Dr. I. Gleason **SS 2024**
Dr. J. Anschütz

Algebraic Geometry II

1. Exercise sheet

Exercise 1 (4 points):

Let A be a ring and M an A-module. Show that the following are equivalent:
(i) M is finite locally free, i.e., there exists an open covering $\mathrm{Spec}(A) = \bigcup_{i\in I} U_i$ such that $\widetilde{M_{|U_i}} \cong \mathcal{O}_{U_i}^{n_i}$ for each $i \in I$ with some $n_i \geq 0$.
(ii) M is finitely presented and flat.
(iii) M is finitely generated and projective.
Hint: First show that (i) is equivalent to (ii). Then use for "(i),(ii)⇒ (iii)" that $\mathcal{H}om_{\mathcal{O}_X}(\widetilde{N}, -) \cong \widetilde{\mathrm{Hom}_A(N, -)}$ *on* $X = \mathrm{Spec}(A)$ *for each finitely presented A-module N.*

Exercise 2 (4 points):

Let X be an integral scheme, which is of finite type over a field k. Show that the normalization $\widetilde{X} \to X$ is flat if and only if it is an isomorphism.
Hint: Use Exercise 1 and finiteness of $\widetilde{X} \to X$.

Exercise 3 (4 points):

Let k be a field.
(i) Show that the normalization $\widetilde{X} \to X$ of $X = V(y^2 - x^3 - x^2) \subseteq \mathrm{Spec}(k[x, y])$ is generalizing, but that each projection $\widetilde{X} \times_X \tilde{X} \to \widetilde{X}$ is not generalizing if $\mathrm{char}(k) \neq 2$.
(ii) Show that the blow-up $\mathrm{Bl}_{(x,y)}(\mathrm{Spec}(k[x, y])) \to \mathrm{Spec}(k[x, y])$ is not flat.

Exercise 4 (4 points):

Let A be a ring. Let $C_\bullet := (\ldots C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \to 0 \to 0 \to \ldots)$ be a complex of A-modules, which is concentrated in (homological) degrees ≥ 0. Set $Z_i := \ker(d_i)$, $B_i := \mathrm{im}(d_{i+1})$ for $i \in \mathbb{Z}$. Assume that C_i is a projective A-module for $i \in \mathbb{Z}$ and that $B_i = Z_i$ for any $i \in \mathbb{Z}$, i.e., $C_\bullet$ is exact. Show that $C_\bullet$ is homotopic to 0, i.e., there exist morphisms $h_i \colon C_i \to C_{i+1}$ such that for each $i \in \mathbb{Z}$

$$d_{i+1} \circ h_i + h_{i-1} \circ d_i = \mathrm{Id}_{C_i}.$$

Hint: Construct h_i inductively starting with $h_i = 0$ for $i \leq -1$. Try to only use the property that if C is a projective A-module and $M \to N$ a surjection of A-modules, then any morphism $C \to N$ of A-modules can be lifted to a morphism $C \to M$ of A-modules.

To be handed in on: Thursday, 25.4.2023 (during the lecture, or via eCampus).