Dr. I. Gleason **SS 2024**
Dr. J. Anschütz

Algebraic Geometry II

3. Exercise sheet

Exercise 1 (4 points):

Let $f\colon Y \to X$ be a dominant morphism of finite type between integral schemes. Let η_Y resp. η_X be the generic points of Y resp. X. Let $d := \mathrm{trdeg}(k(\eta_Y)/k(\eta_X))$. Show that there exists an open non-empty subscheme $U \subseteq X$, such that $\dim(f^{-1}(u)) = d$ for all $u \in U$.
Hint: Reduce to the case that Y is affine. Then use Noether normalization over $k(\eta_X)$ and a spreading out argument to find a non-empty open $U \subseteq X$ and a finite, surjective morphism $f^{-1}(U) \to \mathbb{A}^d_U$.

Exercise 2 (4 points):

Let A be a ring and let $C_\bullet : \ldots \to C_{i+1} \overset{d_{i+1}}{\to} C_i \overset{d_i}{\to} \ldots$ be a complex of A-modules. Assume that each C_i is a finite free A-module. For $i \in \mathbb{Z}$ consider the function

$$\beta_i\colon X := \mathrm{Spec}(A) \to \mathbb{Z}_{\geq 0},\ x \mapsto \dim_{k(x)} H_i(C_\bullet \otimes_A k(x)),$$

where $H_i(C_\bullet \otimes_A k(x)) := \ker(d_i \otimes k(x))/\mathrm{im}(d_{i+1} \otimes k(x))$ is the i-th homology of $C_\bullet \otimes_A k(x)$.
(i) Show that for $i \in \mathbb{Z}$ the function β_i is upper semicontinuous.
(ii) Show that for $i \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$ the set $\beta_i^{-1}(n)$ is constructible in X.
(iii) Give an example showing that β_i need not be locally constant.

Exercise 3 (4 points):

Let R be a discrete valuation ring with fraction field K and uniformizer $\pi \in R$. Set $I_1 := (\pi T - 1) \subseteq R[T, T_1, T_2]$, $I_2 := (T_1, T_2) \subseteq R[T, T_1, T_2]$ and $A := R[T, T_1, T_2]/I_1 \cap I_2$ with natural morphism $f\colon X := \mathrm{Spec}(A) \to S := \mathrm{Spec}(R)$.
i) Show $X \cong X_1 \cup X_2$ with $X_1 \cong \mathbb{A}^2_K$, $X_2 \cong \mathbb{A}^1_R$ and that X is equidimensional of dimension 2.
ii) Show that $X_1 \cap X_2 = \{x\}$ for a closed point $x \in X$ and that $\dim \mathcal{O}_{X_1,x} = 2$ and $\dim \mathcal{O}_{X_2,x} = 1$.
iii) Draw a picture of $f\colon X \to S$.

Exercise 4 (4 points):

Let A be a ring and let $f_\bullet\colon C_\bullet \to D_\bullet$ be a morphism of complexes of A-modules. We define the mapping cone $C(f)_\bullet$ to be the complex with terms $C(f)_n := D_n \oplus C_{n-1}$ and differential $d_n^{C(f)_\bullet} := \begin{pmatrix} d_n^D & f_{n-1} \\ 0 & -d_n^C \end{pmatrix}$, where $d_n^C\colon C_n \to C_{n-1}$, $d_n^D\colon D_n \to D_{n-1}$.
(i) Show that $(C(f)_\bullet, d_\bullet^{C(f)_\bullet})$ is a complex.
(ii) Construct a long exact sequence

$$\ldots \to H_n(C_\bullet) \overset{H_n(f_\bullet)}{\to} H_n(D_\bullet) \to H_n(C(f_\bullet)) \to H_{n-1}(C_\bullet) \to \ldots$$

on homology.
Hint: Consider the natural short exact sequence $0 \to D_\bullet \to C(f_\bullet) \to C_\bullet[1] \to 0$, where $C_\bullet[1]$ denotes a "shift" of $C_\bullet$. Then identify the connecting morphism in the associated long exact sequence.

To be handed in on: Thursday, 02.05.2024 (during the lecture, or via eCampus).