|X_{\mathrm{FF}}\setminus\{\infty\}} = T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$$

according to the B_{dR}^+-sublattice $\Xi \subseteq T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$.

$$\begin{array}{ccc} \Xi =: \iota_\infty^*\mathcal{F}' & \leftarrow\!\cdots\!\mapsto & \exists\mathcal{F}' \\ \downarrow & & \vdots \\ T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}} & \longleftarrow & T \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{\mathrm{FF}}\setminus\{\infty\}} \end{array} \qquad \text{via } \beta.$$

These two constructions are mutually inverse by Beauville–Laszlo.

1)$\Rightarrow$ 3) Given a BKF-module, we can associate to it a pair (T, Ξ) by sending

$$M \mapsto \big(T = (M \otimes_{A_{\mathrm{inf}}} W(C^\flat))^{\varphi_M \otimes \varphi = 1}, \Xi = M \otimes_{A_{\mathrm{inf}}} B_{\mathrm{dR}}^+\big).$$

One can check that $T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}} = M \otimes_{A_{\mathrm{inf}}} B_{\mathrm{dR}}$, so Ξ really defines a lattice (in fact, this is already true over the much smaller ring $\mathbb{A}_{\mathrm{inf}}[\frac{1}{\mu}]$ where $\mu = [\epsilon] - 1$) see [1, Lemma 4.26].

2) $\Rightarrow$ 1) (sketch) Conversely, given $\mathcal{F}'$ we get the associated BKF-module using the adic Fargues–Fontaine curve (which we didn't discuss). There is an analytic adic space

$$\mathcal{Y} = \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})(p[p^\flat] \neq 0)$$

such that $\mathcal{X}_{FF} = \mathcal{Y}/\varphi^{\mathbb{Z}}$. There is also a map $\mathcal{X}_{FF} \to X_{\mathrm{FF}}$. Pulling back along

$$\mathcal{Y} \to \mathcal{X}_{FF} \to X_{\mathrm{FF}}$$

we pick up a φ-action. By a Theorem of Kedlaya, any such a vector bundle on $\mathcal{Y}$ comes from an A_{inf}-module M. The descent datum along $\mathcal{Y} \to \mathcal{X}_{FF}$ is precisely the map φ_M.

The second part follows from the Theorem of Scholze–Weinstein. In fact, it turns out that one has $M \otimes \mathbb{A}_{\mathrm{crys}} = M_{\mathrm{crys}}(G)$. □

In summary, we have discussed functors

$$\left\{\begin{matrix} p\text{-divisible groups} \\ \text{over } \mathcal{O}_C \end{matrix}\right\} \to \left\{\begin{matrix} \text{minuscule modifications} \\ \text{of vector bundes on } X_{\mathrm{FF}} \end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix} \text{pairs } (T,\Xi) \text{ consisting of} \\ \bullet\, T \text{ finite free } \mathbb{Z}_p\text{-module,} \\ \bullet\, \Xi \subseteq T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}^+ \} \end{matrix}\right\}$$

$$G \quad \mapsto (T_pG \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{\mathrm{FF}}}, \mathcal{E}(G), \beta_G, T_pG) \mapsto \quad (T_pG, \Xi).$$

What is the Breuil–Kisin–Fargues module good for? In the case of $G = A[p^\infty]$, it is this thing that sees the crystalline realisation of G, namely $H^1_{\mathrm{crys}}(A_{\mathcal{O}_C/p})$. In this sense, the lattice in part 3 is from the Hodge-de Rham comparison.

In general, we can use BKF to recover the Dieudonné module of the special fibre G_k:

Proposition 12.27 ([22, Cor 14.4.4.]). *The Dieudonné module associated to G_k is $M \otimes_{\mathbb{A}_{\text{inf}}} W(k)$, where $\mathbb{A}_{\text{inf}} \to W(k)$ is the natural map.*

This shows the usefulness of BKF-modules. Without the intermediate step of passing to BKF-modules, it's not clear how to associate a Dieudonné module to (T, Ξ).

We end our discussion with a (very short) glimpse on the important paper [1], also called "BMS 1". In the case of $G = A[p^\infty]$ coming from an abelian variety, the Breuil–Kisin–Fargues module M is an $\mathbb{A}_{\text{inf}}$-module that recovers both the crystalline cohomology

$$H^1_{\text{crys}}(A_{\mathcal{O}_C/p}) = M \otimes_{\mathbb{A}_{\text{inf}}} \mathbb{A}_{\text{crys}}$$

As well as the étale cohomology, via

$$H^1_{\text{ét}}(A_C, \mathbb{Z}_p) = (M \otimes_{\mathbb{A}_{\text{inf}}} W(C^\flat))^{\varphi}.$$

Moreover, it allows for an *integral* comparison between the two.

It is natural to wonder whether a similar thing is possible for A replaced by any proper smooth formal scheme over $\mathcal{O}_C$. Amazingly, this turns out to be possible – it is the starting point of the work of Bhatt–Morrow–Scholze, [1]. The basic idea is to define a cohomology theory that takes values in Breuil–Kisin–Fargues modules and which specializes to étale, crystalline and de Rham-cohomology. This later culminated in the prismatic cohomology of Bhatt–Scholze, [3], which greatly simplified the subject of p-adic cohomology theories.

13. Lecture of 22.01.2020: The classification of vector bundles on X

In this lecture we want to sketch the proof of Theorem 11.14, i.e., that the functor

$$\mathcal{E}(-) \colon \varphi - \text{Mod}_{\breve{E}} \to \text{Bun}_X$$

from φ-modules over $\breve{E}$ to vector bundles on X is essentially surjective. The detailed proof we are following is presented in [10, Section 8.3.]. The proof of Theorem 11.14 can be reduced to the following statement.

Theorem 13.1. *Let $n \geq 0$. Then:*

(1) *If*

$$0 \to \mathcal{E} \to \mathcal{O}_X(1/n) \to \mathcal{F} \to 0$$

is a short exact sequence with $\mathcal{F}$ a torsion sheaf of degree 1, then $\mathcal{E} \cong \mathcal{O}_X^n$.

(2) *If*

$$0 \to \mathcal{E} \to \mathcal{O}_X^n \to \mathcal{F} \to 0$$

is a short exact sequence with $\mathcal{F}$ a torsion sheaf of degree 1, then for some $m \in \{1, \dots, m\}$

$$\mathcal{E} \cong \mathcal{O}_X(-1/m) \oplus \mathcal{O}_X^{n-m}.$$

Note that the assumptions of Theorem 13.1 are implied by the classification of vector bundles on X. The proof Theorem 11.14 for E needs Theorem 13.1 for E_h, $h \geq 1$.

We explain how the assumption of Theorem 13.1 is implied by results on p-divisible groups. The assumptions in both points in Theorem 13.1 are of a similar shape, which takes the following abstract form: Fix some vector bundle $\mathcal{E}' \in \mathrm{Bun}_X$ and consider the set of isomorphism classes of modifications

$$\{\mathcal{E} \subseteq \mathcal{E}' \mid \mathcal{E}'/\mathcal{E} \text{ is a torsion sheaf of degree } 1\}.$$

Then we want to draw a conclusion on the possible isomorphism types of $\mathcal{E}$. The torsion sheaf $\mathcal{E}'/\mathcal{E}$ is by assumption isomorphic to $i_*k(x)$ for the inclusion $i\colon \mathrm{Spec}(k(x)) \to X$ for some closed point $x \in X$. Let us fix such a closed point $x \in X$ and denote by $C := k(\infty)/E$ the corresponding untilt of F. Then we obtain the set

$$\mathcal{M}_{\mathcal{E}'} := \{\mathcal{E} \subseteq \mathcal{E}' \mid \mathcal{E}'/\mathcal{E} \cong i_*C\}.$$

Let $\mathcal{E}'(x)$ be the fiber $\mathcal{E}' \otimes k(x)$ of $\mathcal{E}'$ at x. The map

$$\mathcal{M}_{\mathcal{E}'} \to \mathbb{P}(\mathcal{E}'(x))(C),\ \mathcal{E} \mapsto \mathcal{E}'/\mathcal{E} \otimes C$$

is bijective and the different possible isomorphism types of $\mathcal{E}$ give a (highly interesting) decomposition[53] of the set

$$\mathbb{P}(\mathcal{E}'(x))(C) = \coprod_{[\mathcal{E}] \in \mathrm{Bun}_X/\mathrm{isom}} \mathbb{P}(\mathcal{E}'(x))(C)_{[\mathcal{E}]}.$$

For the rest we will assume that $E = \mathbb{Q}_p$ (although this is not sufficient for a full proof of Theorem 11.14 as we also have to consider unramified extension of E there).[54] We already saw that p-divisible groups over $\mathcal{O}_C$ give rise to minuscule modifications on the Fargues-Fontaine curve, cf. Corollary 12.18.

Fix $n \geq 1$ and a p-divisible group $H/\overline{\mathbb{F}}_p$ of dimension 1 and height n (by Dieudonné theory H is unique up to isomorphism).

We define

$$\mathcal{M}^{\mathrm{ad}}_{H,\eta}(C)$$
$$\{(G,\alpha) \mid G p\text{-divisible group over } \mathcal{O}_C, \alpha\colon G \otimes_R \mathcal{O}_C/p \cong H \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_C/p\},$$

thus $\mathcal{M}^{\mathrm{ad}}_{H,\eta}(C)$ are the "C-points of the adic generic fiber of the Lubin-Tate space". Non-canonically,

$$\mathcal{M}^{\mathrm{ad}}_{H,\eta}(C) \cong \mathfrak{m}_C^{n-1}.$$

On $\mathcal{M}^{\mathrm{ad}}_{H,\eta}(C)$ there exists the Gross-Hopkins period morphism

$$\pi_{\mathrm{GH}}\colon \mathcal{M}^{\mathrm{ad}}_{H,\eta}(C) \to,\ (G,\alpha) \mapsto (M(H) \otimes_{W(Fpbar)} C \overset{\alpha}{\cong} M(G_{\mathcal{O}_C/p}) \otimes_{\mathbb{A}_{\mathrm{crys}}} C \to \mathrm{Lie}(G))$$

where we identified

$$\mathbb{P}^{n-1}_C \cong \mathbb{P}(M(H) \otimes_{W(\overline{\mathbb{F}}_p)} C).$$

The image of Gross-Hopkins period morphism π_{GH} has been identified by Gross-Hopkins, [14], and Hartl, [13].

Theorem 13.2. *The morphism*

$$\pi_{\mathrm{dR}}\colon \mathcal{M}^{\mathrm{ad}}_{\mathrm{LT},\eta}(C) \to \mathbb{P}(\mathcal{E}(G)(x))(C) \cong \mathbb{P}^{n-1}(C)$$

is surjective.

---

[53] actually a stratification

[54] If E is arbitrary, one has to replace the constructions with p-divisible groups by their analogues for divisible $\mathcal{O}_E$-modules and the same arguments go through, cf. [10, Section 8.3.].

Note that this implies the first assumption of Theorem 13.1!

Conversely, the classification of vector bundles on X, for which there exists a proof avoiding p-divisible groups, implies Theorem 13.2. Namely, by Scholze-Weinstein, cf. Theorem 12.20 resp. [24], the category of p-divisible groups over $\mathcal{O}_C$ is equivalent to the data

$$\{(T,W) \mid T \text{ finite free } \mathbb{Z}_p-\text{lattice}, W \subseteq T \otimes_{\mathbb{Z}_p} C(-1) \text{ a subvectorspace}\}$$

(note that [24] needs the classification of vector bundles on X). Hence, it suffices to see that in the Lubin-Tate case the "admissible locus", i.e., the locus where the corresponding modification $\mathcal{E} \subseteq \mathcal{E}'$ is trivial, is the full $\mathbb{P}^{n-1}_C$. Let

$$0 \to \mathcal{E} \to \mathcal{O}_X(1/n) \to i_*(C) \to 0$$

be a short exact sequence. It suffices to see that $\mathcal{E}$ is semistable as $\mathcal{E}$ is of degree 0 and thus, if semistable, necessarily trivial (by the classification Theorem 11.14). Assume that $\mathcal{E}$ is not semistable. Then by the Harder-Narasimhan filtration there exists a subbundle

$$\mathcal{O}_X(\lambda) \subseteq \mathcal{E}$$

with $\lambda > 0$. But as $\mathrm{rk}(E) = n$ we can write $\lambda = \frac{d}{m}$ with $m < n$. But then

$$1/n < \lambda$$

which implies that every morphism $\mathcal{O}_X(\lambda) \to \mathcal{O}_X(1/n)$ is trivial, a contradiction. Thus, $\mathcal{E}$ must be semistable.

Let us now pass to the second assumption in Theorem 13.1, that is $\mathcal{E}' \cong \mathcal{O}_X^n$. Let us describe the decomposition

$$\mathbb{P}(\mathcal{E}'(x))(C) = \coprod_{[\mathcal{F}] \in \mathrm{Bun}_X/\mathrm{isom}} \mathbb{P}(\mathcal{E}'(x))(C)_{[\mathcal{F}]} \tag{6}$$

assuming the classification of vector bundles on X. By Theorem 11.14

$$\mathbb{P}(\mathcal{E}'(x))(C)_{[\mathcal{F}]} \neq 0$$

if and only if $\mathcal{F} \cong \mathcal{O}_X^{n-m} \oplus \mathcal{O}_X(-1/m)$ for some $n \geq m \geq 0$. Let

$$0 \to \mathcal{E} \to \mathcal{E}' \to i_* C \to 0$$

be a short exact sequence. Then $\mathcal{E} \cong \mathcal{O}_X(-\frac{1}{n})$ if and only if $H^0(X, \mathcal{E}) = 0$, or equivalently, that the morphism

$$\mathbb{Q}_p^n \cong H^0(X, \mathcal{E}') \to H^0(X, i_* C) \cong C$$

is injective. This condition defines precisely Drinfeld's upper halfplane.

Definition 13.3. Drinfeld's upper halfplane $\Omega^{n-1} \subseteq (\mathbb{P}^{n-1}_C)^{\mathrm{ad}}$ is defined as the (open) complement of all $\mathbb{Q}_p$-rational hyperplanes in the adic space $(\mathbb{P}^{n-1}_C)^{\mathrm{ad}}$ associated to the scheme $\mathbb{P}^{n-1}_C$ over C.

If $n = 2$, then Ω^1 is the complement of the profinite set $\mathbb{P}^1(\mathbb{Q}_p)$ in $(\mathbb{P}^{n-1}_C)^{\mathrm{ad}}$. This of course resembles the formula

$$\mathbb{H}^{\pm} = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$$

for the classical upper/lower halfplane.

By a small argument, the decomposition in (Equation (6)) is therefore given by a disjoint union into Drinfeld spaces for varying $\mathbb{Q}_p$-rational linear subspaces of $\mathbb{P}(\mathcal{E}'(x))$.

In [30] Drinfeld introduced a certain deformation space

$$\mathcal{M}_{\mathrm{Dr}}$$

of p-divisible groups with additional structure (cf. [19]) and constructed a period morphism

$$\pi_{\mathrm{dR}} \colon \mathcal{M}^{\mathrm{ad}}_{\mathrm{Dr},\eta}(C) \to \mathbb{P}^{n-1}(C).$$

Moreover, he proved the following theorem on the image of π_{dR}.

Theorem 13.4 (Drinfeld [30]). *The image of*

$$\pi_{\mathrm{dR}} \colon \mathcal{M}^{\mathrm{ad}}_{\mathrm{Dr},\eta,C} \to \mathbb{P}^{n-1}_C$$

is precisely Ω^{n-1}.

This implies finally the second assumption of Theorem 13.1 and thus our sketch of proof for Theorem 11.14.[55]

14. Lecture of 29.01.2020: The theorem "weakly admissible implies admissible"

In this final lecture we can present the proof of Theorem 2.11 that "weakly admissible implies admissible" following [10, Section 10.5.3.].

For this lecture let K be a discretely valued non-archimedean extension of $\mathbb{Q}_p$ with perfect residue field $k = k_K$. Let

$$K_0 = W(k)[1/p] \subseteq K$$

be the maximal unramified subextension of K. Moreover, let $\overline{K}$ be an algebraic closure of K, set

$$G_K := \mathrm{Gal}(\overline{K}/K)$$

and define

$$C := \widehat{\overline{K}}$$

and

$$F := C^\flat = \varprojlim_{x \mapsto x^p} C.$$

Note that the action of G_K on $\overline{K}$ extends by continuity to C and then by functoriality to F. Finally, let

$$X = X_{\mathbb{Q}_p,F}$$

be the Fargues-Fontaine curve associated with $\mathbb{Q}_p$ and F, and let $\infty \in X$ be the closed point determined by C, i.e., ∞ is the vanishing locus of the Galois stable line $\mathbb{Q}_p t \subseteq B^{\varphi=p}$ with $t = \log[\varepsilon]$ for $1 \neq \varepsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \in C^\flat$. Note that G_K acts on $\mathbb{Q}_p t$ according to the cyclotomic character

$$\chi_{\mathrm{cycl}} \colon G_K \to \mathbb{Z}_p^*.$$

Write

$$B^+_{\mathrm{dR}} := \widehat{\mathcal{O}}_{X,\infty}$$

for the completion of X at x with fraction field B_{dR} and let

$$B^+_{\mathrm{crys}},\ B_{\mathrm{crys}} = B^+_{\mathrm{crys}}[1/t],\ B_e := B^{\varphi=\mathrm{Id}}_{\mathrm{crys}} = H^0(X \setminus \{\infty\}, \mathcal{O}_X)$$

[55] The lecture ended by a very rough introduction to the relative Fargues-Fontaine curve and local Shimura varieties which is omitted in the notes. We refer to [22], [9] for more material on these fascinating topics.

be the various period rings. The Galois group of G_K acts compatibly on

$$X, B_{\mathrm{dR}}^+, B_{\mathrm{crys}}, B_e, \ldots.$$

First of all let us recall the statement of Theorem 2.11.

Theorem 14.1 (Colmez-Fontaine [7])**.** *The category of crystalline Galois representations of G_K is equivalent to the category $\varphi-\mathrm{FilMod}_{K/K_0}^{\mathrm{wa}}$ of weakly admissible filtered φ-modules for K.*

For the proof we will find fully faithful embeddings

$$\mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to \mathrm{Bun}_X^{G_K},\ \varphi-\mathrm{FilMod}_{K/K_0} \to \mathrm{Bun}_X^{G_K}$$

into the category of G_K-equivariant vector bundles on X. Let $\mathcal{C}$ be the intersection of the essential images. Then $\mathcal{C}$, seen as a full subcategory of $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ identifies with crystalline Galois representations while $\mathcal{C}$, seen as a full subcategory of $\varphi - \mathrm{FilMod}_{K/K_0}$, identifies with the subcategory of weakly admissible filtered φ-modules.

We won't be able to provide many details and refer to [10, Section 10.5.3.] for complete proofs.

We need the following theorem of Tate, which in a weaker form was already mentioned in the first lecture, cf. Theorem 2.3.

Theorem 14.2. *Let $\chi\colon G_K \to \mathbb{Z}_p^*$ be a continuous character and denote by $I_K \subseteq G_K$ the ramification subgroup. Then*

$$H^i_{\mathrm{cts}}(G_K, C(\chi)) = \begin{cases} 0, & \text{if } i \geq 2 \text{ or } i \text{ arbitrary and } \chi(I_K) \text{ is infinite} \\ \cong K, & \text{otherwise} \end{cases}$$

Proof. Cf. [28]. □

With this theorem we can determine the invariants of G_K in the various period rings.

Lemma 14.3. *The following statements hold true:*

(1) $K = B_{\mathrm{dR}}^{G_K}$
(2) $K_0 = B_{\mathrm{crys}}^{G_K}$. *In fact, the canonical morphism $K \otimes_{K_0} B_{\mathrm{crys}} \to B_{\mathrm{dR}}$ is injective.*
(3) $K_0 = B_e^{G_K}$.

Proof. The filtration $\{t^n B_{\mathrm{dR}}^+\}_{n\in\mathbb{Z}}$ of B_{dR} has associated gradeds, as Galois modules, given by

$$\{C(n)\}_{n\in\mathbb{Z}}$$

where $C(n)$ denotes the twist of C by the n-th power χ_{cycl}^n of the cyclotomic character. As these powers for $n \neq 0$ all have infinite image, $C(n)^{G_K} = 0$ for $n \neq 0$ by Theorem 14.2. This easily implies $B_{\mathrm{dR}}^{G_K} = K$. All other claims follow easily from this and the injectivity of $K \otimes_{K_0} B_{\mathrm{crys}} \to B_{\mathrm{dR}}$. This injectivity is proven in [10, Corollaire 10.2.8.]. □

Let

$$\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$$

be the category of continuous representations of G_K on finite dimensional $\mathbb{Q}_p$-vector spaces.

Definition 14.4. A representation $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is called crystalline if the canonical morphism

$$(V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})^{G_K} \otimes_{K_0} B_{\mathrm{crys}} \to V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}$$

is an isomorphism.

For example, if G is a p-divisible group over $\mathcal{O}_K$, then the rational Tate module

$$V_p(G) = T_pG(C)[1/p]$$

is a crystalline G_K-representation by Proposition 12.16. Replacing B_{crys} by B_{dR} in Definition 14.4 one obtains the notion of a de Rham p-adic representation of G_K.

One can show that a representation $V \in \mathrm{Rep}_{\mathbb{Q}_p} G_K$ is crystalline if and only if

$$\dim_{K_0}(V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})^{G_K} = \dim_{\mathbb{Q}_p} V.$$

We will now pass to equivariant vector bundles on X.

Definition 14.5. Let $\mathcal{E} \in \mathrm{Bun}_X$. A G_K-action on $\mathcal{E}$ is the data of isomorphisms

$$c_\sigma \colon \sigma^*\mathcal{E} \cong \mathcal{E}$$

for each $\sigma \in G_K$ such that $c_{\sigma\tau} = c_\tau \circ \tau^*(c_\sigma)$ for all $\sigma, \tau \in G_K$.

Note that we did not demand any continuity of the action, and we will have to fix this. As remarked G_K acts on X leaving the point ∞ fixed. Moreover, the G_K-action on B_{dR}^+ is continuous for the canonical topology on B_{dR}^+.

As there is a G_K-equivariant morphism

$$\mathrm{Spec}(B_{\mathrm{dR}}^+) \to X$$

any G_K-action on a vector bundle $\mathcal{E} \in \mathrm{Bun}_X$ gives rise to a semilinear G_K-action on the finite free B_{dR}^+-module $\mathcal{E}_\infty^\wedge := \mathcal{E} \otimes_{\mathcal{O}_X} B_{\mathrm{dR}}^+$. If R is any topological ring, then by invariance of the product topology on R^n under the group $\mathrm{GL}_n(R)$, any finite free R-module has a canonical topology. This applies for example to $R = \mathbb{Q}_p$ or $R = B_{\mathrm{dR}}^+$.

Lemma 14.6. *Let V be a finite dimensional $\mathbb{Q}_p$-vector space with an action of (the abstract group) G_K. Then the action morphism $G_K \times V \to V$ is continuous*[56]*, i.e., V is a p-adic representation of G_K, if and only if the semilinear action morphism*

$$G_K \times (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+) \to V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+$$

is continuous.

Proof. If the G_K-action on V is continuous, then clearly the G_K-semilinear action on $V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+$ is continuous. Conversely, the canonical topology V is the subspace topology for the canonical topology on the B_{dR}^+-module $V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+$. □

This motivates the following definition of an equivariant G_K-bundle on X.

Definition 14.7. An G_K-equivariant vector bundle on X is a pair $(\mathcal{E}, (c_\sigma)_{\sigma \in G_K})$ of a vector bundle $\mathcal{E} \in \mathrm{Bun}_X$ and an action $(c_\sigma)_{\sigma \in G_K}$ of G_K on it such that the associated semilinear G_K-action on the finite free B_{dR}^+-module $\mathcal{E}_\infty^\wedge$ is continuous. Let us denote the category of equivariant G_K-vector bundles by $\mathrm{Bun}_X^{G_K}$.

[56]Or equivalently, the morphism $G_K \to \mathrm{GL}(V)$ with $\mathrm{GL}(V)$ equipped with the canonical topology, is continuous.

As a corollary of the classification of vector bundles on X, cf. Theorem 11.14 we obtain the following.

Corollary 14.8. *The functor*

$$\mathrm{Rep}_{\mathbb{Q}_p} G_K \to \mathrm{Bun}_X^{G_K},\ V \mapsto V \otimes_{\mathbb{Q}_p} \mathcal{O}_X$$

is fully faithful with essential image all G_K-equivariant vector bundles $(\mathcal{E}, (c_\sigma)_{\sigma \in G_K})$ whose underlying vector bundle $\mathcal{E}$ is semistable of slope 0, i.e., trivial.

Next let us introduce the category of filtered φ-modules over K and see how it embeds into the category $\mathrm{Bun}_X^{G_K}$ of equivariant vector bundles.

Definition 14.9. A filtered φ-module $(D, \varphi_D, \mathrm{Fil}^\bullet)$ over K is a φ_D-module $(D, \varphi_D) \in \varphi - \mathrm{Mod}_{K_0}$ together with a filtration $\mathrm{Fil}^\bullet$ on $D_K := D \otimes_{K_0} K$. We denote by

$$\varphi - \mathrm{FilMod}_{K/K_0}$$

the category of filtered φ-modules over K.

Using Fontaine's formalism of period rings it is not difficult to construct a functor

$$F \colon \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{crys}}(G_K) \to \varphi - \mathrm{FilMod}_{K/K_0}.$$

Namely, let V be a crystalline G_K-representation. Then

$$D := D_{\mathrm{crys}}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})^{G_K}$$

with φ_D induced by φ on B_{crys} is a φ-module over $K_0 = B_{\mathrm{crys}}^{G_K}$ of dimension $\dim_{\mathbb{Q}_p}(V)$. Moreover,

$$K \otimes_{K_0} D \cong D_{\mathrm{dR}}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K}$$

acquires naturally a filtration $\mathrm{Fil}^\bullet$ from B_{dR}. Sending

$$V \mapsto (D, \varphi_D, \mathrm{Fil}^\bullet)$$

is our desired functor F. Using the fundamental exact sequence of p-adic Hodge theory, cf. Theorem 10.1, it is not difficult to see that

$$V \cong \mathrm{Fil}^0((B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V)^{\varphi=1}),$$

which implies that F is fully faithful (but we don't know that its images are weakly admissible yet).

The category of filtered φ-modules is not abelian, but naturally an exact category by declaring that a sequence

$$(D_1, \varphi_{D_1}, \mathrm{Fil}^\bullet) \to (D_2, \varphi_{D_2}, \mathrm{Fil}^\bullet) \to (D_3, \varphi_{D_3}, \mathrm{Fil}^\bullet)$$

is exact if it is exact on the associated gradeds. To each finite dimensional K-vector space W with a filtration $\mathrm{Fil}^\bullet$ one can associate the degree

$$\deg(W, \mathrm{Fil}^\bullet) := \sum_{i \in \mathbb{Z}} i \dim_K \mathrm{gr}^i W \in \mathbb{Z}.$$

Taking as the "generic fiber functor" the functor

$$(W, \mathrm{Fil}^\bullet W) \mapsto W$$

one obtains a Harder-Narasimhan formalism for the category of filtered K-vector spaces, cf. Section 11. Define

$$\deg \colon \varphi - \mathrm{FilMod}_{K/K_0} \to \mathbb{Z},\ (D, \varphi_D, \mathrm{Fil}^\bullet) \mapsto \deg(D_K, \mathrm{Fil}^\bullet) - \deg(D, \varphi_D))$$

and

$$\mathrm{rk}(D, \varphi_D, \mathrm{Fil}^\bullet) := \dim_{K_0} D.$$

This yields a Harder-Narasimhan formalism for the category $\varphi - \mathrm{FilMod}_{K/K_0}$.[57]

In modern terminology the condition for a filtered φ-module to be weakly admissible is just semistability of slope 0.

Definition 14.10. A filtered φ-module $(D, \varphi_D, \mathrm{Fil}^\bullet)$ is weakly admissible if it is semistable of slope 0 (with respect to the above Harder-Narasimhan formalism). We denote by $\varphi - \mathrm{FilMod}^{\mathrm{wa}}_{K/K_0}$ the category of weakly admissible filtered φ-modules.

By the general Harder-Narasimhan formalism the category $\varphi - \mathrm{FilMod}^{\mathrm{wa}}_{K/K_0}$ is abelian.

We will now start to construct a fully faithful functor

$$\varphi - \mathrm{FilMod}_{K/K_0} \to \mathrm{Bun}_X^{G_K}.$$

We start by relating φ-modules with B_e-representations.

Definition 14.11. We denote by $\mathrm{Rep}_{B_e} G_K$ the category of finite locally free B_e-modules M with a semilinear G_K-action such that there exists a G_K-stable B_{dR}^+-lattice $\Xi \subseteq B_{\mathrm{dR}} \otimes_{B_e} M$ on which the G_K-action is continuous (with respect to the canonical topology on Ξ).

Clearly, there is a well-defined functor

$$\mathrm{Bun}_X^{G_K} \to \mathrm{Rep}_{B_e} G_K, \mathcal{E} \mapsto H^0(X \setminus \{\infty\}, \mathcal{E}).$$

Proposition 14.12. *The functors*

$$\mathcal{D}\colon \mathrm{Rep}_{B_e} G_K \to \varphi - \mathrm{Mod}_{K_0},\ W \mapsto (W \otimes_{B_e} B_{\mathrm{crys}})^{G_K}$$

and

$$\mathcal{V}\colon \varphi - \mathrm{Mod}_{K_0} \to \mathrm{Rep}_{B_e} G_K,\ (D, \varphi) \mapsto (D \otimes_{K_0} B_{\mathrm{crys}})^{\varphi_D \otimes \varphi = 1}$$

are adjoint. The functor $\mathcal{V}$ is fully faithful and $M \in \mathrm{Rep}_{B_e} G_K$ is in the essential image if and only if $\mathcal{D}(\mathcal{V}(M)) \cong M$.

Proof. For the proof we refer to [10, Proposition 10.2.12.]. □

Surprisingly, the category $\mathrm{Rep}_{B_e} G_K$ is abelian, cf. [10, Proposition 10.1.3.]. Indeed, by looking at the support of the cokernel[58] of a morphism of B_e-representations this is implied by the fact that the only G_K-invariant non-trivial closed subschemes of X are supported at ∞, cf. [10, Proposition 10.1.1.].

The filtration is brought into the picture by the following lemma.

Lemma 14.13. *Let W be a finite dimensional K-vector space. Then the map*

$$\{\mathrm{filtrations on} W\} \to \{G_K - \mathit{stable}\ B_{\mathrm{dR}}^+ - \mathit{lattices\ in}\ W \otimes_K B_{\mathrm{dR}}\}$$

defined by

$$\mathrm{Fil}^\bullet \mapsto \mathrm{Fil}^0(V \otimes_K B_{\mathrm{dR}})$$

is bijective with inverse $\Xi \mapsto \{(t^n \Xi)^{G_K} \subseteq (B_{\mathrm{dR}} \otimes_{B_{\mathrm{dR}}^+} \Xi)^{G_K} = W\}_{n \in \mathbb{Z}}$.

Proof. Cf. [10, Proposition 10.4.3.]. □

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[57] With generic fiber functor $F(D, \varphi_D, \mathrm{Fil}^\bullet) = D_K$.

[58] As B_e is a principal ideal domain, the kernel of each morphism of B_e-representations is again a B_e-representation, i.e., finite free over B_e.

We now can construct our desired fully faithful functor

$$\varphi - \mathrm{FilMod}_{K/K_0} \to \mathrm{Bun}_X^{G_K}.$$

Proposition 14.14. *The category $\mathcal{M}$ defined as the 2-pull back*

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \varphi - \mathrm{Mod}_{K_0} \\ \downarrow & & \downarrow \mathcal{V} \\ \mathrm{Bun}_X^{G_K} & \longrightarrow & \mathrm{Rep}_{B_e} G_K \end{array}$$

is equivalent to $\varphi - \mathrm{FilMod}_{K/K_0}$.

Proof. By Proposition 14.12 the functor

$$\varphi - \mathrm{Mod}_{K_0} \to \mathrm{Rep}_{B_e} G_K$$

is fully faithful. Therefore the proposition follows from Lemma 14.13 invoking the Beauville-Laszlo lemma, cf. Lemma 12.23. □

Let us denote by

$$\mathcal{E}(-) \colon \varphi - \mathrm{FilMod}_{K/K_0} \cong \mathcal{M} \to \mathrm{Bun}_X^{G_K}$$

the (fully faithful) functor deduced from Proposition 14.14.

Before sketching the proof of Theorem 14.1 we need the following lemma.

Lemma 14.15. *The functor*

$$\mathcal{E}(-) \colon \varphi - \mathrm{FilMod}_{K/K_0} \to \mathrm{Bun}_X^{G_K}$$

preserves degrees and Harder-Narasimhan filtrations.[59]

Proof. See [10, Lemme 10.5.5.] and [10, Proposition 10.5.6.]. □

We can now deduce Theorem 14.1.

Proof. (of Theorem 14.1) By Theorem 11.14 the functor

$$\mathrm{Rep}_{\mathbb{Q}_p} G_K \to \mathrm{Bun}_X^{G_K}, \ V \mapsto V \otimes_{\mathbb{Q}_p} \mathcal{O}_X$$

is fully faithful with essential image given by the category $\mathrm{Bun}_X^{G_K,\mathrm{sst},0}$ of all equivariant vector bundles on X which are semistable of slope 0. Thus by Lemma 14.15 we obtain a cartesian diagram

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \cong \mathrm{Bun}_X^{G_K,\mathrm{sst},0} & \longrightarrow & \mathrm{Bun}_X^{G_K} \\ \uparrow & & \uparrow \\ \varphi - \mathrm{FilMod}^{\mathrm{wa}}_{K/K_0} & \longrightarrow & \varphi - \mathrm{FilMod}_{K/K_0} \end{array}$$

But using Proposition 14.14 we can calculate the fiber product differently, namely

$$\begin{aligned} & \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \times_{\mathrm{Bun}_X^{G_K}} \varphi - \mathrm{FilMod}_{K/K_0} \\ \cong \ & \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \times_{\mathrm{Bun}_X^{G_K}} (\mathrm{Bun}_X^{G_K} \times_{\mathrm{Rep}_{B_e}(G_K)} \varphi - \mathrm{Mod}_{K/K_0}) \\ \cong \ & \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \times_{\mathrm{Rep}_{B_e}(G_K)} \varphi - \mathrm{Mod}_{K/K_0}. \end{aligned}$$

[59]The Harder-Narasimhan on Bun_X yields one on $\mathrm{Bun}_X^{G_K}$ by the canonicity of the Harder-Narasimhan filtration.

But this 2-fiber product is precisely the category $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p} G_K$ of crystalline Galois representations! Namely, by the adjunction in Proposition 14.12 for $V \in \mathrm{Rep}_{\mathbb{Q}_p} G_K$ the B_e-representation $V \otimes_{\mathbb{Q}_p} B_e$ lies in the image of the functor

$$\mathcal{V} \colon \varphi - \mathrm{Mod}_{K_0} \to \mathrm{Rep}_{B_e} G_K$$

if and only if the canonical morphism

$$(V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})^{\varphi=1} \otimes_{B_e} B_{\mathrm{crys}} \to V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}$$

is an isomorphism, i.e., if and only if V is crystalline. This finishes the sketch of proof. □

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