Dr. I. Gleason
Dr. J. Anschütz **SS 2024**

Algebraic Geometry II

4. Exercise sheet

Exercise 1 (4 points):

Let A be a ring. Let $M \xrightarrow{f} N \to Q \to 0$ be a sequence of finitely generated A-modules. Assume that N is finite projective and M finitely generated. Let $x \in X := \mathrm{Spec}(A)$ such that

$$0 \to M \otimes_A k(x) \to N \otimes_A k(x) \to Q \otimes_A k(x) \to 0$$

is an exact sequence. Show that there exists an open, affine neighborhood $U \subseteq X$ of x such that $f \otimes_A \mathcal{O}(U)$ is split injective.
Hint: Choose a morphism $g\colon F \to M$ of A-modules with F finite free and $F \otimes_A k(x) \xrightarrow{\cong} M \otimes_A k(x)$. Localize to represent $f \circ g$ by a matrix.

Exercise 2 (4 points):

Compute $\Omega^1_{B/A}$ for
i) $B = A[X]/(f(X))$ with $f(X) \in A[X]$ a polynomial.
ii) $B = \mathbb{Z}[i]$, $A = \mathbb{Z}$.
iii) $B = k[x,y]/(y^2 - x^3 - x)$ with $A = k$ not of characteristic 2.
iv) $B = k[x,y]/(xy)$, $A = k$.
Hint: Use one of the short exact sequence for $\Omega^1_{-/-}$ that where established in the lecture.

Exercise 3 (4 points):

Let A be a perfect $\mathbb{F}_p$-algebra, i.e., the Frobenius $\mathrm{Fr}_A\colon A \to A,\ x \mapsto x^p$ of A is bijective. Prove that $\mathrm{Spec}(A) \to \mathrm{Spec}(\mathbb{F}_p)$ is formally étale.
Hint: If R is a ring of characteristic p, $I \subseteq R$ an ideal with $I^2 = 0$ and $R_0 = R/I$, then for $x \in R$ the element x^p only depends on the image $x_0 \in R_0$ of x.

Exercise 4 (4 points):

Let A be a ring. Let $f_\bullet\colon C_\bullet \to D_\bullet$ be a morphism of complexes of A-modules. Let $0 \to D_\bullet \xrightarrow{g_\bullet} C(f_\bullet) \to C_\bullet[1] \to 0$ be the natural short exact sequence considered on sheet 3, exercise 4 (note that $d_n^{C_\bullet[1]} = -d_{n+1}^{C_\bullet}$). Show that there exists a natural homotopy equivalence

$$C(g_\bullet) \to C_\bullet[1].$$

Hint/Remark: Write down the definitions and have stamina. For the notion of a homotopy equivalence of chain complexes see Tag 010V in the Stacks project, which is available at `stacks.math.columbia.edu`.

To be handed in on: Thursday, 09.05.2024 (via eCampus).