

# The graded algebra P

$$P, E/\mathbb{Q}_p, \mathcal{O}_E, \pi, F_q, F, \mathcal{O} \in m_F \setminus \{0\}$$

Last time:  $I \subseteq (0, \infty)$  interval

$\Rightarrow B_I =$  completion of  $B^b = A_{\text{inf}}[\frac{1}{\pi}, \frac{1}{[\infty]}]$  for family  $(v_r)_{r \in I}$

$$(" = \mathcal{O}(|Y_I|)")$$

$$P := \bigoplus_{d \geq 0} P_d, \quad P_d = B^{\varphi = \pi^d}$$

Thm (Forques-Fontaine):  $P$  is graded factorial with irreducible elements of degree 1, i.e. the mult. monoid

$$\bigcup_{d \geq 0} (P_d \setminus \{0\})/E^x \text{ is free on } (P_1 \setminus \{0\})/E^x$$

In part, if  $d \geq 1$  and  $x \in P_d = B^{\varphi = \pi^d}$ , then ex.  $t_1, \dots, t_d \in B^{\varphi = \pi}$ , s.t.

$$x = t_1 \cdots t_d$$

Remark: \* This does not imply  $P \simeq \text{Sym}_{\mathbb{Q}_p}^{\bullet}(P_1)$

Need:

Thm: Assume  $I \subseteq (0, \infty)$ . Then  $B_I$  is a <sup>compact</sup> principal ideal domain

$$W \text{ with } \text{Spm } B_I \xrightarrow{\sim} |Y_I|$$

Use

La:  $A$  integral domain. Then  $A$  PID iff  $A$  factorial and each (non-invertible) irreducible element generates a maximal ideal

Pf: " $\Rightarrow$ "  $\checkmark$

" $\Leftarrow$ " Let  $\mathfrak{a} \in I \subseteq A$ , then  $\mathfrak{a} = \pi^{n_1} \cdots \pi^{n_k}$ ,  $\pi$  irreducible

$\Rightarrow I_{(a)} \subseteq A_{(a)}$  gen  $\simeq \prod_{i=1}^k A_{(a_i)}$  and  $I_{(a)}$  gen. by class of some divisor of  $a$  (2)

$\Rightarrow I$  principal

□

Thus sufficient to prove:

1)  $y \in |Y_I| \sim \Theta_y: B^b \rightarrow C_y$  extends to  $\Theta'_y: B_I \rightarrow C_y$  and  $\ker \Theta'_y = (\xi_y)$   
for  $(\xi_y) = \ker \Theta_y$

$(\Rightarrow (B_I)_{\xi_y}^+ \simeq B_{dR, y}^+)$

2)  $\exists f \in B_I \setminus \{0\}$ , s.t.  $\text{Newt}_I(f) = \emptyset$ , then  $f \in B_I^\times$

3)  $\exists f \in B_I \setminus \{0\}$ ,  $r$  slope of  $\text{Newt}_I(f)$ , then ex.  $y \in |Y_I|$ , s.t.  $f = \xi_y \cdot g$ ,  $g \in B_I$

$(\stackrel{1)}{\Rightarrow} \exists y \in |Y_I|$ , s.t.  $f(y) := \Theta'_y(f) = 0$

(Note:  $I$  compact  $\Rightarrow \text{Newt}_I(f)$  has only fin. many slopes  $\neq r$ .)

Moreover, the slopes of  $\text{Newt}_I(f \cdot g)$  are the concatenation of the slopes of  $\text{Newt}_I(f), \text{Newt}_I(g)$ .

Set  $r := v_y(\pi) = d(y, 0)$

For 1):  $\Theta_y: B^b \rightarrow C_y$  is cont. for  $v_r$ -top. on  $B^b$ . Indeed,

$x = \sum_{i \gg -\infty} [x_i] \pi^i \in B^b \Rightarrow \Theta_y(x) = \sum_{i \gg -\infty} \Theta_y([x_i]) \cdot \pi^i$

$\Rightarrow v_y(\Theta_y(x)) \geq \inf_{i \in \mathbb{Z}} \{v(x_i) + i \cdot \underbrace{v_y(\pi)}_{=r}\} = v_r(x)$

General fact:  $\ker \Theta'_y = \overline{(\xi_y)}$   $\leftarrow$  closure in  $B_I$

Let  $f \in \overline{(\xi_y)}$ ,  $f = \lim_{n \rightarrow \infty} f_n$ ,  $f_n = \xi_y \cdot g_n$

$\forall r \in I$   $v_r(g_n - g_m) = v_r(f_n - f_m) - v_r(\xi_y) = v_r(f_n - f_m) - r \neq \infty \Rightarrow (g_n)_n$  Cauchy  $\Rightarrow 1)$

For 2): Let  $I = [a, b]$ ,  $f_n \in B^b$ , s.t.  $f_n \rightarrow f, n \rightarrow \infty$

$\Rightarrow \text{Newt}_I(f_n) = \emptyset$  for  $n \gg 0$

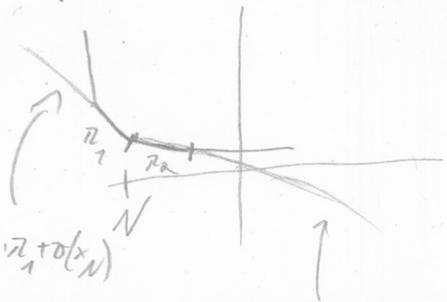
$\text{Newt}_I(f_n) \rightarrow \text{Newt}_I(f)$

$\Rightarrow$  wlog.  $f \in B^b \setminus \{0\}$  (use  $f \neq 0$  to see that  $f_n^{-1}$  converges)

Write  $f = \sum_{n \rightarrow -\infty} [x_n] \pi^n = \sum_{n > N} [x_n] \pi^n + \sum_{n \leq N} [x_n] \cdot \pi^n$

such that each slope of  $\text{Newt}(f)$  on  $(-\infty, N]$  is  $< -b$  (uses  $\text{Newt}_I(f) = \emptyset$ )  
on  $[N, \infty)$  is  $> -a$

Let  $r_1 =$  slope of  $\text{Newt}(f)$  on  $[N-1, N]$  ( $r_1 = \infty$  poss.)  
 $r_2 =$  slope of  $\text{Newt}(f)$  on  $[N, N+1]$



Then  $-r_1 > b \geq a > -r_2$  and

$v(x_n) \geq (n-N)r_1 + v(x_N), n \leq N$

$v(x_n) \geq (n-N)r_2 + v(x_N), n > N$

Write Set  $y := \sum_{n \leq N} [x_n] \pi^n$

$z := \sum_{n > N} [x_n] \pi^n$

Write  $y = [x_N] \cdot \pi^N (1 + \underbrace{\sum_{n < N} [x_n / x_N^{-1}] \pi^{n-N}}_{=: \tilde{y}})$

Claim:  $\tilde{y}$  top. nilp in  $B_I$ .

Let  $r \in I$ . Then

$v_r(\tilde{y}) = \inf_{n < N} \{v(x_n) - v(x_N) + r(n-N)\} \geq \inf_{n \leq N} \{(r_1 + r)(n-N)\} = -r_1 + r > 0$

Thus  $\eta = [x_N] \cdot \pi^N (1 + \tilde{\eta}) \in B_I$

Similarly write  $f = \eta(1 + \eta^{-1}z)$ . Note  $v_r(\eta) = v_r([x_N] \cdot \pi^N)$  for  $r \in I$

Similarly as above  
~~But~~  $\eta^{-1}z$  top. nilp.  $\Rightarrow 2)$

3) Use that  $|Y_{-n}|$  is complete and approximation.

Finished!  
#1  $B_I$  PID for  $I$  compact

Def:  $I \subseteq (0, \infty)$  interval

$\text{Div}^+(|Y_I|) =$  max partially ordered monoid of formal sums  $\sum_{y \in |Y_I|} n_y y$ , s.t.

for each  $J \subseteq I$  compact the set  $\{y \in |Y_J| \mid n_y \neq 0\}$  is finite

E.g.  $I$  compact  $\Rightarrow \text{Div}^+(|Y_I|) = \mathbb{N} \cdot |Y_I|$

Def:  $f \in B_I \setminus \{0\}$ .

$\text{div}(f) := \sum_{y \in |Y_I|} \text{ord}_y(f) y$ , where  $\text{ord}_y: B_{dR, y}^+ \rightarrow \mathbb{Z} \cup \{\infty\}$  is the "vanishing order at  $y$ ".

Prop:  $I \subseteq (0, \infty)$  any interval

$\Rightarrow \text{div}: (B_I \setminus \{0\}) / B_I^\times \rightarrow \text{Div}^+(|Y_I|)$

is injective. (bijective if  $I$  compact)

Moreover,  $\text{div}(f) \geq \text{div}(g)$  if and only if  $f \in g B_I$

Prof: Clear if  $I$  compact, in general write

$$B_I = \varinjlim_{\substack{J \subseteq I \\ \text{compact}}} B_J, \quad \text{Div}^+(|Y_I|) = \varinjlim_{\substack{J \subseteq I \\ \text{compact}}} \text{Div}^+(|Y_J|)$$

Recall:  $P = \bigoplus_{d \geq 0} P_d$ ,  $P_d = B^{\varphi = \pi d}$

(5)

La:  $B^{\varphi = \pi d} = \begin{cases} B^{\varphi = \pi d} & , d > 0 \\ E & , d = 0 \\ 0 & , d < 0 \end{cases}$  (~~last~~ last time  $B^{\varphi = \pi}$  big!)

Proof:  $d \leq 0 \Rightarrow$  Similar as for  $B^b$  (last time). Using

$$A_{\text{inf}} = \{ f \in B \mid \text{Newt}_{(0, \infty)}(f) \subseteq \mathbb{R}_{\geq 0}^2 \}$$

Def:  $\text{Div}^+(|Y|_{\varphi Z}) := \text{Div}^+(|Y|)^{\varphi Z} = \{ \sum_{\gamma \in |Y|} n_{\gamma} \gamma \mid \# \text{ invariant under } \varphi \}$

$$(\varphi^* \gamma := \varphi^{-1}(\gamma))$$

If  $I = \bigcap_{i=1}^r \langle \alpha_i \rangle \subset [a, \infty)$ , then  $\text{Div}^+(|Y|_{\varphi Z}) \cong \text{Div}^+(|Y|_I)$   
restr.

Thm:

$\text{div} : \bigcup_{d \geq 0} (P_d \setminus \{0\}) / E^{\times} \rightarrow \text{Div}^+(|Y|_{\varphi Z})$  is isom. of monoids

( $\Rightarrow$   $P$  graded factorial)

For  $x \in P_d, d \geq 0$

$$\varphi^*(\text{div}(x)) = \text{div}(\varphi(x)) = \text{div}(\pi^d x) = \text{div}(x)$$

$\Rightarrow$   $\text{div}$  well-dfn.

Proof: Let  $x \in P_d, y \in P_{d'}$  s.t.  $\text{div}(x) = \text{div}(y)$ . (6)

Wlog  $d' \geq d$ .

Then  $x = uy$  with  $u \in B^x, u \in B^{\varphi = \pi^{d-d'}} = \{0\}$   
 $\uparrow$   
 if  $d' > d$

$\Rightarrow d = d'$  and  $u \in (B^{\varphi = 1}) = E^x$

Let  $y \in |Y|$ . It suffices to show that the divisor

$$\sum_{n \in \mathbb{Z}} \varphi^{n*}(\varphi^{n*}(y)) \text{ in image of } \bigcup_{d \geq 0} (P_d / \{0\}) / E^x$$

Write  $\xi_y = \pi - [a]$

$$\text{Set } x := \prod_{n \geq 0} \left( 1 - \frac{[a]^{q^n}}{\pi} \right) = \prod_{n \geq 0} \frac{\varphi^n(\xi_y)}{\pi}$$

Then  $x$  converges in  $B$  (as  $[a]^{q^n} \rightarrow 0, n \rightarrow \infty$ )

$$\text{and } \text{div}(x) = \sum_{n \geq 0} \varphi^{n*}(y)$$

La: Let  $b \in B^b \cap W_{0,E}(F)^x$ . Then

$$\dim_E B^{\varphi=b} = 1$$

First Apply this to  $\xi_y$

$$\Rightarrow \exists z \in B, \text{ s.t. } \varphi(z) = b \xi_y$$

$$\Rightarrow \text{div}(z) = \text{div}(\varphi^{-1}(b \xi_y)) = \text{div}(\varphi^{-1}(b)) + \text{div}(\varphi^{-1}(\xi_y)) = (\varphi^{-1}(b)) + (\varphi^{-1}(\xi_y)) + \dots$$

$$\Rightarrow \text{div}(x \cdot z) = \sum_{n \in \mathbb{Z}} \varphi^{n*}(y)$$

Moreover,  $\varphi(x \cdot z) = \pi \cdot \frac{1}{\pi} \cdot x \cdot b \cdot \xi_y = \pi \cdot x \cdot z, \text{ i.e. } x \cdot z \in B^{\varphi=\pi}$

Remains proof of (a):

$$(W_{\mathcal{O}_E}(F) \begin{bmatrix} 1 \\ \pi \end{bmatrix})^{\varphi=1} = E \Rightarrow \dim_E B^{\varphi=b} \leq 1.$$

Wlog  $b \in A \setminus \pi A$  (multiply with Teichmüller lift)

Construct <sup>converging</sup>  $x_n \in A$ , s.t.  $x_n \notin \pi A$  and  $\varphi(x_n) \equiv bx_n \pmod{\pi^n}$ .

For  $n=1$ , take  $x_1 = [a]$  with  $a \in \mathcal{O}_F \setminus \{0\}$  sol.

$$\text{of } \varphi(x) \equiv bx \pmod{\pi} \quad (\text{i.e. } a^{q-1} \equiv b \pmod{\pi})$$

Assume  $x_n$  constructed.

$$\text{Write } \varphi(x_n) \equiv bx_n + \pi^n [z] \pmod{\pi^{n+1}}$$

For  $u \in \mathcal{O}_F$

$$\begin{aligned} \varphi(x_n + \pi^n [u]) &\equiv b(x_n + \pi^n [u]) - b[u] \cdot \pi^n + \pi^n [u^q] + \pi^n [z] \\ &\equiv -\pi^n (b \cdot u - [u^q] - z) \pmod{\pi^{n+1}} \end{aligned}$$

has solution in  $\mathcal{O}_F$  □