

## The vector bundles $\mathcal{O}_X(n)$

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$E/\mathbb{Q}_p$  fin.,  $\pi \in \mathcal{O}_E^\times \rightarrow \mathbb{F}_q$ ,  $F/\mathbb{F}_q$  non-arch, alg. cl.

$$X = \text{Proj} \left( \bigoplus_{d \geq 0} B^{\Psi=\pi} \right), \quad B = \text{completion of } B^b = A \text{ if } \left[ \frac{1}{n}, \frac{1}{\infty} \right] \\ \text{w.r.t. } (v_r)_{r \in (0, \infty)}$$

Today: 2) Construct vectorbundles  $\mathcal{O}_X(n)$ ,  $n \in \mathbb{Q}$ , on  $X$   
 (1) discuss Harder-Narasimhan formalism

## HN-formalism

$\mathcal{C}$  exact category (e.g.  $\mathcal{C} = \text{Bun}_{X/\mathbb{F}_1}, \text{Bun}_{\mathbb{P}^1_K}$ )

additive + notion of s.e.s.

with facts       $\deg: \text{Ob}(\mathcal{C}) \rightarrow \mathbb{Z}$       "degree"    ( $E \in \text{Bun}_X \rightsquigarrow \deg E := \deg L^{\text{rk } E}$ )  
 additive  
 in s.e.s.       $\text{rk}: \text{Ob}(\mathcal{C}) \rightarrow \mathbb{N}_{\geq 0}$       "rk"  
 $P_{\mathbb{Z}}(x) \stackrel{\deg}{\cong} \mathbb{Z}^1$

+  $F: \mathcal{C} \rightarrow \mathcal{A}$  exact, faithful  
 $\sim$ , abelian categories      last time  
 $(F = \text{"generic fiber"})$

such that  $F$  induces bijection for each  $E \in \mathcal{E}$

$\{\text{strict subobjects of } \mathcal{E}\} \xrightarrow{\sim} \{\text{subobjects of } F(\mathcal{E})\}$

2) If  $u: \mathcal{E} \rightarrow \mathcal{E}'$ , such that  $F(u)$  is an isom, then  $\deg \mathcal{E} \leq \deg \mathcal{E}'$   
 with equality iff  $u$  isom.

Def: 1)  $\mathcal{E} \in \mathcal{C}$   
 $\Rightarrow m(\mathcal{E}) := \frac{\deg \mathcal{E}}{\operatorname{rk} \mathcal{E}} \in \mathbb{Q} \cup \{\infty\}$  is the slope of  $\mathcal{E}$

2)  $\mathcal{E} \in \mathcal{C}$  semistable if  $\mu(F) \leq m(\mathcal{E})$  for all strict subobjects  $F \neq 0$ .

Ex:  $\mathcal{E} = \text{Bun}_X \cong \mathcal{O}_X(n)$  semistable,  $\mathcal{O}_X(n) \oplus \mathcal{O}_X(m)$  sst ( $\Rightarrow n=m$ )

Similarly,  $\text{Bun}_{\mathcal{O}_X^n}$

La:  $\mathcal{E}, \mathcal{E}' \in \mathcal{C}$ ,  $\mathcal{E}, \mathcal{E}'$  sst of slopes  $n, n'$ . If  $n > n'$ , then

$$\operatorname{Hom}_{\mathcal{C}}(\mathcal{E}, \mathcal{E}') = 0$$

Prf (Exercise, in the cases of interest)

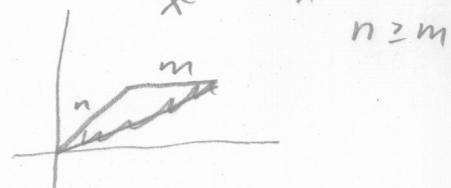
Prop: Each  $\mathcal{E} \in \mathcal{C}$  has a unique functorial filtration, the HN-filtr,

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \dots \subsetneq \mathcal{E}_r = \mathcal{E}$$

such that  $\mathcal{E}_i/\mathcal{E}_{i-1}$  is sst. for each  $1 \leq i \leq r$  and the sequence of slopes  $\mu(\mathcal{E}_i/\mathcal{E}_{i-1})$  is strictly decreasing

The HN-polygon is unique concave polygon in  $\mathbb{R}^2$  with origin  $(0,0)$  and slopes  $\mu(\mathcal{E}_i/\mathcal{E}_{i-1})$  with mult.  $\operatorname{rk}(\mathcal{E}_i/\mathcal{E}_{i-1})$ ,  $i=1, \dots, r$ .

$\mathcal{E}$  sst. ( $\Rightarrow$  HN-polygon is a line



Proof: If  $F(\mathcal{E})$  simple in  $A_n \Rightarrow \mathcal{E}$  sst.

Thus assume  $0 \rightarrow F \rightarrow \mathcal{E} \rightarrow g \rightarrow 0$  wt  $\operatorname{rk} F, \operatorname{rk} g < \operatorname{rk} \mathcal{E}$

By ind.  $F, g$  admit HN-filtr.

$\stackrel{(1a)}{\Rightarrow}$  slopes of strict subobjects of  $\mathcal{E}$  are bdd

Take  $F' \subseteq \mathcal{E}$  strict subobj of maximal slope whose rk is maximal.

Then can take  $\mathcal{E}_1 = F'$  (excuse). Uniqueness and naturality follow from La.

(3)

Prop Let  $\pi \in \mathbb{Q}_{\text{ess}}$ , then the subcategory

$$\mathcal{C}_\pi^{\text{sst}} = \{E \in \mathcal{C} \mid \mu \in \text{Ess}, \mu(E) \in \{\pi, \infty\}\}$$

is abelian and of finite length

D.F. (Exercise, in case of interest)

Ex:  $\mathcal{C} = \text{Bun}_{\mathbb{P}^1_K}$ ,  $E \simeq \bigoplus_{i=1}^{n_f} \mathcal{O}(d_i)^{n_{d_i}}$ ,  $n_i \in \mathbb{N}$  (almost all  $\neq 0$ )  
 $d_1 \neq d_2 \neq \dots \neq d_f$ ,  $n_i > 0$

$$= \text{End } \mathcal{O} \subseteq \mathcal{O}(d_1)^{n_1} \subseteq \mathcal{O}(d_1) \oplus \mathcal{O}(d_2)^{n_2} \subseteq \dots \subseteq E$$

HN-filtr.

$$\mathcal{C}_\pi^{\text{st}} \simeq \text{Vec}(\mathbb{k}) \quad \text{for } \pi \in \mathbb{Z}$$

$$\langle \overset{\circ}{\mathcal{O}(n)} \rangle$$

Def:  $\check{E} := W_E(\bar{\mathbb{F}}_q)[\frac{1}{\pi}]$  ~~DR~~,  $\bar{\mathbb{F}}_q = \text{alg. cl. of } \mathbb{F}_q \text{ in } F$

Note  $\check{E} \leq B$

Def: let  $A$  bearing with an endo,  $\varphi: A \rightarrow A$ . A  $\varphi$ -module over  $A$   
 is a fin. proj.  $A$ -module  $M$  with an isom.

$$\varphi_M: \varphi^*M \xrightarrow{\sim} M$$

$\varphi$ -Mod $_A$  = cat. of  $\varphi$ -mod. over  $A$

If  $M$  is free,  $e_1, \dots, e_n \in M$  basis

$$= \varphi_M(e_i \otimes 1) = \sum_{j=1}^n a_{ij} e_j \quad a_i := (a_{ij}) \in GL_n(A)$$

Changing  $e_1, \dots, e_n$  according to  $g \in GL_n(A)$  changes  
 $a$  to  $g a g^{-1}$ .

Thus isom. cl of free,  $\varphi$ -n  $\varphi$ -modules  $\xrightarrow{\text{1.3}} \text{GL}_n(E)/\varphi\text{-conjugacy}$  (4)

Assume  $A = E$ ,  $\varphi = \text{Frobenius on } E$

~~Well known~~ Have  $\deg: \{\varphi\text{-modules of rk } 1\}_{\text{tors}} \xrightarrow{\sim} E^\times/\varphi\text{-conj} \xrightarrow{\sim} \mathbb{Z}$   
val. on  $E$

Set  $\mathcal{C} = A = \varphi\text{-Mod}_E$  ("isocrystals if  $E/\mathbb{Q}_p$  unramified")

$$\text{rk } M := \dim_E M$$

$$\deg(M) := \deg(M^{\text{rk } M})$$

$\Rightarrow$  HN-formalism available

similarly for  $(\mathcal{C}, \text{rk}, -\deg)$

$\Rightarrow$  HN-filtr. can. split and  $\text{Hom}_{\mathcal{C}}(\mathcal{E}, \mathcal{E}') = 0$  if  $\mathcal{E}, \mathcal{E}'$  st of different slopes

For  $n \in \mathbb{Q}$ ,  $n = \frac{d}{r}$ ,  $d \in \mathbb{Z}, r > 0$  coprime, define

~~as in the  $\varphi$ -module~~  $D(n) := E^r$  with associated matrix

$$\varphi_{D(n)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

Thm (Dieudonné-Manin classification)  $\varphi\text{-Mod}_E$  is semisimple with simple objects given up to isom, by  $D(n)$  for  $n \in \mathbb{Q}$ . For  $n \in \mathbb{Q}$

the dir. alg.  $\text{End}_{\varphi\text{-Mod}_E}(D(n))$  over  $E$  is central of invariant  $\{[x] \in \text{Br}(E) \mid \exists \alpha \in \mathbb{Q}/\mathbb{Z}$   
depends on normalization

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Prf(Sketch): HN-formalism, + passage to unramified coverings of  $E$

( $\rightarrow$  replace  $\varphi$  by  $\varphi^h$ )

reduce proofs to

ess. pt: each semistable  $\varphi$ -module over  $\breve{E}$  at slope 0 is a direct sum  
of  $D(0) = (\breve{E}, \varphi)$ .

$$\text{By insp. } \mathrm{Ext}^1_{\varphi\text{-Mod}_{\breve{E}}} (D(0), D(0)) \cong \breve{E}/(\varphi - \mathrm{Id})\breve{E}$$

$$\bar{F}_q = \mathcal{O}_{\breve{E}}/\pi \text{ alg. closed} \stackrel{\cong}{\rightarrow} \mathcal{O}_{\breve{E}}/(\varphi - \mathrm{Id})\mathcal{O}_{\breve{E}} = 0 \Rightarrow \breve{E}/(\varphi - \mathrm{Id})\breve{E} = 0$$

check mod  $\pi$

Jnd, suff. ex. non-zero morph.  $D(0) \rightarrow D$

Write  $\varphi_D = a$ ,  $a \in \mathrm{GL}_n(\breve{E})$ . ~~BD of Repres~~  ~~$\mathrm{Id} \otimes a$~~   $\in \mathcal{O}_{\breve{E}}^\times$

$\Rightarrow$  After row operations a triangular with  $a_{11} \in \mathcal{O}_{\breve{E}}^\times$  ( $D$  isssst. pf. l. 0)

As  $\bar{F}_q$  alg. cld,  $a_{11} = \varphi^{(k)}/x$  for some  $x \in \mathcal{O}_{\breve{E}}$ .

$\Rightarrow$   $(\breve{E}, a_{11}, \varphi) \cong D(0)$  as desired

$\eta_1$   
 $D$

Note,  $\breve{E} \subseteq B$

Def:  $\mathcal{E}(-) : \varphi\text{-Mod}_{\breve{E}} \rightarrow \mathrm{Bun}_X$  in ~~BD~~,  $(D, \varphi_D) \mapsto \bigoplus_{d \geq 0} (B \otimes_{\breve{E}} D)^{\varphi^d} = \pi^{ad}$

E.g.,  $n \in \mathbb{Z} \rightsquigarrow D(n)$  is sent to  $\mathcal{O}_X(\# - n)$

At later,  $\mathcal{E}(-)$  well-defined (apriori  $\mathcal{E}(D, \varphi_D) \in \mathrm{Coh}_X$ )

For  $h \geq 0$  let  $\breve{E}_h$  = unramified ext. of  $E$  at day.  $h$ ,

$$X_h = \text{corresp. FF-arr} = \mathrm{Proj} \left( \bigoplus_{d \geq 0} B^{\varphi^h = \pi^d} \right)$$

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La:  $(D, \varphi_0) \in \varphi\text{-Mod}_{\mathbb{E}}$ ,  $d=0$

$$\stackrel{?}{=} \underset{\mathbb{E}}{\underset{E}{E_h \otimes (B \otimes D)}}^{\varphi \otimes \varphi_0 = \pi^d} \simeq (B \underset{\mathbb{E}}{\otimes D})^{\varphi^h \otimes \varphi_0^h = \pi^{hd}}$$

$$\#2) X \underset{\mathbb{E}}{\otimes E_h} \simeq X_h$$

$$\begin{array}{ccc} 3) & \varphi\text{-Mod}_{\mathbb{E}} & \xrightarrow{\mathcal{E}_{\mathbb{E}}(-)} \text{Coh}_{X_{\mathbb{E}}} \\ (D, \varphi_0) & \downarrow & \downarrow \text{--} \otimes E_h \\ (D, \varphi_0^h) & \varphi^h\text{-Mod}_{\mathbb{E}}^h & \xrightarrow{\mathcal{E}_{\mathbb{E}}(-)} \text{Coh}_{X_h} \end{array}$$

Pf (sketch): 1)  $\mathbb{Z}/h\mathbb{Z} \simeq \text{Gall}(E_h/E)$  acts on  $(B \underset{\mathbb{E}}{\otimes D})^{\varphi^h \otimes \varphi_0^h = \pi^{hd}}$

$E_h$ -semilinearly via  $\pi^{-d} \varphi \otimes \varphi_0$  with invariants

$$(B \underset{\mathbb{E}}{\otimes D})^{\varphi \otimes \varphi_0 = \pi^d} \stackrel{?}{=} \text{claim by Hilbert's 90}$$

= 2, 3)

~~sketch~~

=)  $\mathcal{E}_{\mathbb{E}}(-)$  takes values in v.b., because for  $n \in \mathbb{Q}$

$$(D(n), \varphi_{D(n)}) \simeq \bigoplus_{i=1}^{\frac{d}{r}} (\mathbb{E}, \pi^d \varphi)$$

La:  $n \in \mathbb{Q}$ ,  $n = \frac{d}{r}$ ,  $d \in \mathbb{Z}$ ,  $r > 0$ ,  $d, r$  coprime

$$\stackrel{?}{=} \mathcal{E}(D(n)) \simeq (f_r)_*(\mathcal{O}_{X_r}(d)) \quad , f_r: X_r \rightarrow X \text{ can propt.}$$

look at

Pf ~~sketch~~: Exercise (pullback to  $X_r$  and classify descent data)

In part (exercise):  $f_{r*}, f^*$  preserve semistability,  $\mathcal{O}_X(n)$  is semistable of slope  $n$

Main thm (Fargues Fontaine):  $\mathcal{E}(-)$  induces bij.

$$\Phi \text{-Mod}_{\mathbb{E}/\text{Isom}}^{\vee} \simeq \text{Bun}_{\mathbb{E}/\text{Isom}}.$$

(not equivalence, LHS ab., RHS not)

Proof (sketch): two lectures.

Modifications of v.b.

Prop: (Noetherian case of Beauville-Laszlo)

$A$  noth. ring,  $f \in A$  any element.

$$= \Phi: \text{Mod}_A \xrightarrow{\sim} \text{Mod}_{A[\frac{1}{f}]} \times_{\text{Mod}_{A[\frac{1}{f}]}} \text{Mod}_{A_f^1} \text{ can. functor.}$$

Similarly, for finite proj. modules

Concretely, RHS = cat. of triples  $(N_1, N_2, \alpha)$ ,  $\alpha$  ison.  $N_1 \otimes A[\frac{1}{f}] \simeq N_2 \otimes A[\frac{1}{f}]$

$$\Phi(M) = (M \otimes A[\frac{1}{f}], M \otimes A_f^1, \text{can})$$

In part,  $X$  FF (or any Dedekind scheme),  $\mathcal{E}$  v.b. on  $X$ ,  $x \in X$  cl.

$\lambda \in \text{Frac}(\widehat{\mathcal{O}}_{X,x}) \otimes \mathcal{E} = \bigoplus_x \widehat{\mathcal{O}}_{X,x}$  - lattice, obtain

"modification of  $\mathcal{E}$  at  $x$  via  $\lambda$ " as the v.b. with ass. to

$$\text{triple } (\mathcal{E}|_{X \setminus \{x\}}, \lambda, \text{can}: \lambda \otimes \text{Frac}(\widehat{\mathcal{O}}_{X,x}) \simeq \text{Frac}(\widehat{\mathcal{O}}_{X,x}, \mathcal{E}_x))$$

Prf (sketch): [SP, Tag 05ET]

Inverse functor:  $\mathcal{Y}(N_1, N_2, \alpha) \xrightarrow{\text{ker}} \text{ker}(N_1 \otimes N_2 \xrightarrow{(\text{can}, -\text{can})} N_2 \otimes A_f^1[\frac{1}{f}])$

Use of flatness of  $A \rightarrow A[\frac{1}{f}] \times A_f^1$  ( $A$  Noetherian), to check  $\Phi \circ \mathcal{Y} \circ \mathcal{J}_{\text{cl}} \simeq \mathcal{J}_{\text{cl}} \circ \Phi \circ \mathcal{Y}$