Dr. I. Gleason
Dr. J. Anschütz **SS 2024**

Algebraic Geometry II

1. Exercise sheet

Exercise 1 (4 points):

Let R be a Dedekind ring. Show that an R-module M is flat if and only if M is torsion free.
Hint: For the "if" statement reduce to R being a principal ideal domain by localizing at maximal ideals. Then reduce to the case that M is finitely generated.

Exercise 2 (4 points):

Let R be a Dedekind ring and A an integral domain. Show that $R \to A$ is flat if and only if $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ sends the generic point to the generic point.

Exercise 3 (4 points):

Let $f\colon Y \to X$ be a non-constant holomorphic map of connected, compact Riemann surfaces. Let $\mathbb{D} \subseteq \mathbb{C}$ be the open unit disc.
i) Let $g\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic map with $g(z) \neq 0$. Show $\mathbb{D} \to \mathbb{D},\ z \mapsto z \cdot g(z)$ is biholomorphic in a neighborhood of 0 and that for $n \geq 1$ there exists an open neighborhood $U \subseteq \mathbb{D}$ of 0 and a holomorphic function $h\colon U \to \mathbb{D}$ with $h^n = g_{|U}$.
Remark: Using that $\mathbb{D}$ is simply connected, one can even arrange that $U = \mathbb{D}$.
ii) Show that for $y \in Y$ there exists $n_y \geq 1$, open neighborhoods $U \subseteq Y$ of y, $V \subseteq X$ of $f(y)$ with $f(U) \subseteq V$ and isomorphisms $U \cong \mathbb{D}, V \cong \mathbb{D}$ such that $f_{|U}$ identifies with the morphism $\mathbb{D} \to \mathbb{D},\ z \mapsto z^{n_y}$.
iii) Deduce that for $y \in Y$ the number n_y depends only on y (and not the chosen data in (ii)) and that the branching number $\sum_{y \in Y}(n_y - 1)$ is finite.

Exercise 4 (4 points):

Let A be a ring.
(i) Show that the A-module $I := \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ is injective, i.e., the functor $\mathrm{Hom}_A(-, I)$ sends short exact sequences of A-modules to short exact sequences.
(ii) Show that for each A-module M, there exists an injective A-module J and an injection $M \to J$ of A-modules.
Hint: You may use that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module (by Baer's criterion). Show that products of injective A-modules are again injective.

To be handed in on: Thursday, 18.4.2023 (during the lecture, or via eCampus).