

The curve

(1)

Aim: Show $X = \text{Proj}(P)$, $P = \bigoplus_{d \geq 0} B^{\varphi = \pi^d}$

is a "curve".

(X is a bit similar to $\mathbb{P}_{\mathbb{E}}^1$)

~~Q~~ $E \rightsquigarrow \mathbb{A}$, $\mathbb{P}_{\mathbb{A}}^1$ analog of X

$\rightsquigarrow \mathbb{R}$, $\widetilde{\mathbb{P}}_{\mathbb{R}}^1 = V^+(x^2 + y^2 + z^2) \subseteq \mathbb{P}_{\mathbb{R}}^2$ analog of X)

$y \in |\mathbb{Y}| \rightsquigarrow \text{ex. } t \in B^{\varphi = \pi} \text{ (unique up to } E^x\text{), s.t.}$

$$\text{div}(t) = \sum_{n \in \mathbb{Z}} (\varphi^n)^*(y)$$

$\Pi(\xi_y) = t \left(= \frac{\prod_{n \geq 0} (\varphi^n)^*(\xi_y)}{\pi^n} \cdot \mathbb{Z} \right) \text{, with } \varphi(z) = \bar{z}$

$$\text{not } \frac{\prod_{n \geq 0} (\varphi^n)^*(\xi_y)}{\pi^n}$$

Thm ("Fundamental exact sequence of p-adic Hodge theory")

$y \in |\mathbb{Y}|$, $t := \Pi(\xi_y) \in B^{\varphi = \pi}, d \geq 0$

$$\Rightarrow 0 \rightarrow Et^d \rightarrow B^{\varphi = \pi^d} \rightarrow B_{dR, y}^+ / \xi_y^d B_{dR}^+ \rightarrow 0$$

is exact

$$\text{Prf: } x \in B^{\varphi = \pi^d} \cap \xi_y^d B_{dR}^+ \Rightarrow \text{div}(x) \geq d \cdot y$$

$$\Rightarrow \text{div}(x) \geq \text{div}(t) \Rightarrow x \in Et^d$$

$\text{div}(x)$ φ -equiv.

(2)

Ind. + C_y alg. cld \Rightarrow reduce Sury. of $B^{\varphi=\pi^d} \rightarrow B_{\text{dR}}^+ /_{\zeta_y^d} B_{\text{dR}}^+$

to case $d=1$

For simplicity, $E = \mathbb{Q}_p$ (in general use Lubin-Tate theory)

Then next la

$\text{La}(E = \mathbb{Q}_p)$, $\varepsilon = (1, \zeta_p, \dots) \in \mathcal{O}_{C_y}^\times \simeq \mathcal{O}_F^\times$, $t := \log[\varepsilon] \in B^{\varphi=p}$

$$\begin{aligned} \text{ex. } 1 &\rightarrow \varepsilon^{\otimes p} \rightarrow 1+m_F \rightarrow C_y \rightarrow \mathbb{Q} \\ &\quad \downarrow \quad \downarrow \log[-] \quad \| \\ 1 &\rightarrow \mathbb{Q}_p \cdot t \rightarrow B^{\varphi=p} \xrightarrow{G_y} C_y \rightarrow 0 \end{aligned}$$

Prf: $\log[-]$ is well-dfn., $1+m_F \rightarrow B$, $x \mapsto \log[x] = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{([x]-1)}{n}^n$

$$(\varphi(\log[x]) = \log([x]^p) = p \log[x] \Rightarrow \text{image in } B^{\varphi=p})$$

~~surjective~~

~~continuous~~ ~~closed~~ ~~containing~~ ~~$\log[B]$~~ ~~if $\varphi(\log[B])$~~

Have to show surjectivity.

Have comm. diag

$$1+m_F \xrightarrow{\log([x]-1)} B^{\varphi=p}$$

$$\text{surj. as } \rightarrow \downarrow (-)^{\#}$$

$$\text{alg. closed} \quad 1+m_C \xrightarrow{\quad} C_y$$

surj. as C_y alg. cld, and exp. local inverse

S:

Corollary: $P/\mathfrak{t}P \simeq \{f \in C_y[T] \mid f(0) \in E\}$

In particular, $\text{Proj}(P/\mathfrak{t}P) = \{(0)\} \hookrightarrow \text{Proj}(P)$, \propto_t image.

Prf: $\mathcal{O}_Y: B \rightarrow C_Y$ can. morph. (3)

$$\Rightarrow P/\epsilon P \cong S, \sum_{d \geq 0} x_d \mapsto \sum_{d \geq 0} \mathcal{O}_Y(x_d) T^d \text{ isom. bsp in deg. 0, surj.}$$

If $x \in P_d$, s.t. $\mathcal{O}_Y(x) = 0$

$\Rightarrow x \equiv t \cdot t' \pmod{\bigoplus_y^d B_y^+}$ with $t' \in B_{dR}^{++}$

$\Rightarrow x - t \cdot t' \in B \cap t^d$

Let $P \subseteq B/S$ be a graded prime ideal. If $cT^d \in P$, some $d \geq 1$

$$\Rightarrow cT^{d+1} = c^2 T \cdot cT^d \in P \Rightarrow P \notin \text{Proj}(RS) \quad \square$$

$\times P$ gen. by $P_1 \Rightarrow$ have line bds. $\mathcal{O}_X(n)$ (ass. to graded module $P[n]_d = P_{d+n}$)
 $d \in \mathbb{Z}$)

La: $H^0(X, \mathcal{O}_X(n)) = B^{\varphi=\pi^n}$

Prf: Have can. morph. $B^{\varphi=\pi^n} = P_n \rightarrow H^0(X, \mathcal{O}_X(n))$

P graded factorial \Rightarrow iso
 Exercise

Main result:

Thm (Forstner-Fontaine): $t \in P_1 = B^{\varphi=\pi}$ non-zero, then

$B_t := P[\frac{1}{t}]_0 = B[\frac{1}{t}]^{\varphi=1}$ is a principal ideal domain, and

$\text{Proj}(P) \cong \text{Spec } B_t \cup \{\infty_t\}$.

In part, X is noetherian and regular of Krull dim. 1

Prf: Show B_t is factorial and each (non-inv.) irreduc. elt. generates a maximal ideal

If $x \in B_t^{\varphi=\pi}$ then for some $d \geq 0$, $x = \frac{t'}{t^d}$ with $t' \in B^{\varphi=\pi^d} \setminus \{0\}$

\Rightarrow Can factor $t' = t_1 \cdots t_d$ with $t_i \in B^{\varphi=\pi}$

Prev. cor. each $\frac{t}{t'}$ either a unit or $(\frac{t}{t'}) \subseteq B_{t'}$ maximal. (7)

Pick $t, t' \in B^{\varphi=\pi} \setminus \{0\}$, non-colinear

$\Rightarrow X = \text{Spec}(B_t) \cup \text{Spec}(B_{t'}) \Rightarrow$ Thus \square

$|X| = \text{clsd. points of } X$

La: ex. bijections as for \mathbb{P}_E^1

$$B_{dR}/\mathcal{O}_{\mathbb{A}^1/\mathbb{Z}} = |X| \xrightarrow{\sim} (P_1 \setminus \{0\})/E^\times$$

If $y \in |Y|$ with image $x \in |X|$, then

$$B_{dR,x}^+ := \mathcal{O}_{X,x} \simeq B_{dR,y}^+$$

Prf: Exercise

Def

i) deg: $\text{Div}(x) \rightarrow \mathbb{Z}, \sum_{x \in |X|} n_x x \mapsto \sum_{x \in |X|} n_x$

ii) $f \in k(x)^*$ $\rightsquigarrow \text{div}(f) := \sum_{x \in |X|} \text{ord}_x(f) \cdot x$

function field
of X val. for $B_{dR,x}^+$

Prop: $f \in k(x)^* \Rightarrow \deg(\text{div}(f)) = 0$

and $\deg: \text{Pic}(X) \xrightarrow{\sim} \mathbb{Z}$, inverse $n \mapsto \mathcal{O}_X(n)$

Prf: Wlog $f = \frac{t'}{t}, t, t' \in B^{\varphi=\pi}$

$\Rightarrow \text{div}(f) = \infty_{t'} - \infty_t$ is of deg 0

Have s.e.s.

Ω

$0 \rightarrow \mathbb{Z}[\mathcal{O}_X(1)] \rightarrow \text{Pic}(X) \rightarrow \text{Pic}(\text{Spec } B_t) \rightarrow 0$

Proposition

$$\text{Prop: } H^i(X, \mathcal{O}_X(n)) = \begin{cases} B^{q=\pi^n}, & i=0 \\ 0, & i \geq 2 \\ 0, & i=1, n \geq 0 \end{cases}$$

$$B_{dR, X}^+ / \mathbb{F}\ell^n B_{dR}^+ + E, \quad i=1, n < 0 \quad (\ell \in X \text{ any pt})$$

Prf: In particular,

$$H^1(X, \mathcal{O}_X(-1)) \neq 0$$

Prf: $i=0: \checkmark$
 ~~$i \geq 2: \mathcal{O} \rightarrow \mathcal{O} \rightarrow 0$~~ : \checkmark (as X covered by two affine opens)

$i=1$: For $n \geq 0$, suff.

$$H^1(X, \mathcal{O}_X) = 0 \quad (\text{look at } \mathcal{O} \rightarrow \mathcal{O} \xrightarrow{\ell^n} \mathcal{O}_X(n) \rightarrow B_{dR, \mathcal{O}_X}^+ / \ell^n B_{dR}^+)$$

Put $t \in B^{q=\pi^n} \setminus \{0\}$, $x := \mathcal{O}_t$ $\rightsquigarrow H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X(n))$

Have $\mathcal{O} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{\text{Spec } B_t} \rightarrow B_{dR, X}^+ / B_{dR, X}^+ \rightarrow 0$

$\mathfrak{f}: \text{Spec } B_t \hookrightarrow X$ open immersion

Fundamental exact seq \Rightarrow

$$\mathcal{O} \rightarrow E \rightarrow B_t \rightarrow B_{dR, X}^+ / B_{dR, X}^+ \xrightarrow{\cong} H^1(X, \mathcal{O}_X) \rightarrow H^1(\text{Spec } B_t, \mathcal{O}_U) \rightarrow H^1(U, \mathcal{O}_U) = 0$$

$X \hookrightarrow \text{affine}$

$$\Rightarrow H^1(X, \mathcal{O}_X) = 0$$

For $n = 0$ use

$$\mathcal{O} \rightarrow \mathcal{O}_X(-n) \xrightarrow{\ell^n} \mathcal{O}_X \rightarrow B_{dR, X}^+ / \ell^n B_{dR, X}^+ \rightarrow 0$$

$$\Rightarrow H^1(X, \mathcal{O}_X(-n)) \cong B_{dR, X}^+ / \ell^n B_{dR}^+ + E$$

$$H^1(X, \mathcal{O}) = 0$$

Assume $E = \mathbb{Q}_p \rightsquigarrow$ can express X in terms of $B_{\text{crys}}^+ = A_{\text{crys}}[\frac{1}{p}]$ (6)

Def: 1) $B^{b,+} := A_{\text{inf}}[\frac{1}{p}]$

2) $I \subseteq (0, \infty)$ interval, $B_I^+ = \text{compl. of } B^{b,+}$ for $(r_i)_{i \in I}$
 = closure of $B^{b,+}$ in B_I^-

3) For $E = \{r\}$ set $B_r^+ := B_{\{r\}}^+$

4) $B^+ = B_{(0, \infty)}^+$ (= "functions on $\{Y\}$ which extend to boundary")

If $r' \leq r$, then

$$v_{r'}(x) \geq \frac{r'}{r} v_r(x)$$

$$\Rightarrow B_{r'}^+ \subseteq B_r^+ \quad \text{and} \quad B_r^+ = B_{(0, r]}^+$$

φ on $B^{b,+}$ induces

$$\varphi: B_r^+ \xrightarrow{\sim} B_{qr}^+ \subseteq B_r^+$$

La: $a \in m_F \setminus \{0\}$, $r := v(a)$. Then $B_r^+ = A_{\text{inf}}[\frac{[a]}{p}] \hat{\wedge} [\frac{1}{p}]$

Dot Fix C/\mathbb{Q}_p alg. closed, $P = (P, P^{\frac{1}{p}}, \dots)$

Recall: $A_{\text{crys}} = A_{\text{inf}}[\frac{[P^b]}{n!}]_P^1$

La: $r = v_C(P)$

$$\Rightarrow B_P^+ \subseteq B_{\text{crys}}^+ \subseteq B_{\text{crys}}^+|_r$$

$$\text{and } B^+ = \bigcap_{i=1}^{\infty} \varphi^n(B_{\text{crys}}^+) = \bigcap_{i=1}^{\infty} \varphi^n(B_r^+)$$

$$\text{Prf: } A_{\text{inf}}[\frac{[P^b]^p}{p}] \subseteq A_{\text{inf}}[\frac{[P^b]^n}{n!} |_{n \geq 0}] \subseteq A_{\text{inf}}[\frac{[P^b]}{p}],$$

Moreover $\varphi^n: B_r^+ \rightarrow B_r^+$ has image B_{nr}^+ and $B^+ = \bigcap_{n \geq 1} B_{nr}^+$ □

$$\text{Prop: } P = \bigoplus_{d \geq 0} B^{\varphi=p^d} \cong \bigoplus_{d \geq 0} (B^+)^{\varphi=p^d} = \bigoplus_{d \geq 0} (B_{\text{crys}}^+)^{\varphi=p^d} \quad (7)$$

Pf: Check if $x \in B^{\varphi=p^d} \Rightarrow \text{Neut}_{(0,0)}(x) \geq 0$

$$\& B^+ = \{x \in B \mid \text{Neut}_{(0,0)}(x) \geq 0\}$$

$$\varepsilon = (1, \zeta_p, \dots) \in \mathcal{O}_C^\flat, t := \log[\varepsilon]$$

$$\Rightarrow B_{\text{crys}} := B_{\text{crys}}^+ [\frac{1}{t}], B_e = (B_{\text{crys}})^{\varphi=1}$$

$$X = " \text{Spec}(B_e) \cup \text{Spec } B_{dR}^+"$$