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SS 2024

Algebraic Geometry II

5. Exercise sheet

Exercise 1 (4 points):

Let A be a ring with $2 \in A^\times$ and let $g(x) = x^d + a_1 x^{d-1} + \ldots + a_d$ be a monic polynomial of degree $d \geq 2$. Consider the vanishing locus $X := V^+(z^{d-2}y^2 - z^d g(x/z)) \subseteq \mathbb{P}^2_A$.
i) Show that the morphism $X \to \mathbb{P}^1_A,\ [x, y, z] \mapsto [y, z]$ is flat. Deduce that $f\colon X \to S := \operatorname{Spec}(A)$ is flat.
ii) Let $X^{\mathrm{sm}} := \{x \in X \mid f \text{ is smooth at } x\}$. Show that $\{s \in S \mid f^{-1}(s) \subseteq X^{\mathrm{sm}}\}$ is the complement of the discriminant $\Delta_g \in A$ of g.
iii) If $g(x) = x^2 + ax + b$ or $g(x) = x^3 + ax + b$ write down Δ_g.

Exercise 2 (4 points):

In the situation of sheet 9, exercise 1 from Algebraic Geometry 1 assume that $s \in \mathcal{L}^n(S)$ is a generator, i.e., the morphism $\mathcal{O}_S \xrightarrow{s} \mathcal{L}^n$ is an isomorphism, and that $n \in \mathcal{O}_S(S)^\times$.
i) Show that the morphism $f\colon X \to S$ constructed in that exercise is finite, étale and surjective.
ii) Let k be an algebraically closed field and $n \geq 1$ invertible in k. Deduce from i) and results from the lecture that each unit $s \in k[[u]]^\times$ admits an n-th root.

Exercise 3 (4 points):

Let k be an algebraically closed field and let $f\colon Y \to X$ be a non-constant morphism of smooth curves over k. Let $y \in Y(k)$ with image $x := f(y)$ and assume that $f_y^*\colon \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is tamely ramified, i.e., $\mathfrak{m}_{X,x}\mathcal{O}_{Y,y} = \mathfrak{m}_{Y,y}^e$ with $e \geq 1$ invertible in k. Show that there exists isomorphisms $k[[u]] \cong \widehat{\mathcal{O}_{X,x}}$, $k[[v]] \cong \widehat{\mathcal{O}_{Y,y}}$ of k-algebras such that f_y^* identifies with the morphism $u \mapsto v^e$.
Remark/Hint: The statement is analogous to Exercise 3 on Sheet 1. Use exercise 2.

Exercise 4 (4 points):

Let $R \to A$ be a morphism of rings. For a surjection $\varphi\colon B \to A$ with kernel J from a polynomial R-algebra $B = R[T_i \mid i \in I]$ we define the *naive (or truncated) cotangent complex*

$$L_\varphi := [J/J^2 \xrightarrow{d} A \otimes_B \Omega^1_{B/R}]$$

placed in (homological) degrees $0, 1$. Let $\varphi\colon B \to A, \psi\colon C \to A$ be two surjections from polynomial R-algebras.
i) Show that there exists a morphism $f\colon B \to C$ with $\psi \circ f = \varphi$, and that f induces a natural morphism $h_f\colon L_\varphi \to L_\psi$ of complexes.
ii) Show that if $f, f'\colon B \to C$ are two morphisms as in i), then the morphisms $h_f, h_{f'}$ are homotopic.
iii) Show that L_φ and L_ψ are homotopy equivalent.
Hint: In ii) show that $f - f'$ defines an R-linear derivation from B into the A-module J_C/J_C^2, where $J_C := \ker(C \to A)$.

To be handed in on: Thursday, 16.05.2024 (during the lecture or via eCampus).